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The Value of Mediated Communication[☆]

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Abstract

Kamenica and Gentzkow (2011) consider a model in which a sender chooses a public communication device for signaling his information to an uninformed receiver, who then takes an action that affects the welfare of both individuals. In their model, the sender is fully committed to truthfully communicate the signal to the receiver, so that they abstract from incentive compatibility issues. By considering mediated communication, we provide an analytical framework overcoming this overly restrictive assumption. We use the concept of virtual utility to develop a geometric approach to the sender's optimization problem. We characterize the value of persuasion from the concavification of a non-revealing function over beliefs. We apply our approach to a model of information transmission based on Crawford and Sobel (1982). In this setting, we provide necessary and sufficient conditions for the sender to benefit from his private information.

Keywords: Bayesian persuasion, mediated communication, incentive compatibility, virtual utility.

JEL Classification: D82, D83.

1. Introduction

This paper provides an analytical framework for studying *Bayesian persuasion* problems in which the sender cannot commit himself to truthfully communicate the signal to the receiver, so that incentive compatibility becomes one of the major issues for communication to be meaningful. By allowing the two players to communicate with a neutral third party who makes non-binding recommendations, we are able to solve two analytical problems that could possibly prevent a tractable analysis of incentive compatibility: first, truthful revelation of information when communication is direct may considerably limit the ability of the sender to credibly signal his information by himself. In particular, Forges (1985) and Farrell (1993) propose some examples in which no substantive communication can occur between the players. However, it is well known that the set of implementable outcomes may be strictly larger when players use mediated rather than direct communication (see for instance, Forges (1985, 1990)). Second, revelation of

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influential information with direct communication requires the sender to be indifferent between all signals he sends with positive probability. This is a strong form of incentive compatibility whose formulation is difficult to deal with¹. In contrast, when the players use a mediator to perform the randomization of the signals on behalf of the sender, incentive compatibility will only demand each type of the sender to prefer the expected allocation designated for him.

On a recent pioneering work, Kamenica and Gentzkow (2011) offer a general approach to Bayesian persuasion under full commitment on the part of the sender. They consider a sender-receiver game in which before learning his type (ex-ante stage), the sender publicly chooses a signaling strategy, i.e., a conditional distribution of signals for each of his types², that he will use for transmitting his information to an uninformed receiver (signaling stage). The sender produces a signal according to his true type and the corresponding distribution of signals. He cannot distort the signal realization, nor he can misrepresent his information. The receiver observes the signal realization and then takes an action that affects the welfare of both individuals³. Kamenica and Gentzkow characterize the sender's problem as a constrained splitting of the total prior probability into a distribution over posterior beliefs. Proceeding as in Aumann and Maschler (1995), they derive a non-revealing payoff function and then compute both the sender's optimal expected payoff (value of persuasion) and the optimal signaling strategy from the *concavification* of that function⁴. Although this model provides good insights into the problem of finding optimal information structures, in most of the relevant social, economic and political settings, the full commitment assumption is overly restrictive. By allowing players to communicate through a mediator, we overcome this difficulty. Specifically, we are able to characterize incentive constraints by a set of linear inequalities. As a result, we can use an alternative geometric approach based on duality theory. This enables us to transform the sender's problem into a simplified problem without incentive constraints, that can be solved using the concavification of some non-revealing payoff function.

Under mediation, the sender communicates with a neutral trustworthy mediator who then recommends an action to the receiver. The mediator's recommendation is not binding, that is, the receiver is free to choose any action different from the recommended one. The mediator can only create value by controlling the flow of information between both players. He introduces noise in the communication, which may relax the incentive constraints faced by the sender. The revelation principle applies, so that, without loss of generality we can restrict attention to mediation protocols in which the sender reports his type truthfully and the receiver obeys the prescribed recommendation. A mediation protocol in which the sender always reports the truth and the receiver always follows the recommendation is called a *communication equilibrium* (see Myerson (1986) and Forges (1986)). In this setup, the problem of the sender is to select a communication equilibrium maximizing his ex-ante payoff. The set of communication equilib-

¹For instance, solving principal-agent problems with adverse selection and limited commitment in which communication is direct may require a laborious manipulation of incentive constraint (see Bester and Strausz (2001)).

²Kamenica and Gentzkow (2011) refer to a signaling strategy as a signal. In order to distinguish the conditional distribution from its realizations, they call this latter *signal realization*. We deliberately use the terminology developed in the literature of communication games.

³Brocas and Carrillo (2007), Rayo and Segal (2010), and Tamura (2014) study some related models.

⁴The concavification of a function f is the least concave function that is larger or equal to f .

ria is defined by a system of linear inequalities from which the dual variables (shadow prices) can be used to define the sender's *virtual utility* (see Myerson (1991, ch. 10)). These virtual utility scales incorporate into the utility function the signaling costs associated with incentive compatibility. Using the concept of virtual utility we construct a fictitious game in which there are no incentive constraints and the sender's payoffs are in the virtual scales. For this game, the optimal splitting of the prior probability guarantees the sender an expected virtual payoff equal to the concavification of a certain non-revealing payoff function. Our main result is that the sender's optimal expected payoff equals the value of persuasion in the fictitious game with virtual scales defined by the optimal signaling costs (Theorem 1). We also characterize the optimal mediation protocol: we show that the optimal splitting of the prior probability is constrained by the Bayes plausibility (martingale property) together with a *complementary slackness* condition (Proposition 1).

Unlike in the full commitment model, in which the optimal number of signals does not exceed the number of types, when misrepresentation is problematic, incentive compatibility may increase the amount of information the sender needs to signal in order to attain the optimal value (a detailed analysis is provided in section 3.4). The idea is that the sender might require to signal as much information as when he is fully committed, but also he needs to make such revelation credible to the receiver. This is illustrated by means of some eloquent examples (see section 3.1). It is also shown that the optimal value of persuasion may not be a concave function as in the full commitment model. In fact, it may contain convex segments lying strictly below the concavification of the (full commitment) non-revealing function.

We conclude the paper with an application of our approach to a model of information transmission with the same qualitative features as the one of Crawford and Sobel (1982). In this setting, we provide necessary and sufficient conditions for the sender to benefit from his private information.

Related literature. Our analytical framework is the same as in Kamenica and Gentzkow (2011), except that we consider a more general interaction situation in which communication is mediated and the sender may distort his private information. In that respect, our paper relates to the recent literature on information design known as Bayesian persuasion. To our best knowledge, this literature so far has been rather unsuccessful in developing a tractable approach for an explicit analysis of the sender's informational incentive compatibility. Perez-Richet (2014), Alonso and Câmara (2014) and, Hedlund (2014) study Bayesian persuasion when the sender privately observes some information *before* choosing a public signaling strategy⁵. When the sender fully commits to a public signaling strategy prior to learning any private information, the interaction is ultimately nonstrategic: the sender solves a constrained decision problem. In contrast, when the sender is privately informed, the persuasion game becomes a signaling game whose Bayesian equilibria are implicitly characterized by some informational incentive constraints. It is worth mentioning that these contributions assume that communication is direct. On the other hand, Kolotilin, Li, Mylovanov and Zapechelnyuk (2015) study Bayesian persuasion

⁵These papers are connected to a strand of literature in mechanism design that considers the choice of a mechanism by an *informed principal* (Myerson (1983), Maskin and Tirole (1990, 1992)).

with a privately informed receiver. In their framework, the sender designs a communication device that gathers information from the receiver and then sends a recommendation to the receiver conditional on her report and the sender's type. In addition to the strategic incentive constraints ensuring that the receiver will follow the recommendation, the sender is also led to consider informational incentive constraints guaranteeing that the receiver finds it optimal to report truthfully her information. Here the communication device is a mediation rule unable to verify the receiver's private information, but capable of identifying the sender's type.

This paper also relates to the literature on contracting with limited commitment. This literature considers a principal-agent setup in which the principal (receiver) is imperfectly informed about the agent's (sender) type. The principal cannot contractually commit herself to chose any action, however, she may extract information from the agent by using a communication protocol. Bester and Strausz (2001) study direct communication in which the agent simply sends a single message to the principal. In contrast, Bester and Strausz (2007) allow the principal to use general communication devices which may enlarge the set of implementable contracts. Contrary to Bayesian persuasion, here the communication device (mechanism) is designed by the uninformed party, i.e., the principal. However, because the agent cannot commit to truthfully transmit his information, informational incentive compatibility is a relevant matter.

Mitusch and Strausz (2005) and, Golstman, Hörner, Pavlov and Squintani (2009) compare different communication protocols in the framework of Crawford and Sobel (1982). In this regard, our paper is also connected with this literature. They study the conditions under which mediation improves upon direct communication. As with contracting problems with adverse selection and limited commitment, it is also assumed that the communication procedures are designed to maximize the ex-ante welfare of the receiver.

Finally, by using the concept of virtual utility, we borrow some analytical tools that were developed by Myerson (1984a,b) in order to extend the Nash bargaining solution and the Shapley value to cooperative games with incomplete information.

This paper is organized as follows. Section 2 is devoted to formally describing the basic sender-receiver scenario. The concept of communication equilibrium is also defined. Section 3 introduces the mediated persuasion problem and the virtual utility approach. The main results are presented. It also contains some eloquent examples illustrating our findings. Section 4 provides an application of our analytical framework to a model of information transmission. Finally, Section 5 contains concluding remarks. All proofs are relegated to Section 6.

2. Basic Game

Our basic framework is a two-person finite Bayesian game in which player 1 has no decision to make, but he is the only player to have private (nonverifiable) information. Let K be the (finite) set of types of player 1. A type $k \in K$ is chosen according to⁶ $p \in \Delta(K)$, and only player 1 is informed about k . We assume that $p^k > 0$ for every $k \in K$. Player 2 chooses an action in a (finite) set J . When action j is chosen by player 2 and player 1 is of type k , then player 1 and player 2 get respective payoffs a_j^k and b_j^k . We refer to this game as $\Gamma(p)$.

⁶For any finite set A , $\Delta(A)$ denotes the set of probability distributions over A .

