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# A learning trajectory of the accumulation function in multiple-linked representational environment

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*Our study was designed to identify a learning trajectory for the accumulation function as it learned with multiple representational technology based artifact. Thirteen pairs of 17-year-old students were asked to explain the connections between multiple-linked representations and to conjecture about the mathematical relationship embedded in the artifact. The study was guided by the semiotic mediation theory and the theory of knowledge objectification. We include two rounds of analysis: one to detect mathematical elements involved in learning the accumulation function, another to identify the progression process of learning the accumulation function. The data analysis identified four phases in the processes of objectification along which we suggest a trajectory for learning the accumulation function concept.*

**Keywords:** Accumulation function, multiple-linked representation, semiotic mediation, objectification, learning path.

## INTRODUCTION

Constructing a learning trajectory for students is one of the most daunting and urgent issues facing mathematics education today (Steffe, 2004). The main issue concerns the promotion of students' development of new mathematical concepts, especially concepts the development of which is challenging. The accumulation concept, which central to the idea of integration, is one of such concept. Learning the integral concept based on the accumulation function may create opportunity for the student to make the connection between the derivative and the integral which is essential for meaningful learning of calculus (Thompson et al., 2013). Simon (1995) offered the Learning trajectory as a way to explicate an important

aspect of pedagogical thinking involved in teaching mathematical understanding. According to Simon, learning trajectory consists of three components: (a) the students' learning goal, (b) a hypothesis about the process of the students' learning of mathematical concept, and (c) tasks to be used to promote the students' learning (Simon, 1995).

Studies regards the learning of the accumulation function concept (Thompson, Byerley, & Hatfield, 2013), have examined the role of graphic and numeric artefacts in learning the accumulation function. In these studies, technology was used mostly in attempts to relate a function with its accumulation function in a simulated-physical situation such that of time graphs involving velocity and position (e.g., Noble, Nemirovsky, Wright, & Tierney, 2001). The present study examines the learning of the integral concepts analytically by high school students as they learn it with dynamic, interactive, and multiple-representation artifacts.

Multiple-representational artifacts enable the development of learning environments that present in separate linked windows the accumulation and its rate of change functions. In an attempt to identify a learning trajectory which help students conceptualizing the accumulation function, our study was designed to explore the objectification processes of the accumulation function when it learn by multiple-linked representations and interactive artifact. To do so, we explore the ways in which high school students learn the accumulation function with multiple-linked representation artifact.

## THEORETICAL FRAMEWORK

The study was carried out using the theoretical framework of semiotic mediation (Bartolini Bussi

& Mariotti, 2008) and the theory of knowledge objectification (Radford, 2003). The semiotic mediation theory considers learning as an alignment between the personal meanings arising from the use of a certain artifact for the accomplishment of a task and the mathematical meanings that are deposited in the artifact. Personal meanings are to refer “to a state in which a learner believes/feels/thinks (tacitly or explicitly) that he has grasped the cultural meaning of an object (whether he has or has not),” and a mathematical meanings are to refer, “to the extent that its usage is congruent with its usage by the mathematical community” (Berger, 2005, p. 83). In the context of using artifacts, the semiotic mediation theory describes the relations between personal meanings and mathematical meanings as a double semiotic relationship. On one hand, concentrating on the use of the artifact for accomplishing a task, recognizing the construction of knowledge within the solution of the task. On the other hand, analysing the use of the artifact, distinguishing between the personal meanings arising from the use of the artifact in accomplishing the task and meanings that an expert recognizes as mathematical when observing the students’ use of the artifact in order to complete the task. Radford (2003) suggests a semiotic tool to analyse the dynamically evolving relationship between personal and mathematical meanings. The basic components of the semiotic tool are the students’ progressive attention and awareness of the mathematical object. Varieties of semiotic means of objectification that have a representational function attract the students’ attention to mathematical objects. Furthermore, the properties of the artifact can help students attend to the mathematical objects related to the activity under consideration. Paying attention to the necessary aspects of the mathematical phenomenon and using various semiotic means of objectification, students become aware of the attributes of mathematical objects within that phenomenon. Being aware, students attain objectification of the mathematical objects, which then become apparent to them through various devices and signs.