A (*mediated*) *communication device* δ is a mapping $\delta : K \rightarrow \Delta(J)$, namely a vector of probability distributions $(\delta^k)_{k \in K}$ over J for every $k \in K$. By adding a communication device δ to the game $\Gamma(p)$, one generates an extended game $\Gamma_\delta(p)$, which is played as follows:

1. A type $k \in K$ is randomly chosen according to p .
2. Player 1 learns his type $k \in K$.
3. Player 1 sends a confidential report $k' \in K$ to a mediator.
4. The mediator chooses an action $j \in J$ with probability $\delta_j^{k'}$.
5. The mediator recommends the action j to player 2.
6. Player 2 chooses an action and both players receive payoffs as in $\Gamma(p)$.

For obvious reasons, we refer to player 1 in $\Gamma_\delta(p)$ as the sender, and player 2 as the receiver. In this game, a strategy for the sender is a transition probability $\tau : K \rightarrow \Delta(K)$ where $\tau(k' | k)$ is the probability to report k' if his type is k . A strategy τ is called *sincere* if $\tau(k | k) = 1$ for every $k \in K$, namely, if the sender always reveals honestly his type to the mediator. A strategy for the receiver in $\Gamma_\delta(p)$ is a transition probability $\varsigma : J \rightarrow \Delta(J)$ where $\varsigma(i | j)$ is the probability to choose i when j is recommended by the mediator. A strategy ς is called *obedient* if $\varsigma(j | j) = 1$ for every $j \in J$, i.e., if the receiver always follows the recommendation made by the mediator. When both players are sincere and obedient, respectively, the expected payoff of the sender of type k is

$$a^k(\delta) = \sum_{j \in J} \delta_j^k a_j^k$$

and the expected payoff of the receiver is

$$b(\delta) = \sum_{k \in K} p^k \sum_{j \in J} \delta_j^k b_j^k$$

The communication device δ is *incentive-compatible for the sender* if and only if the sincere strategy is a best response for the sender in $\Gamma_\delta(p)$ whenever the receiver is obedient, that is,

$$\sum_{j \in J} \delta_j^k a_j^k \geq \sum_{j \in J} \delta_j^{k'} a_j^k, \quad \forall k, k' \in K \quad (2.1)$$

The *informational incentive constraints* in (2.1) reflect the fact that neither the receiver nor the mediator can verify the private information of the sender (adverse selection problem).

Suppose action j is recommended to the receiver. Then, she computes posterior probabilities $p_j = (p_j^k)_{k \in K}$ given by

$$p_j^k = \frac{\delta_j^k p^k}{\sum_{k' \in K} \delta_j^{k'} p^{k'}}$$

The communication device δ is *incentive-compatible for the receiver* if and only if the obedient strategy is a best response for the receiver in $\Gamma_\delta(p)$ whenever the sender is sincere, namely,

$$\sum_{k \in K} \delta_j^k p^k b_j^k \geq \sum_{k \in K} \delta_j^k p^k b_i^k, \quad \forall i, j \in J \quad (2.2)$$

The *strategic incentive constraints* in (2.2) characterize the receiver's inalienable right to control her action in J (moral hazard problem). The two sides of (2.2) differ from the expected payoffs w.r.t. the posterior p_j because we have not divided by the total probability of receiving the recommendation to play j ; however, this factor can be ignored, since it is the same on both sides.

Definition 1.

We denote as $\mathcal{D}(p)$ the set of communication devices satisfying the strategic incentive constraints for a given prior p .

We define $F(q)$ as the set of receiver's optimal actions at belief $q \in \Delta(K)$, i.e.,

$$F(q) = \left\{ \sigma \in \Delta(J) \mid \sum_{k \in K} q^k \sum_{j \in J} \sigma_j b_j^k = \max_{j \in J} \sum_{k \in K} q^k b_j^k \right\}$$

Let $\pi_j(\delta) = \sum_{k \in K} p^k \delta_j^k$ be the probability of sending the recommendation j when δ is implemented. Then, δ is incentive compatible for the receiver if and only if $\pi_j(\delta) > 0$ for some $j \in J$ only if $j \in F(p_j)$.

We introduce some additional notation. For any conditional distribution $\delta^k \in \Delta(J)$, $\text{supp}(\delta^k)$ denotes its support. Also, for any communication device δ we define its support as $\text{supp}(\delta) \equiv \{j \in J \mid \pi_j(\delta) > 0\}$.

Definition 2 (Communication equilibrium).

A communication device δ is a *communication equilibrium* of $\Gamma(p)$ if and only if the sincere and obedient strategies form a Nash equilibrium of $\Gamma_\delta(p)$, that is, δ is incentive compatible for both the sender and the receiver. We let $\mathcal{D}^*(p)$ denote the set of communication equilibria of $\Gamma(p)$.

A communication equilibrium δ is *fully revealing* (FR) if $\cap_{k \in K} \text{supp}(\delta^k) = \emptyset$, so that the receiver can infer the true type of the sender by looking at the prescribed recommendation. It is *non-revealing* (NR) if $\delta^k = \delta^{k'}$ for every $k, k' \in K$, so that no useful information is revealed to the receiver. It is *partially revealing* (PR) if it is not NR nor FR.

REMARKS:

1. Assume that the game $\Gamma(p)$ is extended by allowing the players to communicate for a possibly infinite number of stages through a *general* communication device, sending signals for every player at every stage but also receiving messages from them. Such devices may involve preplay communication, before player 1 learns his type, but also interplay communication, after player 1 has learnt his type but before player 2 chooses his action. Then a *general revelation principle* holds: the set of all Nash equilibrium payoffs of all extensions of $\Gamma(p)$ by general communication devices coincides with the set of all communication equilibrium payoffs (see Forges (1986) and Myerson (1991, ch. 6)).
2. Plain conversation between the players seems to be more natural than mediated communication. Is it possible to achieve any communication equilibrium payoff by means of cheap-talk? Forges (1990) shows that there may exist communication equilibrium payoffs

that cannot be implemented as Nash equilibrium payoffs of any long cheap-talk extension of $\Gamma(p)$. However, communication equilibrium payoffs can be implemented as *correlated* equilibrium payoffs (in the sense of Aumann (1974)). Then a *feasibility theorem* holds: the set of all correlated equilibrium payoffs of all cheap-talk extensions of $\Gamma(p)$ coincides with the set of all communication equilibrium payoffs. In fact, any communication equilibrium payoff can be achieved as a correlated equilibrium payoff of the cheap-talk extension of $\Gamma(p)$ with one stage of information transmission (see Forges (1985)).

3. Communication equilibria are defined by a set of linear inequalities, hence the set $\mathcal{D}^*(p)$ is a convex polyhedron. Furthermore, this set is closed, bounded and non-empty. Indeed, the communication device δ defined by $\delta^k = \sigma$ for all $k \in K$ with $\sigma \in F(p)$ is a NR communication equilibrium of $\Gamma(p)$. As a consequence, the set of all communication equilibrium payoffs is also a convex polytope.

3. Mediated Persuasion

In the basic game $\Gamma(p)$, player 1 has the option to remain silent and let player 2 choose an action given p . He can also design a communication system to signal his private information, and try to persuade player 2 to change her action. We assume that player 1 publicly chooses a mediated communication device δ (i.e., a mediator) *before* learning his type. Then both players interact as in $\Gamma_\delta(p)$. Because the selection of the communication device is done at the ex-ante stage, this choice is by itself uninformative. By the revelation principle, player 1 can restrict attention to the mediated communication equilibria of $\Gamma(p)$. Then, the problem of player 1 is to choose a communication device $\delta \in \mathcal{D}^*(p)$ to solve

$$\max_{d \in \mathcal{D}^*(p)} \sum_{k \in K} p^k a^k(d) \quad (3.1)$$

We shall refer to this linear programming problem as the *primal problem for p*.

Definition 3 (Value of persuasion).

The optimal value of the primal problem for p will be called the *value of persuasion at p* and is denoted $a^*(p)$.

3.1. Mediated Persuasion Under Verifiable Information

Before studying the primal problem in detail, let us consider the more simplified persuasion game in which the type of the sender is verifiable by the mediator but not by the receiver. In such a situation, the informational incentive constraints are not relevant, so that our framework reduces to Kamenica and Gentzkow's (2011) model. The problem for the sender is thus

$$\max_{d \in \mathcal{D}(p)} \sum_{k \in K} p^k a^k(d) \quad (3.2)$$

Notice that the domain of maximization in (3.2) is the set of communication devices satisfying only the strategic incentive constraints, i.e., $\mathcal{D}(p)$. Under this verifiability assumption, the sender has nothing to communicate to the mediator. The only thing he has to do is to choose a communication device that will recommend an action to the receiver depending on his true

type. We let $j(q) \in F(q)$ denote the receiver's optimal action at belief q that maximizes the sender's ex-ante expected utility⁷. When both sender and receiver hold the same belief q , the maximal ex-ante utility that the sender can expect is

$$\hat{a}(q) \equiv \sum_{k \in K} q^k a_{j(q)}^k$$

We refer to the function \hat{a} as the *non-revealing payoff function*. Let $\text{cav } \hat{a}$ be the *concavification* of \hat{a} , i.e., the least concave function that is larger or equal to \hat{a} . As observed by Kamenica and Gentzkow (2011), the optimal value of the relaxed primal problem (3.2) is $\text{cav } \hat{a}(p)$. Also there exist a subset $I \subseteq J$ of actions with $|I| \leq |K|$ and posterior probabilities $(p_i)_{i \in I}$ with $p_i \in \Delta(K)$ for every $i \in I$, such that there exist a unique probability vector $\rho \in \Delta(I)$ satisfying

$$\sum_{i \in I} \rho_i \hat{a}(p_i) = \text{cav } \hat{a}(p) \quad \text{and} \quad \sum_{i \in I} \rho_i p_i = p \quad (3.3)$$

Then, it is possible to design a communication device δ so that the receiver will find that the conditional distributions over the types of the sender that she computes after receiving a recommendation are always elements of the set $\{p_i \mid i \in I\}$. Since the distribution ρ is unique, the receiver will infer with probability ρ_i the conditional distribution p_i . Then, by (3.3), the communication device δ guarantees the sender an expected payoffs of $\text{cav } \hat{a}(p)$. Such an optimal communication device can be easily computed from the following formula:

$$\delta_j^k = \begin{cases} \frac{p_j^k \rho_j}{p^k}, & \text{if } j \in I \\ 0, & \text{otherwise} \end{cases}, \quad \forall k \in K \quad (3.4)$$

Condition (3.3) on the left is thus an optimality condition, while condition on the right is called *Bayes plausibility*.

Notice that in the previous result the number of signals required for achieving the optimal value of (3.2) is bounded by the number of types of the sender. Namely, the most the sender needs to transmit to the receiver is just k , which has $|K|$ possible values.

Finally, we observe that for any $p \in \Delta(K)$, we have that

$$\hat{a}(p) \leq a^*(p) \leq \text{cav } \hat{a}(p) \quad (3.5)$$

The first inequality follows from the fact that $\hat{a}(p)$ can always be achieved by the NR communication equilibrium δ defined by $\delta_{j(p)}^k = 1$ for every k . The second inequality is due to the fact that $\mathcal{D}^*(p) \subseteq \mathcal{D}(p)$.

3.2. The Virtual Persuasion Game

As we have seen, when there are no informational incentive constraints, the solution to the primal problem can be easily characterized. Informational incentive constraints complicate matters

⁷If there is more than one action in $F(q)$ that maximizes sender's expected utility, we let $j(q)$ denote an arbitrary element of that set.

by interconnecting the signals in different states. However, we can remove the informational incentive constraints using duality theory. The idea is that the set of communication equilibria is defined by a system of linear inequalities (see remark 3) for which the dual variables can be used to define the sender's *virtual utility*. These virtual utility scales incorporate into the utility function the signalling costs associated with the incentive compatibility. Using the concept of virtual utility we can transform the original primal problem into a simplified problem without informational incentive constraints.

Let $\gamma(k' | k) \geq 0$ be the dual variable for the constraint that the type k of the sender should not gain by reporting k' in the primal problem for p . Following Myerson (1984a,b, 1991), we define the *virtual utility* of the sender from the action j , when his type is k , w.r.t. the prior p and the duals γ to be

$$\alpha_j^k(p, \gamma) = \frac{1}{p^k} \left[\left(p^k + \sum_{k' \in K} \gamma(k' | k) \right) a_j^k - \sum_{k' \in K} \gamma(k | k') a_j^{k'} \right] \quad (3.6)$$

Virtual utility of the sender is a distorted utility that magnifies the difference between his true type and the types that would be tempted to imitate him. In a situation where information is not verifiable, so that misrepresentation is possible, some types of the sender may adopt a different behavior in an effort to distinguish themselves from the types that try to mimic them. This new compromise in the payoff maximization goal of the sender may generate some signaling costs that are mathematically measured by the dual variables. The dual variables correspond to the “shadow prices” of strengthening incentive compatibility.

Economists have widely recognized the prominent character of the shadow prices (dual variables) of the resource constraints for measuring the marginal costs of scarcity. Also, it has been used for measuring the costs of distortionary taxation, regulations and market failures among others. In the same way, the dual variables of the informational incentive constraints together with the concept of virtual utility enhances our understanding of the ex-post inefficiencies derived from the signaling costs associated with incentive compatibility.

Let us assume that, as a consequence of the pressure that a type might feel in getting the receiver to trust him, the sender begins to act as if he were maximizing his virtual utility (Myerson (1991) refers to this idea as the *virtual utility hypothesis*). Thus, for some fixed prior probability p and signaling costs γ , consider the (p, γ) -virtual persuasion problem, a fictitious game that differs from the original persuasion game in the following. First, the sender's types are verifiable by the mediator (but not by the receiver), so that there are no informational incentive constraints. Second, the sender's payoffs are in the virtual utility scales $(\alpha_j^k(p, \gamma))_{j \in J, k \in K}$ instead of $(a_j^k)_{j \in J, k \in K}$. Let $i(q) \in F(q)$ denote the receiver's optimal action at belief q that maximizes the sender's ex-ante expected virtual utility⁸. When both sender and receiver hold the same belief q in the (p, γ) -virtual game, the non-revealing (virtual) payoff function is given by

$$\hat{\alpha}(q; p, \gamma) \equiv \sum_{k \in K} q^k \alpha_{i(q)}^k(p, \gamma)$$

⁸If there is more than one action in $F(q)$ that maximizes sender's expected virtual utility, we let $i(q)$ denote an arbitrary element of that set.

Then, as already observed in section 3.1, the value of persuasion in the (p, γ) -virtual game is given by $\text{cav } \hat{a}(p; p, \gamma)$. Although the (p, γ) -virtual game gives us some insights on how to simplify the sender's problem by removing the informational incentive constraints, it does not say anything about the "dual" relationship between the optimal value of the primal problem for p , $a^*(p)$, and the value of persuasion in the virtual game. Furthermore, it leaves open the question of determining the optimal signaling costs incurred by the sender in order to distinguish himself from the types that are tempted to imitate him. However, there exist an answer to the first question that will make the second question redundant.

Theorem 1.

For any prior $p \in \Delta(K)$ we have that

$$a^*(p) = \min_{\gamma \geq 0} \text{cav } \hat{a}(p; p, \gamma) \quad (3.7)$$

We refer to the minimization problem in (3.7) as the dual problem for p .

Fix a prior $p \in \Delta(K)$ and let $\gamma^*(p)$ be an optimal solution of the dual problem for p . Then, the key implication of theorem 1 is that the value of persuasion in the original game coincides with the value of persuasion in the $(p, \gamma^*(p))$ -virtual game, namely,

$$a^*(p) = \text{cav } \hat{a}(p; p, \gamma^*(p))$$

Thus, instead of saying that incentive compatibility restricts the sender's ability to signal his information, we may say that he is compelled to modify his actual preferences from the real to the virtual scales $(p, \gamma^*(p))$.

The virtual game is more than just a convenient construct for computing the value of persuasion. It gives us insights into the problem of finding optimal signals when misrepresentation is problematic. Myerson (1991) provides an extended discussion of the meaning and significance of the virtual utility.

Example 1. In the following game, player 1 has a privately known type that may be H or L and player 2 must choose an action from the set $J = \{j_1, j_2, j_3\}$. Payoffs for both players depend on 1's type and 2's action as follows:

a, b	j_1	j_2	j_3
H	1,3	3,1	-5,-3
L	-1,-3	2,-1	0,0

The receiver's optimal action at belief q is

$$F(q) = \begin{cases} j_3, & \text{if } 0 \leq q < \frac{1}{5} \\ \Delta(\{j_2, j_3\}), & \text{if } q = \frac{1}{5} \\ j_2, & \text{if } \frac{1}{5} < q < \frac{1}{2} \\ \Delta(\{j_1, j_2\}), & \text{if } q = \frac{1}{2} \\ j_1, & \text{if } \frac{1}{2} < q \leq 1 \end{cases}$$

The non-revealing value function \hat{a} joint with its concavification $\text{cav } \hat{a}$ are depicted in figure 1.

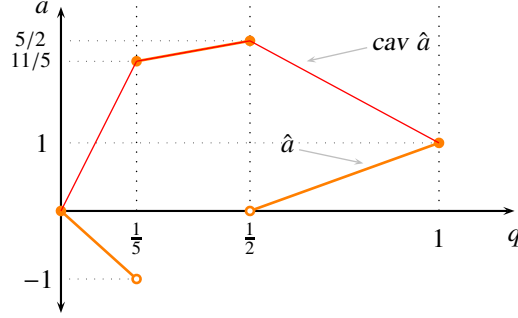


Figure 1: Function \hat{a} and its concavification

Let p be the prior probability of type H and consider the case in which $p < \frac{1}{5}$. In the absence of incentive constraints, an optimal communication device will lead to the posteriors $p_3 = 0$ and $p_2 = \frac{1}{5}$ with probabilities $1 - 5p$ and $5p$ respectively. However, this splitting of the total probability p cannot be made incentive compatible for the sender. The reason is that action j_2 is strictly preferred to action j_1 for the sender, hence type L would have incentives to report H in order to influence the posterior probabilities towards $\frac{1}{5}$. As a consequence, the sender cannot achieve the expected payoff of $\text{cav } \hat{a}(p)$ in any extension of the game $\Gamma(p)$, yet he can do better than $\hat{a}(p)$. Consider for instance the communication device $\delta_1^H = \delta_3^L = 1$ leading to the posterior probabilities $p_3 = 0$ and $p_1 = 1$. Straightforward algebra reveals that this device is a FR communication equilibrium giving an expected payoff to the sender equal to p . Although the sender cannot get $\text{cav } \hat{a}(p)$, he can guarantee at least p which is strictly better than $\hat{a}(p)$. Indeed, he can do even better as we will see in the sequel.

Tedious but easy computations show that, for any $p \leq \frac{1}{5}$, the optimal value of the dual variables solving the dual problem for p is

$$\gamma^*(H | L) = \frac{10p(1-p)}{3-11p} \equiv \gamma^*(p), \quad \gamma^*(L | H) = 0$$

Hence the (p, γ^*) -virtual utility game is

α, b	j_1	j_2	j_3
H	$1 + \frac{\gamma^*(p)}{p}, 3$	$3 - \frac{2\gamma^*(p)}{p}, 1$	$-5, -3$
L	$-(1 + \frac{\gamma^*(p)}{1-p}), -3$	$2(1 + \frac{\gamma^*(p)}{1-p}), -1$	$0, 0$

Figure 2 illustrates the non-revealing value function \hat{a} of the virtual game and its concavification. We conclude that the optimal value of the primal problem for $p < \frac{1}{5}$ is

$$a^*(p) = \text{cav } \alpha(p; p, \gamma^*(p)) = p + \gamma^*(p) < 11p = \text{cav } \hat{a}(p)$$

When $p < \frac{1}{5}$ both types of the sender will try to persuade the receiver to change her action from j_3 to j_2 . For this, the sender needs to induce the receiver to believe that he is of type H with a probability lying on $[1/5, 1/2]$. When the true state is H , the receiver is better by choosing j_2 ;

however, when the true state is L she would prefer to play j_3 . Thus, in state H the incentives of both players are aligned, while in state L they are diametrically opposed. Consequently, the receiver will face difficulties to trust in any claim about the sender's private information. So we have that ex-post inefficiencies (signaling costs) are incurred in a communication equilibrium (recall that $a^*(p) < \text{cav } \hat{a}(p)$). Such signaling costs are mathematically formalized and measured by the dual variable $\gamma^*(p)$.