Given the cultural knowledge deposited in any designed artifact, we consider the linked graphs, the table of values, and the control tools that are the focus of the present study as culturally determined signs that convey knowledge regard the accumulation function. The accumulation function is represented symbolically

as  $\sum_{i=1}^n f(x_i) \cdot \Delta x_i$ . The mathematical structure of the accumulation function involve variety of mathematical elements. Such as  $\Delta x$ , product  $f(x)\Delta x$ , and sum of products  $\sum f(x)\Delta x$ . These elements are deposited in the artifact used in this study.

The students’ learning goal is to come up with conjectures about the mathematical relationship of the accumulation function. We hypothesize the students should become aware of the mathematical elements of the accumulation function as they interact with the artifact. Our aim is to explore the ways in which students become aware of the idea of accumulation as it is represented in the artifact. We propose to answer the following research question:

How and what are the mathematical elements used by the students in the process of becoming aware of the mathematical meanings of the accumulation function when it is learned with multiple-linked representational artifact?

### SEMIOTIC POTENTIAL OF THE ARTIFACT

The interactive integral and accumulation artifacts at the heart of our study are part of the calculus unlimited (CUL) artifact. As a multi-representational artifact, CUL contains different types of tools that we grouped into four categories.

- I) *Graphing tools*: Two vertically aligned Cartesian systems, coordinated vertically. The trajectory in the upper Cartesian system signifies a function. The function is defined symbolically by the free input of a single variable expression. The trajectory in the bottom system presents the values of Riemann sums  $\sum_{i=1}^n f(x_i) \cdot \Delta x_i$ .
- II) *Numeric tools*: The associated table of values contains three columns. The left column presents the upper boundary values, the middle column presents the delta x values and the right column shows the accumulated values.
- 3) *Accumulation tools*: Because our study focused on the computation of rectangles, the rectangles that appear on request in the upper Cartesian system (the function system) represent the product of  $f(x_i + \Delta x_i)\Delta x_i$ . Rectangles are color-coded to reflect the product sign (positive or negative) (Figure 1).

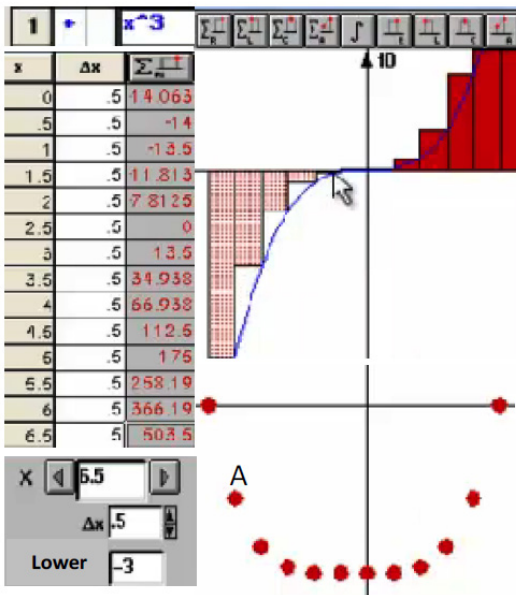


Figure 1: CUL interface

IV) *Boundaries tool*: After the function  $f$  and the method of accumulation are specified, three parameters determine the value of the accumulation at a given point: the lower and upper boundaries of a bounded region and the width of each interval into which the region should be divided. The design of CUL attempts to direct the attention of the user to these parameters and emphasizes especially the control over the upper boundary value, with its immediate visual feedback of a value of accumulation points. Students control the bounded area by using arrows to move a marker in intervals of  $\Delta x$  to the left or to the right of the lower boundary ( $\Delta x$  is an absolute measure of the interval, thus always achieving a positive value both right and left of the lower boundary). The upper boundary value is represented in the accumulation function graph by a marked coloured point and in the table of values.