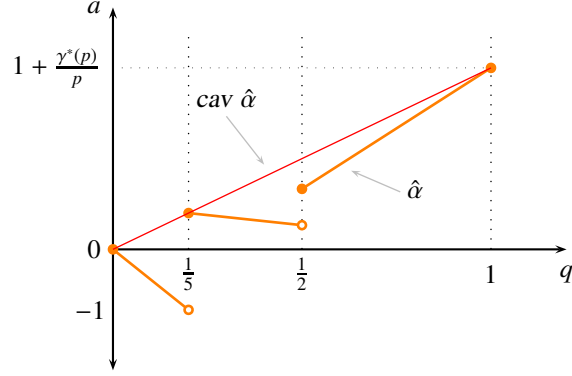


Figure 2: Function \hat{a} and its concavification

In order to achieve the optimal value $a^*(p)$, the sender requires to induce a split of the total probability p into the posteriors $p_3 = 0$, $p_2 = \frac{1}{5}$ and $p_1 = 1$. Hence, an optimal communication device requires to signal at least $3 (> |K| = 2)$ different messages (actions). Unlike the case in which the sender's information is verifiable by the mediator, when incentive constraints matter, the sender needs to transmit to the receiver more information. The idea is that the sender requires to signal as much information as when information is verifiable (i.e., he needs to induce posterior beliefs $p_3 = 0$, $p_2 = \frac{1}{5}$), but also he needs to make such revelation credible to the receiver. Because type L has incentives to imitate type H , this latter will need to find a way to separate himself from type L . To do this, type H can commit himself to recommend action j_1 (with some positive probability), something that is too costly for type L . In this manner, the receiver can discriminate between both types, so that whenever j_1 is recommended, she deduces that this message can only come from type H , i.e., $p_1 = 1$, and thus, she follows the recommendation. More generally, Forges (1994, corollary 4.3) shows that for any game in which the sender has two types, the sender may need at most 4 messages. We will further discuss this issue in section 3.4.

Let us consider now the case $p > \frac{1}{2}$. A similar analysis shows that the optimal value of the dual variables is

$$\gamma^{**}(L | H) = \frac{2p(p-1)}{5p-1} \equiv \gamma^{**}(p), \quad \gamma^{**}(H | L) = 0$$

and the optimal value of the primal problem for $p > \frac{1}{2}$ is thus

$$a^*(p) = \text{cav } \alpha(p; p, \gamma^{**}(p)) = p + 6\gamma^{**}(p) < 4 - 3p = \text{cav } \hat{a}(p)$$

Finally, for any $p \in [1/5, 1/2]$, the optimal action of the receiver is j_2 , which is the preferred action for both types of the sender. Thus, he would prefer not to signal any information, thereby, $a^*(p) = \hat{a}(p) = \text{cav } \hat{a}(p)$. Thus, the value function a^* looks like in figure 3.

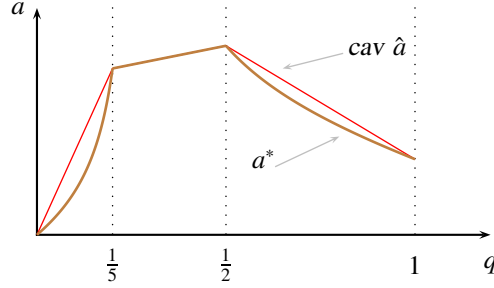


Figure 3: Functions a^* and $\text{cav } \hat{a}$ for example 1

When the sender's types are verifiable by the mediator, the optimal value function is concave, since it equals $\text{cav } \hat{a}$. This example shows that informational incentive compatibility may lead to an optimal value function a^* that is neither concave nor convex. In fact, it can contain strictly convex segments. We would like to mention that analogous examples can be easily constructed (see section 4 and example 2).

Example 2. The following game, proposed by Forges (1990), shares some features with our previous example. Payoffs for both players depend on the sender's type and the receiver's action as follows:

a, b	j_1	j_2	j_0	j_3	j_4
H	3,0	4,4	0,7	10,9	6,10
L	6,10	10,9	0,7	4,4	3,0

This example has a natural interpretation in terms of a job assignment scenario. An employer must decide whether to hire a candidate and, if so, to assign the employee to one of four possible jobs. The candidate may be one of two types. Type L performs better in job 1 but prefers job 2; he is bad at job 3, and even worse at job 4. Type H is similar but with jobs reversed. Sender's prior probability of type H is $p \in (0, 1)$.

By performing a similar analysis as in example 1, it can be shown that the value of persuasion in this game (depicted in figure 4) is given by

$$a^*(p) = \begin{cases} 6 + 14p + \frac{28p(5p-1)}{4-15p}, & \text{if } p < \frac{1}{5} \\ \frac{44}{5}, & \text{if } \frac{1}{5} \leq p \leq \frac{4}{5} \\ 20 - 14p - \frac{28(1-p)(5p-4)}{15p-11}, & \text{if } p > \frac{4}{5} \end{cases}$$

In particular, we have that for any $p < 1/5$ (resp. $p > 4/5$) only type L (resp. H) has incentives to lie, so that $\gamma(H | L) > 0$ (resp. $\gamma(L | H) > 0$). Also, at least 3 actions are required to be recommended in order to achieve $a^*(p)$ for any $p \notin [1/5, 4/5]$.

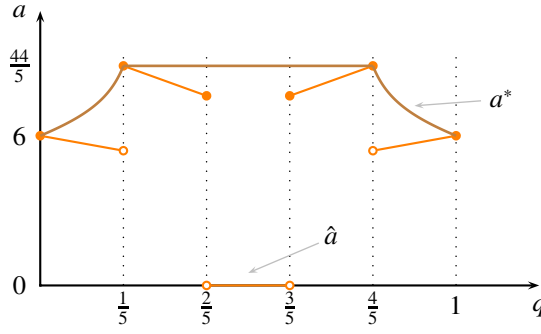


Figure 4: Functions a^* and \hat{a} for example 2

3.3. Optimal Mediators

Whenever the optimal value of the dual variables is zero for some prior p , so that incentive constraints do not matter, the value of persuasion at p coincides with $\text{cav } \hat{a}(p)$. Then, conditions in (3.3) identify the posterior beliefs induced by any optimal communication device; these beliefs correspond to the points on \hat{a} whose convex combination yields $\text{cav } \hat{a}(p)$. The corresponding distribution of posteriors ρ is the (unique) solution of a well determined system of linear equations given by (3.3). Hence, the optimal communication device can be easily computed using formula (3.4). As we have shown in examples 1 and 2, when there are binding informational incentive constraints, the sender may require more messages than his number of types. Then, the number of unknowns in (3.3) increases while the number of equations remains the same. The system in (3.3) may become underdetermined and therefore infinitely many distributions of posteriors may be consistent with the same prior probability. Thus, additional conditions are required for characterizing any optimal communication device.

In order to understand this issue, let us return to the example 1. Let $p < \frac{1}{5}$ and consider the $(p, \gamma^*(p))$ -virtual game. According to figure 2 and conditions (3.3), the optimal value of the primal problem can be achieved by splitting the total probability p in either of the following collection of posteriors:

- (i) $p_3 = 0, p_2 = \frac{1}{5}$.
- (ii) $p_3 = 0, p_1 = 1$.
- (iii) $p_3 = 0, p_2 = \frac{1}{5}, p_1 = 1$.

In case (i), Bayes plausibility implies that $(\rho_2, \rho_3) = (5p, 1 - 5p)$ and thus formula (3.4) yields $\delta_2^H = 1 - \delta_3^H = 1$ and $\delta_2^L = 1 - \delta_3^L = \frac{4p}{1-p}$. But this communication device is not incentive compatible for the sender. In case (ii) we have that Bayes plausibility implies that $(\rho_1, \rho_3) = (p, 1 - p)$ and therefore formula (3.4) yields $\delta_1^H = \delta_3^L = 1$. As we already mentioned above, this communication device is a FR communication equilibrium giving an expected payoff to the sender which is strictly lower than $a^*(p)$. Finally, in case (iii), Bayes plausibility does not uniquely identify a distribution of posteriors. In particular, any probability vector (ρ_1, ρ_2, ρ_3) satisfying $\rho_1 + \frac{\rho_2}{5} = p$ is a feasible distribution of posteriors. An additional condition is thus required in order to identify the correct distribution of posteriors.

Duality theory implies a relationship between the primal and dual problems that is known as the *complementary slackness*. Specifically, it says that if a dual variable is positive, then the

associated informational incentive constraint must be binding. Conversely, if a constraint fails to bind, then the associated dual variable must be zero. Complementary slackness provides us the additional equations we needed. Consider again example 1. As we have already shown, the optimal value of $\gamma^*(H | L)$ is strictly positive. Then, according to the complementary slackness, the constraint asserting that the type L should not gain by reporting H is binding, i.e.,

$$2\delta_2^L - \delta_1^L = 2\delta_2^H - \delta_1^H$$

The previous equality joint with formula (3.4) yield the additional restriction

$$5\rho_1(1 - p) = 2\rho_2(1 - 5p)$$

Then, Bayes plausibility implies that the optimal distribution of posteriors is

$$\rho_1 = \frac{2p(1-5p)}{3-11p}, \quad \rho_2 = \frac{5p(1-p)}{3-11p}, \quad \text{and} \quad \rho_3 = 1 - \rho_1 - \rho_2$$

Given these posteriors, formula (3.4) gives the optimal communication device solving the primal problem for p . Complementary slackness together with the optimality condition in theorem 1 fully characterize the optimal solutions to the primal problem.

Proposition 1.

Suppose δ is a communication device satisfying the informational incentive constraints for the sender. Then, δ is an optimal solution of the primal problem for p if and only if there exists a vector $\gamma \geq 0$ such that

$$\gamma(k' | k) \left[\sum_{j \in J} (\delta_j^k - \delta_j^{k'}) a_j^k \right] = 0, \quad \forall k, k' \in K \quad (3.8)$$

and

$$\sum_{k \in K} p^k \alpha^k(\delta; p, \gamma) = \text{cav } \hat{\alpha}(p; p, \gamma) \quad (3.9)$$

Condition (3.9) is the counterpart of condition (3.3) in the general model with nonverifiable information. It says that the optimal communication device induces a distribution of posterior beliefs giving the sender an ex-ante expected virtual payoff equal to the concavification of the non-revealing virtual function $\alpha(\cdot; p, \gamma)$ evaluated at the prior distribution p . This distribution of posteriors is Bayes plausible, that is, the expected posterior probability equals the prior probability. As we already illustrated before, when informational incentive constraints matter, (3.9) is not the only restriction imposed on the optimal communication device. According to proposition 1, the complementary slackness condition in (3.8) is also required.

Following Myerson (1991), we say that a type k *jeopardizes* another type k' , in the optimal communication device δ , if the constraint that the type k should not gain by reporting k' is binding (i.e., $\sum_{j \in J} \delta_j^k a_j^k = \sum_{j \in J} \delta_j^{k'} a_j^k$) and the associated dual variable $\gamma(k' | k)$ is positive.

3.4. Extreme Communication Equilibria and the Number of Signals

When incentive constraints are not essential, for instance because information is verifiable by the mediator, the number of signals with positive probability do not exceed the number of types of the sender (see section 3.1). Yet this number may be increased with every binding informational incentive constraint (see examples 1 and 2). The reason is that the optimal splitting of the prior probability is constrained not only by the Bayes plausibility but also by the complementary slackness (3.3). Intuitively, the sender requires to signal as much information as when he is not constrained by the incentive compatibility, but also he might need to transmit additional information in order to make such revelation credible.

Exploiting the geometric properties of the set of communication equilibria it is possible to identify an upper bound on the minimum number of actions that can have positive probability in a solution of the primal problem. Recall that for any fixed $p \in \Delta(K)$, the feasible set of the primal problem for p is a convex polytope (bounded polyhedron). Then, the sender's expected payoff achieves its maximum at an extreme point of $\mathcal{D}^*(p)$ (or a convex combination of them).

Definition 4 (Extreme communication equilibrium).

We say that $\delta \in \mathcal{D}^*(p)$ is an extreme communication equilibrium of $\Gamma(p)$ if it is an extreme point of $\mathcal{D}^*(p)$.

Let us introduce some notation. For any $j \in J$, let $\Delta_j \subseteq \Delta(K)$ be the set of beliefs for which the action j is optimal for the receiver, that is,

$$\Delta_j = \{q \in \Delta(K) \mid j \in F(q)\} \quad (3.10)$$

The set Δ_j is a closed and convex polytope; we denote its boundary by $\partial\Delta_j$.

The following proposition, due to Forges (1994), states that either an extreme communication equilibrium is NR or it induces posterior probabilities lying on the boundary of Δ_j .

Proposition 2.

Let $\delta \in \mathcal{D}^*(p)$ an extreme communication equilibrium of $\Gamma(p)$. Then, for all $j \in J$ with $\pi_j(\delta) > 0$ we have that,

- (i) Either $p_j = p$ implies that $\delta_j^k = 1$ for all $k \in K$,
- (ii) or $p_j \in \partial\Delta_j$.

Using a basic result from the theory of linear programming, it is possible to characterize the cardinality of the support of any extreme communication equilibrium. A solution of a system of linear inequalities is an extreme point of the corresponding feasible set if and only if it can be obtained as the unique solution to a system of equations derived from equality constraints by setting a subset of variables to zero. Therefore, the number of non-zero components in any extreme point is no greater than the number of binding constraints. Thus, a way to identify an upper-bound on the number of actions with positive probability in an extreme communication equilibrium is to determine how many incentive constraints can be binding. For instance, Forges (1994) shows that whenever the sender has only two types (i.e., $|K| = 2$), proposition 2 implies that for every action with positive probability, there is a binding strategic incentive constraint. Using this insight she establishes the following result.

Corollary 1.

Assume $|K| = 2$ and let δ be an extreme communication equilibrium of $\Gamma(p)$. Then, no more than 4 actions can have positive probability in δ , i.e., $|\text{supp}(\delta)| \leq 4$.

According to this corollary, the cardinality of the support of any extreme communication equilibrium is bounded by the number of types ($|K| = 2$) plus the maximum number of binding informational incentive constraints. Unfortunately, the reasoning in the proof of corollary 1 relies strongly on the fact that $|K| = 2$. However, a similar result can be proved for the general case $|K| \geq 2$ by modifying the sender's problem⁹. Given a solution of the primal problem for $p, \bar{\delta}$, we replace each $\bar{\delta}_j^k$ by $\theta_j \bar{\delta}_j^k$, with $\theta_j \geq 0$, and then we add the constraint $\sum_j \theta_j \bar{\delta}_j^k = 1$ for all $k \in K$. By keeping fixed $\bar{\delta}$ and p , we obtain a linear programming problem on θ . For this problem, the strategic incentive constraints are redundant, thus applying the previous insights gives us that at most $|K| + |K|(|K| - 1)$ actions have positive probability. Given an optimal solution $\tilde{\theta}$ of this linear program, the communication device $\tilde{\delta}$ defined by $\tilde{\theta}_j \bar{\delta}_j^k$ is also an optimal solution of the primal problem for p . Since, all actions j for which $\theta_j = 0$ have zero probability in $\tilde{\delta}$, we are able to find an upper bound on the number of signals.

Proposition 3.

Let $p \in \Delta(K)$ and assume that at most m informational incentive constraints can be binding at any solution of the primal problem for p . Then, there is a solution of the primal problem for p for which the number of actions with positive probability does not exceed $|K| + m$.

Since the number of informational incentive constraints is at most $|K|(|K| - 1)$, the key implication of this proposition is that the minimum number of actions that can have positive probability in a solution of the primal problem is bounded above by $|K|^2$. We notice that this bound is uniform on the number of actions.

3.5. The Value of Private Information

The *value of information* for the sender at the prior p is the difference between the value of persuasion at p and the non-revealing value at p , i.e., $a^*(p) - \hat{a}(p)$. We say that the sender benefits from his private information at p if $a^*(p) > \hat{a}(p)$. Theorem 1 provides a necessary and sufficient condition for the sender to benefit from his information.

Corollary 2.

Let $\gamma^*(p)$ be an optimal solution to the dual problem for p . Then, the sender benefits from his information at p if and only if

$$\text{cav } \hat{a}(p; p, \gamma^*(p)) > \hat{a}(p)$$

We say that *there is information the sender would like to signal* if $\hat{a}(p) < \text{cav } \hat{a}(p)$. In words, if the sender were not constrained by the incentive compatibility, he would prefer to persuade the receiver to change his action rather than letting her to choose an action based on p . From the inequalities in (3.5) we deduce the following observation.

⁹The same method is also applied by Bester and Strausz (2007).

REMARK 4. If there is no information the sender would like to signal, the sender does not benefit from his information.

Hence, owning any information the sender would like to share is a necessary condition, yet not sufficient for him to benefit from his information, as the following examples shows.

Example 3. Consider the following sender-receiver game due to Forges (1985). Player 1 has a privately known type that may be H or L and player 2 must choose an action from the set $J = \{j_1, j_2, j_3\}$. Payoffs for both players depend on 1's type and 2's action as follows:

a, b	j_1	j_2	j_3
H	6,0	0,4	4,6
L	0,6	6,4	4,0

The receiver's optimal actions at belief q is

$$F(q) = \begin{cases} j_1, & \text{if } 0 \leq q < \frac{1}{3} \\ \Delta(\{j_1, j_2\}), & \text{if } q = \frac{1}{3} \\ j_2, & \text{if } \frac{1}{3} < q < \frac{2}{3} \\ \Delta(\{j_2, j_3\}), & \text{if } q = \frac{2}{3} \\ j_3, & \text{if } \frac{2}{3} < q \leq 1 \end{cases}$$

The non-revealing value function \hat{a} joint with its concavification $cav \hat{a}$ are depicted in figure 5.

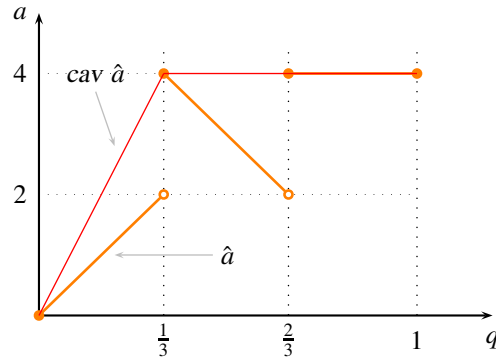


Figure 5: Value function

Let p be the prior probability of type H . Notice that for any prior $p \in (0, 1/3)$ the sender has information he would like to signal, i.e., $cav \hat{a}(p) > \hat{a}(p)$. In particular, the sender would like to choose a communication device leading to the posteriors $p_1 = 0$ and $p_2 = \frac{1}{3}$ with probabilities $1 - 3p$ and $3p$ respectively, thus achieving the expected payoff $cav \hat{a}(p)$. However, such a communication device cannot be constructed without violating the informational incentive constraints. In fact, no communication equilibrium of $\Gamma(p)$ can give the sender an expected payoff larger than $\hat{a}(p)$. Because the receiver has difficulties to trust any claim about the unverifiable sender's private information, the sender cannot benefit from any information he would like to transmit.

For sake of completeness, let us show that for any communication equilibrium δ of $\Gamma(p)$ with $p \in (0, 1/3)$, the ex-ante expected payoff of the sender is $\hat{a}(p)$. Let p_j be the conditional probability of type H that the receiver will infer after receiving the recommendation to play the action j . Assume that δ is not NR. Then, $p\delta_1^H + (1-p)\delta_1^L > 0$. Suppose that $p\delta_2^H + (1-p)\delta_2^L > 0$. Then, incentive compatibility for the receiver requires that $p_1 \leq \frac{1}{3} \leq p_2$, which implies that

$$\begin{aligned} 2p\delta_1^H &\leq (1-p)\delta_1^L \\ 2p\delta_2^H &\geq (1-p)\delta_2^L \end{aligned}$$

Subtracting the first inequality from the second, we have that

$$\delta_1^H - \delta_2^H = \frac{1-p}{2p} (\delta_1^L - \delta_2^L) < \delta_1^L - \delta_2^L$$

However, incentive compatibility for the sender implies that

$$\delta_1^H - \delta_2^H \geq \delta_1^L - \delta_2^L$$

Then, $\delta_2^H = \delta_2^L = 0$. Now suppose that $p\delta_3^H + (1-p)\delta_3^L > 0$. Then, incentive compatibility for the receiver requires that $p_3 \geq \frac{2}{3}$, implying that

$$\delta_3^L \leq \frac{p}{2(1-p)} \delta_3^H < \delta_3^H$$

However, incentive compatibility for the type L of the sender requires that $\delta_3^L \geq \delta_3^H$. Therefore $\delta_3^H = \delta_3^L = 0$. Thus, only j_1 is recommended with positive probability in equilibrium. Hence, the unique communication equilibrium is $\delta_1^H = \delta_1^L = 1$. We conclude that $a^*(p) = \hat{a}(p)$ for any $p \in [0, 1/3)$.