**STUDY DESIGN**

The present study explores approximately 12 hours of learning by 13 pairs of 17-year-old students from

The task is to explore and explain possible connections between graphs in the upper and the lower Cartesian systems and the linked table of values. You should be able to input function expressions and obtain a pair of graphs such as  $f(x) = x^2, x^2-9, x^3, (x+3)(x-1)(x+4)$ . While working on the task you may use the CUL artifact to generate graphs by means of symbolic expression, and change the value of X by pressing the  $\blacktriangleleft$   $\blacktriangleright$  arrows in the upper part of the value box, as needed. As default values you can ask to specify 0.5 in  $\Delta x$  box, -3 in the lower box, and -3 in the X box.

Figure 2

two different schools in Israel. The students volunteered to participate in five after-school meetings. At the time the meetings took place, the students had already learned the concepts of function and derivative. The Author introduced them briefly to the interface and illustrated how to use it. He explained, for example, how to input the symbolic expression, how to change the controlling parameters. In particular, the students were told about the technical functionality of the artifact.

To promote the processes of objectification of the accumulation function, we asked the students to explain and explore the possible connection between two given function graphs and the table of values. They were given the instructions in Figure 2.

The learning took place in the computer lab at school. Students were video-recorded and the corresponding computer screens were captured.

Data Analysis. The data analysed through two levels: (a) the macro level: identifies the mathematical elements involved in objectifying the accumulation function. Mathematical elements were defined as segments of discourse in which the students sought to discover the mathematical relationship inherent the accumulation function, for example, delta x, product, sum of product, and the positions of the accumulation function graph. (b) at the meso level, we analyzed the data in two stages: (a) we distinguished personal meanings that are accepted mathematically from those that are not. Statements that in the context of the accumulation function are mathematically wrong were defined as personal meanings that are not accepted mathematically. By contrast, statements that are mathematically correct in the context of the accumulation function were defined as personal meanings that are accepted mathematically. For example, statements like “the y-value of the initial point in the accumulation function is equal to the height of the rectangle,” which refers to the objectification product

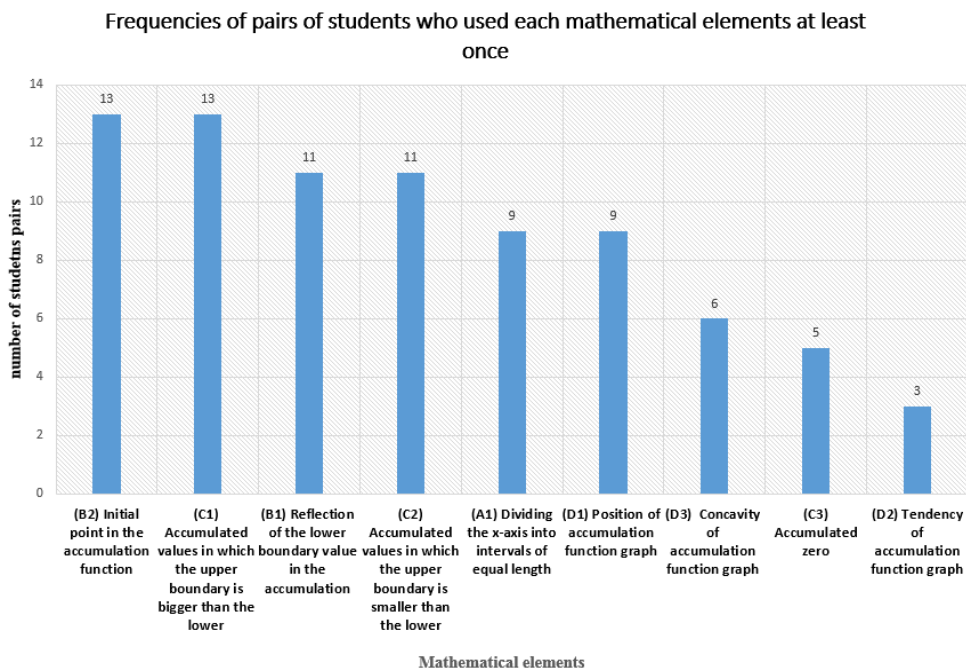


Figure 3: Frequencies of pairs of students who used each elements at least once

were defined as personal meanings that are not accepted mathematically. For each value of  $\Delta x$  other than 1, the initial point in the accumulation function is equal to the product of the rectangle dimensions, but not equal to the height of the rectangle. For this reason, we defined this statement as a personal meaning that is not accepted mathematically. Statements like “the y-value of the initial point in the accumulation function is the same as the first rectangle area” were defined as personal meanings that are accepted mathematically. (b) we applied the strategy of evolution over time to each pair of students, to analyze the evolution of personal meanings into mathematical ones over time.

**FINDINGS**

The detected elements were grouped into four categories: (a) objectifying  $\Delta x$  as splitting the x-axis into equal-length segments, (b) objectifying the product, (c) objectifying the sum of product, and (d) objectifying the properties of the accumulation function graph. Categories (b), (c), and (d) were divided into subcategories based on the mathematical element that is under consideration. In the graph below, we summarized the frequencies of pairs of students who used each element at least once in the learning process.

We found that all the students tried to objectify the product  $f(x)\Delta x$ , which are represented by the initial value point in the accumulation function graph (point

A, Figure 1). Furthermore, all of them tried to objectify the sum of product in a domain in which the upper boundary values are bigger than the lower boundary values (B2, C1). The majority of students (11 pairs) considered the reflection of the lower boundary value in the accumulation function graph (B1). Eleven pairs of students also tried to identify the accumulation function in a domain in which the upper boundary values are smaller than the lower boundary values (C2). Nine pairs tried implicitly to identify  $\Delta x$  as dividing the x-axis into segments of equal length (A1). Nine pairs also considered the position of the accumulation function graph relative to the x-axis (D1). Six pairs tried to objectify the concavity of the accumulation function graph (D3). Five pairs considered the accumulated zero (C3), and three pairs considered the tendency of the accumulation function graph (D2)

**Paths followed to learn the accumulation function**

In Table 1 we present a distinction between personal meanings that are accepted mathematically and those that are not. The number of episodes associated with each category is also shown in Table 1. For example, in row B2 the number 41 indicates that 41 of all the episodes are associated with the mathematical element ‘product’ represented by the “Initial value point in the accumulation function.” Eighteen of these were considered to be accepted mathematically and 23 were not.

We counted a total of 211 episodes containing mathematical elements. Five percent (10/211) of the episodes were devoted to identifying delta x as dividing the x-axis into intervals of equal length. Thirteen percent (27/211) were devoted to identifying the reflection of the lower boundary value in the accumulation function graph. All of these but one were coded as meanings accepted mathematically. These findings may indicate that the objectification of these two mathematical elements was not challenging for the students. It is possibly that the design of the artifact helped students objectify these mathematical elements. Twenty one percent of the episodes (41/211) were devoted to the objectification of the initial value point. Twenty-three of these were coded as personal meanings that are not accepted mathematically. This finding may indicate that objectification of the product was challenging for the students.

Twenty seven percent of the episodes (57/211) were devoted to objectification of the accumulated values when the upper boundary value is bigger than the lower one. Forty-four episodes of these were coded as personal meanings that are accepted mathematically. In contrast, among the 31 episodes that were devoted to objectification of the accumulated values when the upper boundary value is smaller than the lower one, only four were coded as personal meanings that are accepted mathematically. This finding indicates that the students were better able to objectify the sum of product when the upper boundary is bigger than the lower boundary than the other way around.