On the other hand, for any $p \in [1/3, 2/3)$, it can be easily seen that the communication device

$$\delta_2^H = 1 - \delta_3^H = \frac{2-3p}{3p}, \quad \delta_2^L = 1 - \delta_3^L = \frac{2(2-3p)}{3(1-p)}$$

is a communication equilibrium inducing the posteriors $p_2 = \frac{1}{3}$ and $p_3 = \frac{2}{3}$ with probabilities $2-3p$ and $3p-1$, respectively, hence achieving the expected payoff $\text{cav } \hat{a}(p)$.

Finally, for any $p \in [2/3, 1]$, $\text{cav } \hat{a}(p) = \hat{a}(p) = a^*(p)$. Summarizing,

$$a^*(p) = \begin{cases} \hat{a}(p), & \text{if } 0 \leq p < \frac{1}{3} \\ \text{cav } \hat{a}(p), & \text{if } \frac{1}{3} \leq p \leq 1 \end{cases}$$

The problem of determining whether the sender benefits from his private information can be equivalently formulated as the existence of solutions to a system of linear inequalities. Most of the results on the feasibility of linear inequalities are based on linear duality theory. It exploits the geometry of the linear inequalities and it is equivalent to separation theorems, theorems of linear alternative inequalities or the well known Minimax theorem. Theorem 1 (and a fortiori

corollary 2) makes use of these results for establishing the positive value of private information. Without more specific assumptions on the components of the game $\Gamma(p)$, it would be difficult to find general sufficient conditions for the sender to benefit from his information at p . In the next section we focus on a model of information transmission in which informational and strategic incentive constraints can be easily characterized by some monotonicity conditions. In such a framework, we derive necessary and sufficient conditions under which the value of private information is strictly positive.

4. Persuasion in a Simple Model à la Crawford-Sobel

In this section we study the value of private information in a discrete model of information transmission with the same qualitative features as the one of Crawford and Sobel (1982)¹⁰. Consider a setting in which the actions of the receiver can be ranked, so we can write $J = \{1, 2, \dots, J\}$. The payoffs for both players depend on the action implemented by the receiver and on the sender's private information about his type. For simplicity, we assume that the type of the sender can take one of two possible values¹¹ on $K = \{H, L\}$. With probability p he is of type H and with probability $1 - p$ he is of type L .

We assume that both players have type-dependent single-peaked preferences, so that each player has a unique most preferred action in J given every type L or H of the sender. Formally, let j_S^k be the most preferred action for the type $k \in K$ of the sender. Likewise, we denote j_R^k the most preferred action of the receiver given the type $k \in K$ of the sender. Then, for every $k \in K$, the preferences of both players satisfy

$$i < j \leq j_S^k \Rightarrow a_i^k < a_j^k \quad \text{and} \quad i > j \geq j_S^k \Rightarrow a_i^k < a_j^k$$

$$i < j \leq j_R^k \Rightarrow b_i^k < b_j^k \quad \text{and} \quad i > j \geq j_R^k \Rightarrow b_i^k < b_j^k$$

We suppose that $j_R^H \neq j_R^L$, so that the information the receiver can acquire from the sender is useful. Additionally, we assume the following *sorting condition*:

$$a_j^L - a_{j-1}^L < a_j^H - a_{j-1}^H, \quad \forall j \geq 2$$

This condition is similar to the usual single crossing property (or Spence-Mirrlees condition) assumed in models of screening and signaling. It implies the following result.

Lemma 1.

For any two actions $i, j \in J$ with $j < i$, we have that $a_j^H \geq a_i^H$ implies that $a_j^L > a_i^L$.

An immediate consequence of lemma 1 is that the ideal choices of both types of the sender satisfy the ordering $j_S^H \geq j_S^L$.

¹⁰It is worth mentioning that a continuous (in actions) version of the present model is also analyzed by Mitusch and Strausz (2005) in order to study mediation in a situation in which a principal needs information from an agent to implement an action.

¹¹We restrict attention to two types as in Mitusch and Strausz (2005); this substantially simplifies the results while conveying all the main insights.

Lemma 2.

Given any belief $p \in [0, 1]$, we have that for all $j \in F(p)$

$$\min\{j_R^L, j_R^H\} \leq j \leq \max\{j_R^L, j_R^H\}$$

Because the sender has only two possible types, the prior distribution is fully described by the probability p . Moreover, the set of beliefs for which any action $j \in J$ is optimal for the receiver (as defined in (3.10)) can be described by an interval $\Delta_j = [p_j^-, p_j^+]$, where $p_j^- \leq p_j^+$. Then, the family of sets $\{\Delta_j\}_{j \in J}$ define a partition of the interval $[0, 1]$. In particular, lemma 2 implies that $\Delta_j \neq \emptyset$ only if $\min\{j_R^L, j_R^H\} \leq j \leq \max\{j_R^L, j_R^H\}$.

When the ideal points of the receiver exhibit the ordering $j_R^H > j_R^L$, the sets Δ_j are increasingly ordered, so that, for any two actions $i, j \in J$ such that $j < i$, we have that $p_j^+ \leq p_i^-$. Therefore, incentive compatibility for the receiver implies that any revealing communication equilibrium δ induces increasing posterior probabilities, namely, $p_j < p_i$ for any $j < i$ with $\pi_i(\delta) > 0$ and $\pi_j(\delta) > 0$. It can be easily seen that this is equivalent to the following monotone likelihood ratio property:

$$\delta_i^H \delta_j^L \geq \delta_i^L \delta_j^H, \quad \forall i > j$$

with strict inequality for all $i, j \in \text{supp}(\delta)$. This in turn implies that the distribution δ^H (first-order) stochastically dominates the distribution δ^L , i.e.,

$$\sum_{l=1}^j \delta_l^L \geq \sum_{l=1}^j \delta_l^H, \quad \forall j \in J \quad (4.1)$$

with strict inequality for all $j \in \text{supp}(\delta)$. The inequality in (4.1) is reversed in case the ideal points of the receiver satisfy the ordering $j_R^H < j_R^L$.

On the other hand, observe that the gains to the sender from reporting any type L or H when he is actually of type k can be expressed as

$$\sum_{j \in J} (\delta_j^L - \delta_j^H) a_j^k = \sum_{j=1}^{|J|-1} \left(\sum_{l=1}^j \delta_l^L - \sum_{l=1}^j \delta_l^H \right) [a_j^k - a_{j+1}^k] \quad (4.2)$$

Hence, the order of the receiver's ideal points plays an important role in the alignment of the incentives of both players. Using this insight we are able to prove the following proposition.

Proposition 4.

Assume that $j_R^H < j_R^L$. Then the sender does not benefit from his private information at any prior p . If $j_R^H > j_R^L$, any communication equilibrium inducing some revelation of information leaves at most one informational constraint binding.

When $j_R^H < j_R^L$ the incentives of both players are diametrically opposed, so that any revelation of information is ineffective. According to proposition 4, a necessary condition for the value of information to be positive is that $j_R^H > j_R^L$. Thus, in the remainder of this section we will focus on this ordering. Then, in view of lemma 2, we can assume without loss of generality that the lowest action is j_R^L and the highest action is j_R^H . We consider the following assumption.

Assumption 1 (A.1). The ideal actions of the receiver satisfy $j_R^L = 1$ and $j_R^H = J$.

Also, we will assume that all actions are “essential” in the sense that for any action $j \in J$ there exist an open set $Q_j \subseteq [0, 1]$ such that $j \in F(q)$ for all $q \in Q_j$.

Assumption 2 (A.2). For any action j , the interior of Δ_j is not empty.

We notice that under assumption A.1, propositions 3 and 4 imply that at most 3 actions have positive probability in any solution of the primal problem.

Another extreme case in which the incentives of both players are antipodal is when $j_S^L = J$. In such circumstances, type L of the sender will always like to persuade the sender that he is type H . Then, incentive compatibility for type L cannot be satisfied unless both types send the same message. A similar situation is reproduced when $j_S^H = 1$.

Proposition 5.

Suppose that A.1-A.2 hold. If either $j_S^L = J$ or $j_S^H = 1$, the sender does not benefit from his private information at any prior p .

Proposition 5 implies that a necessary condition for the value of private information to be positive is that $j_S^L < J$ and $j_S^H > 1$. We thus consider the following assumption.

Assumption 3 (A.3). The ideal actions of the sender satisfy $j_S^L < J$ and $j_S^H > 1$.

Because preferences are single-peaked, whenever a prior probability p leads the receiver to choose an action $j \in [j_S^L, j_S^H]$, the incentives of both players are aligned: the sender would like to induce the receiver to choose a higher (resp. lower) action in state H (resp. L), the recommendation is truthful since the preferences of both types are opposed, and the receiver will find optimal to follow the recommendation. Thus, informational incentive constraints are not essential and the optimal value of the primal problem for p is precisely $\text{cav } \hat{a}(p)$.

Proposition 6.

Suppose that A.1-A.3 hold and let $p \in [p_{j_S^L}^-, p_{j_S^H}^+]$. Then $a^(p) = \text{cav } \hat{a}(p)$. Hence, the sender benefits from his private information at p if and only if $\text{cav } \hat{a}(p) > \hat{a}(p)$.*

Kamenica and Gentzkow (2011) provide sufficient conditions for $\text{cav } \hat{a}(p)$ to be strictly larger than $\hat{a}(p)$. In particular, the sender benefits from his information at any p in $(p_{j_S^L}^-, p_{j_S^H}^+)$ if \hat{a} is convex on this interval. Notice also that if $j_S^L = j_S^H \equiv j_S$, so that $[p_{j_S^L}^-, p_{j_S^H}^+] = \Delta_{j_S}$, the optimal action of the receiver given any $p \in \Delta_{j_S}$ is to choose j_S . Hence, it is optimal for the sender not to modify the receiver’s belief. The unique solution of the primal problem is thus the non-revealing communication equilibrium $\delta_{j_S}^H = \delta_{j_S}^L = 1$. Therefore, in view of proposition 6, for any $p \in \Delta_{j_S}$, we have that $a^*(p) = \text{cav } \hat{a}(p) = \hat{a}(p)$.