Seventeen percent of the episodes (35/211) were devoted to objectification of the properties of the accumulation function. Only three episodes of these were coded as personal meanings that are not accepted mathematically. Twenty-two episodes were devoted to objectifying the position of the accumulation function graph. Two of these were coded as personal meanings that are not accepted mathematically. Four episodes were devoted to the tendency of the accumulation function, and all of them were coded as accepted mathematically. Nine episodes were devoted to the concavity of the accumulation function graph and coded as accepted mathematically. This finding indicates that the students who noticed the last two elements also became aware of their mathematical meaning.

Below we present evolutionary path in the learning process, which we identified through the data analysis. A learning path refers to the organization of learning activities in a proper order composed of knowledge elements, designed task, and artifact such that students can effectively study a subject area.

The chart in Figure 4 show the order in which students were engaged with various mathematical elements, showing also the time elapsed. The data are color-coded. The mathematical elements appear in the left column. The numbers at the bottom represent the time in minutes. We divided each minute into four squares, each one representing 15 seconds. The colours distinguish between personal and mathematical meanings. The grey colour represents personal

<i>Elements</i>	<i>Accepted</i> $\Delta$	<i>Not accepted</i>	<i>Total</i>
<i>A: Delta x</i>			
<i>A1: Dividing the x-axis into intervals of equal length</i>	10	0	10
<i>B: Objectifying the product <math>f(x) \cdot \Delta x</math></i>			
<i>B1: Reflection of the lower boundary value in the accumulation function</i>	26	1	27
<i>B2: Initial value point in the accumulation function graph</i>	18	23	41
<i>C: Objectifying the accumulation function as sum of products</i>			
<i>C1: Accumulated value in which the upper boundary is bigger than the lower</i>	44	13	57
<i>C2: Accumulated value in which the upper boundary is smaller than the lower</i>	4	27	31
<i>C3: Accumulated zero</i>	10	0	10
<i>D: Objectifying properties of the accumulation function graph</i>			
<i>D1: Position of the accumulation function graph</i>	20	2	22
<i>D2: Tendency of the accumulation function graph</i>	4	0	4
<i>D3: Concavity of the accumulation function graph</i>	8	1	9

**Table 1:** Frequency of mathematical elements distinguishing personal meanings that are and are not accepted mathematically



understanding the product are not necessarily related to performing calculations but rather to how the product is formed. Our findings indicate that the complexity of becoming aware of the product was not related to performing the calculation but to the semiotic structure of the multi-representations. Although, the complexity that accompany the objectification of the product, eventually the students were able to overcome this complexity. The properties of the function graph and controlling the upper boundary afford the students to objectify the product. Varying the upper boundary parameter in a discrete manner and its reflection on the multi-representations drew the students' attention to the connection between the emerging rectangles and the accumulation function graph. The semiotic potential of the artifact encourages the emergence of personal meanings and that the control tools promote evolution toward the accepted mathematical meaning. Concerning objectifying the sum of product, the students have partially become aware of the sum of product. On one hand, objectifying the sum of product in which the upper boundary is bigger than the lower was less challenging for them (Table 1). On the other hand, despite our attempt to provide students with a wide variety of graphs to help them objectify the meaning of the accumulation function for each value of  $x$ , they were not able to objectify the accumulation graph in which the upper boundary is smaller than the lower boundary value. This finding is not completely consistent with Sealey's findings that "None of the students spent much time explicitly discussing the concepts represented in the summation layer" (p. 240).

In spite of, the properties of the accumulation function graph is not mentioned implicitly in the mathematical structure. However, becoming aware to the properties of the accumulation function graph is epistemologically important since the transition from local to global observation means an evolution of the meanings because of theoretical reasons. Hence, we believe that this phase should be included in the learning path for objectifying the accumulation function.

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