When a prior probability p leads the receiver to choose an action $j < j_S^L$, both types of the sender would like to persuade the receiver to change her action to a higher action. But then, the receiver will have difficulties preventing type L from claiming to be type H . Because the receiver will only move to play a higher action in state H , then type L of the sender jeopardizes type H . Thus,

according to the virtual utility hypothesis, type H of the sender should adopt a behavior that exaggerates the difference from type L . This new payoff maximization compromise generates ex-post inefficiencies (signaling costs) which are reflected in a value of persuasion $a^*(p)$ which is strictly lower than $cav \hat{a}(p)$. A symmetric reasoning also holds in the case the prior p leads the receiver to choose an action $j > j_S^H$. We will study these two cases under the simplifying assumption that $j_S^H = j_S^H \equiv j_S$. The conclusions remain the same when $j_S^L < j_S^H$.

Lemma 3.

Suppose A.1-A.3 hold. Let δ be a solution of the primal problem for p . Then,

- (i) If $p < p_{j_S}^-$ type L jeopardizes type H at δ .
- (ii) If $p > p_{j_S}^+$ type H jeopardizes type L at δ .

In any of the two cases enumerated in lemma 3, according to the virtual utility hypothesis, incentive compatibility forces the sender to adopt a behavior that magnifies the difference between his true type and the type that jeopardizes him. For this new compromise to be effective in separating his both types, he needs to find an action that contributes little or nothing to the welfare of his true type (he gains only virtual utility) and that results costly enough to the other type, so that this latter would not be tempted to impersonate the true type.

Proposition 7.

Assume A.1-A.3 are satisfied. The following assertions hold¹²:

- (i) Let $j < j_S$ and suppose that $a_j^L \geq a_j^L$. Then, for any $p \in [p_j^-, p_j^+]$, the optimal value of the primal problem for p is¹³

$$a^*(p) = \hat{\rho}_j^- \hat{a}(p_j^-) + \hat{\rho}_j^+ \hat{a}(p_j^+) + (1 - \hat{\rho}_j^- - \hat{\rho}_j^+) a_j^H$$

where

$$\hat{\rho}_j^+ = \frac{(p - p_j^-)(1 - p) [a_j^L - a_j^L]}{(p_j^+ - p_j^-)(1 - p) [a_j^L - a_j^L] + (p_j^+ - p)(1 - p_j^-) [a_{j+1}^L - a_j^L]},$$

and

$$\hat{\rho}_j^+(1 - p_j^+) + \hat{\rho}_j^-(1 - p_j^-) = 1 - p$$

Furthermore, the sender benefits from his information at any $p \in (p_j^-, p_j^+)$.

- (ii) Let $j > j_S$ and suppose that $a_j^H \geq a_1^H$. Then, for any $p \in [p_j^-, p_j^+]$, the optimal value of the primal problem for p is¹⁴

$$a^*(p) = \tilde{\rho}_j^- \hat{a}(p_j^-) + \tilde{\rho}_j^+ \hat{a}(p_j^+) + (1 - \tilde{\rho}_j^- - \tilde{\rho}_j^+) a_1^L$$

where

$$\tilde{\rho}_j^- = \frac{p(p_j^+ - p) [a_j^H - a_1^H]}{p(p_j^+ - p_j^-) [a_j^H - a_1^H] + p_j^+(p - p_j^-) [a_{j-1}^H - a_j^H]},$$

¹²Explicit formulas for the optimal communication device can be found in the proof of the proposition.

¹³In case $a_j^L = a_j^L$, p is restricted to $[p_j^-, p_j^+]$. This is to avoid the denominator in the definition of $\hat{\rho}_j^+$ to vanish.

¹⁴In case $a_j^H = a_1^H$, p is restricted to $(p_j^-, p_j^+]$. This is to avoid the denominator in the definition of $\tilde{\rho}_j^-$ to vanish.

and

$$\tilde{\rho}_j^+ p_j^+ + \tilde{\rho}_j^- p_j^- = p$$

Furthermore, the sender benefits from his information at any $p \in (p_j^-, p_j^+)$.

In order to interpret the previous result, consider the case in which $p \in (p_j^-, p_j^+)$ for some $j < j_s$. Observe that in state H the preferences of both players are aligned; both the sender and the receiver would prefer j_s rather than j . However, in state L the sender would like to persuade the receiver that he is type H (see lemma 3). Therefore, the receiver will have difficulties to trust in any claim coming from type H . In an effort to distinguish himself from type L , type H can commit himself to recommend the action J , with some positive probability. This separating strategy may be moderately pleasant (if $a_j < a_j^H < a_{j_s}^H$) or unpleasant (if $a_j > a_j^H$) for type H , but in any case it is hurtful for type L ($a_j^L \leq a_j^L < a_{j_s}^L$). Thus any recommendation to play J cannot come from type L , so that the receiver will be able to discriminate between both types. Incentive compatibility compels type H to incur in a costly signal (from which he only gets virtual utility) in order to reduce the misrepresentation of type L . The idea behind the proof of proposition 7 is as follows: by inducing a splitting of the prior probability $p \in (p_j^-, p_j^+)$ into posterior beliefs $\hat{\rho}$, the sender can guarantee the same expected payoff in both the real game and the (p, γ) -virtual game with $\gamma(H | L) > 0$ and $\gamma(L | H) = 0$. Because this split is incentive compatible (by construction of $\hat{\rho}$), weak duality implies that such split is an optimal solution of the primal problem for p . A symmetric reasoning applies for any $p \in (p_j^-, p_j^+)$ with $j > j_s$.

The following corollary is deduced immediately.

Corollary 3.

Suppose that A.1-A.3 hold.

- (i) Assume that $a_j^H \geq a_{j_s-1}^H$ and $a_1^L \geq a_j^L$. Then the sender benefits from his information at almost any prior $p < p_{j_s}^-$.
- (ii) Assume that $a_1^H \leq a_j^H$ and $a_1^L \geq a_{j_s+1}^L$. Then the sender benefits from his information at almost any prior $p > p_{j_s}^+$.

Example 4. The general behavior of the value of persuasion in this model will be illustrated in the following example. Payoffs for both players are:

a, b	j_1	j_2	j_3	j_4	j_5
H	2,0	5,4	9,7	10,9	8,10
L	1,10	3,9	6,7	4,4	1,0

For any $p \in [0, 1/5)$, the receiver will optimally play action j_1 . According to proposition 7-(i), the optimal distribution of posteriors is given by $\tilde{\rho}_1^- = 1 - p$ and $\tilde{\rho}_1^+ = 0$, thus the value of persuasion is

$$a^*(p) = 1 + 7p$$

Similarly, for any prior $p \in [1/5, 2/5)$ the receiver's optimal action is j_2 . Again, by proposition 7-(i), the optimal distribution of posteriors is given by

$$\tilde{\rho}_2^- = \frac{25(2 - 5p)(1 - p)}{2(17 - 35p)}, \quad \tilde{\rho}_2^+ = \frac{10(5p - 1)(1 - p)}{2(17 - 35p)}$$

Hence, the value of persuasion is

$$a^*(p) = \frac{535p^2 - 197p - 50}{70p - 34}$$

For any prior $p \in [2/5, 4/5]$, the receiver optimally chooses an action in $[j_S^L, j_S^H]$. Then, proposition 6 implies that the value of persuasion equals $\text{cav } \hat{a}(p)$. We have that

$$a^*(p) = \frac{4 - 5p}{2} \hat{a}\left(\frac{2}{5}\right) + \frac{5p - 2}{2} \hat{a}\left(\frac{4}{5}\right) = \frac{28}{5} + 4p$$

Finally, for any $p \in (4/5, 1]$ the receiver's optimal action is j_5 . Proposition 7-(ii) implies that the optimal distribution of posteriors is given by

$$\tilde{\rho}_5^- = \frac{15p(1-p)}{4(2p-1)}, \quad \tilde{\rho}_5^+ = \frac{p(5p-4)}{2p-1}$$

Then, the value of persuasion is

$$a^*(p) = \frac{23p^2 + 13p - 4}{8p - 4}$$

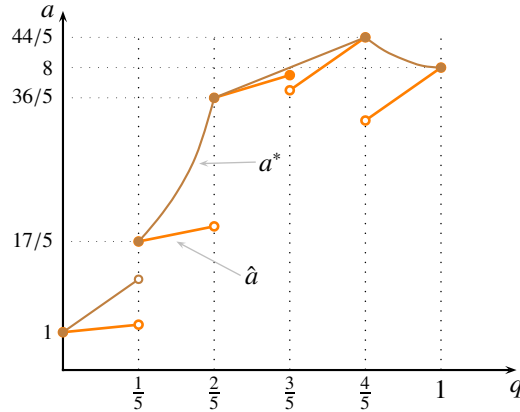


Figure 6: Functions a^* and \hat{a} for example 4

Figure 6 depicts the non-revealing function \hat{a} together with the value of persuasion a^* . We notice the following general facts:

- a^* is linear increasing (resp. decreasing) in Δ_j for any $j < j_S^L$ (resp. $j > j_S^H$), whenever $a_j^L = a_j^L$ (resp. $a_j^H = a_1^H$).
- a^* is increasing (resp. decreasing) and strictly convex in Δ_j for any $j < j_S^L$ (resp. $j > j_S^H$), provided that $a_j^L > a_j^L$ (resp. $a_j^H > a_1^H$).
- a^* is linear and equals $\text{cav } \hat{a}$ on the interval $[p_{j_S^L}^-, p_{j_S^H}^+]$.
- a^* may fail to be continuous, but it is always upper-semicontinuous.

5. Concluding Remarks

This paper provides an analytical framework for studying Bayesian persuasion problems in which the sender cannot commit himself to truthfully communicate the signal to the receiver, so that incentive compatibility becomes one of the major issues for communication to be meaningful. We demonstrate that by allowing the players to communicate through a neutral trustworthy mediator who makes non-binding recommendations to the receiver, one can transform the sender's problem into a simplified problem without incentive constraints. The key idea is that under mediated communication incentive compatibility is characterized by a system of linear inequalities. The duals of these constraints are used to define virtual utility scales which incorporates into the sender's utility function the signaling costs associated with incentive compatibility. With this mathematical insight, we are able to extend Kamenica and Gentzkow's (2011) original persuasion framework to problems with limited commitment on the part of the sender. This is not new. Indeed, this virtual utility approach has proved to be very fruitful for extending cooperative solution concepts to general games with incomplete information (Myerson (1984a,b, 2007) and Salamanca (2016)). The rationale for the virtual utility in these cooperative solutions is complex, which blurs its significance. In our approach, however, virtual utility arises in a simpler and more intuitive way.

6. Proofs

6.1. Lemmas

Proof of Lemma 1. The result follows from the following inequality:

$$a_i^L - a_j^L = \sum_{l=j+1}^i (a_l^L - a_{l-1}^L) < \sum_{l=j+1}^i (a_l^H - a_{l-1}^H) = a_i^H - a_j^H$$

□

Proof of Lemma 2. Assume $j_R^H > j_R^L$. Notice that for any $j < j_R^L$ we have that $b_j^H < b_{j_k}^H$ because preferences are single-peaked. Also, $b_j^L < b_{j_k}^L$ since j_R^L is the receiver's most preferred action in state L . Then, action j_R^L strictly dominates any action $j < j_R^L$ for the receiver given any belief. In a like manner, we can also see that any action $j > j_R^H$ is strictly dominated by j_R^H for the receiver given any belief. A similar reasoning applies for the case $j_R^H < j_R^L$. □

Proof of Lemma 3. Fix $p < p_{j_s}^-$ and let δ be a solution of the primal problem for p . Suppose that both informational incentive constraints are slack. Then, δ cannot be NR, since at any NR communication equilibrium both informational constraints are binding. Hence, $|\text{supp}(\delta)| > 1$. By the complementary slackness condition (3.8) $\gamma(H | L) = \gamma(L | H) = 0$ and thus $a^*(p) = \text{cav } \hat{a}(p)$. Therefore, $\delta_j^H = \delta_j^L = 0$ for all $j > j_s$. Notice that

$$\begin{aligned} \sum_{j \in J} (\delta_j^L - \delta_j^H) a_j^k &= \sum_{j=1}^{|J|-1} \left(\sum_{l=1}^j \delta_l^L - \sum_{l=1}^j \delta_l^H \right) [a_j^k - a_{j+1}^k] \\ &= \sum_{j=1}^{j_s-1} \left(\sum_{l=1}^j \delta_l^L - \sum_{l=1}^j \delta_l^H \right) [a_j^k - a_{j+1}^k] < 0 \end{aligned}$$

where the inequality follows from (4.1), since $j_R^L < j_R^H$, and the fact that $a_j^k < a_{j+1}^k$ for all $j < j_S$. Thus, incentive compatibility for the type L is not satisfied, which contradicts the fact that δ is a solution of the primal problem for p . Therefore, by proposition 3 only one informational incentive constraint is active at δ , and it must be the constraint claiming that type L should not gain by reporting H . A similar reasoning applies for $p > p_{j_S}^+$. \square

6.2. Proof of Theorem 1

The Lagrangian of primal problem for p is

$$\begin{aligned} \mathcal{L}(\delta, p, \gamma) &= \sum_{k \in K} p^k \sum_{j \in J} \delta_j^k a_j^k + \sum_{k \in K} \sum_{k' \in K} \gamma(k' | k) \left[\sum_{j \in J} \delta_j^k a_j^k - \sum_{j \in J} \delta_j^{k'} a_j^{k'} \right] \\ &= \sum_{k \in K} p^k \alpha^k(\delta; p, \gamma) \end{aligned} \quad (6.1)$$

where $\gamma \geq 0$ and $\delta \in D(p)$.

Then, the dual problem for p , associated to the primal problem for p , is given by

$$\begin{aligned} \min_{\alpha \geq 0} \max_{\delta \in D(p)} \mathcal{L}(\delta, p, \gamma) &= \min_{\alpha \geq 0} \max_{\delta \in D(p)} \sum_{k \in K} p^k \alpha^k(\delta; p, \gamma) \\ &= \min_{\alpha \geq 0} \text{cav } \hat{\alpha}(p; p, \gamma) \end{aligned}$$

Thus, strong duality implies that $a^*(p) = \min_{\alpha \geq 0} \text{cav } \hat{\alpha}(p; p, \gamma)$.

6.3. Propositions

Proof of Proposition 1. This follows from standard results in optimization theory (using the duality theorem of linear programming). \square

Proof of Proposition 3. Let $p \in \Delta(K)$ and assume that at most $m (\leq |K|(|K| - 1))$ informational incentive constraints can be binding at any solution of the primal problem for p . Then, there exist a set $M \subseteq K \times K$ such that $|M| = m$ and any solution of the programming problem

$$\begin{aligned} \max_{\delta \geq 0} \sum_{k \in K} p^k \sum_{j \in J} \delta_j^k a_j^k & \quad (6.2) \\ \text{s.t. } \sum_{j \in J} \delta_j^k a_j^k & \geq \sum_{j \in J} \delta_j^{k'} a_j^{k'}, \quad \forall (k, k') \in M \\ \sum_{k \in K} \delta_j^k p^k b_j^k & \geq \sum_{k \in K} \delta_j^k p^k b_i^k, \quad \forall i, j \in J \\ \sum_{j \in J} \delta_j^k & = 1, \quad \forall k \in K \end{aligned}$$

is a solution of the primal problem for p , and viceversa. Let $\bar{\delta}$ be a solution of (6.2). Let $N = \text{supp}(\bar{\delta})$ and $n = |N|$. If $n \leq |K| + m$, there is nothing to prove. Then, assume that

$n > |K| + m$. Consider the linear programming problem

$$\begin{aligned}
& \max_{\theta \geq 0} \sum_{k \in K} p^k \sum_{j \in J} \theta_j \bar{\delta}_j^k a_j^k & (6.3) \\
& \text{s.t.} \quad \sum_j \theta_j \bar{\delta}_j^k a_j^k \geq \sum_j \theta_j \bar{\delta}_j^{k'} a_j^{k'}, \quad \forall k, k' \in K \\
& \quad \quad \sum_{j \in J} \theta_j \bar{\delta}_j^k = 1, \quad \forall k \in K
\end{aligned}$$

Because $\bar{\delta}$ is a solution of (6.2), the vector $\bar{\theta} \geq 0$ defined by $\bar{\theta}_j = 1$ for all $j \in N$ and $\bar{\theta}_j = 0$ for all $j \in J \setminus N$ solves the linear program (6.3). By a fundamental result of linear programming, we can always find a solution of (6.3) among the extreme points of its feasible set. Therefore, since (6.3) has $|K| + m$ constraints, it has a basic feasible solution $\tilde{\theta} \geq 0$ with no more than $|K| + m$ strictly positive components. For every $j \in J$ and $k \in K$, we define $\tilde{\delta}_j^k = \tilde{\theta}_j \bar{\delta}_j^k \geq 0$. Then, $\tilde{\delta}$ satisfies $|\text{supp}(\tilde{\delta})| \leq |K| + m$,

$$\begin{aligned}
& \sum_{j \in J} \tilde{\delta}_j^k = \sum_{j \in J} \tilde{\theta}_j \bar{\delta}_j^k = 1, \quad \forall k \in K \\
& a(\tilde{\delta}) = \sum_{k \in K} p^k \sum_{j \in J} \tilde{\theta}_j \bar{\delta}_j^k a_j^k = \sum_{k \in K} p^k \sum_{j \in J} \bar{\theta}_j \bar{\delta}_j^k a_j^k = a(\bar{\delta})
\end{aligned}$$

and

$$\sum_{k \in K} \tilde{\delta}_j^k p^k b_j^k = \tilde{\theta}_j \sum_{k \in K} \bar{\delta}_j^k p^k b_j^k \geq \tilde{\theta}_j \sum_{k \in K} \bar{\delta}_j^k p^k b_i^k = \sum_{k \in K} \tilde{\delta}_j^k p^k b_i^k, \quad \forall i, j \in J$$

In summary, $\tilde{\delta}$ is a solution of the primal problem for p with $|\text{supp}(\tilde{\delta})| \leq |K| + m$. \square

Proof of Proposition 4. We start by proving the first statement. Assume that $j_R^H < j_R^L$ and let δ be a communication equilibrium inducing some revelation of information. Then, $|\text{supp}(\delta)| > 1$. Since $j_R^H < j_R^L$, $\sum_{l=1}^j \delta_l^L \leq \sum_{l=1}^j \delta_l^H$ for all $j \in J$, with strict inequality for every $j \in \text{supp}(\delta)$. Therefore, the sorting condition implies that

$$\begin{aligned}
\sum_{j \in J} (\delta_j^L - \delta_j^H) a_j^L &= \sum_{j=1}^{|J|-1} \left(\sum_{l=1}^j \delta_l^H - \sum_{l=1}^j \delta_l^L \right) [a_{j+1}^L - a_j^L] \\
&< \sum_{j=1}^{|J|-1} \left(\sum_{l=1}^j \delta_l^H - \sum_{l=1}^j \delta_l^L \right) [a_{j+1}^H - a_j^H] \\
&= \sum_{j \in J} (\delta_j^L - \delta_j^H) a_j^H
\end{aligned}$$

Thus, incentive compatibility for type H implies that incentive compatibility for type L cannot be satisfied. Therefore, the unique communication equilibria are non-revealing.

Now we prove the second statement. Assume that $j_R^H > j_R^L$ and let δ be a communication equilibrium inducing some revelation of information. Again, because $|\text{supp}(\delta)| > 1$, the sorting

condition together with (4.1) imply that

$$\begin{aligned}
\sum_{j \in J} (\delta_j^L - \delta_j^H) a_j^L &= \sum_{j=1}^{|J|-1} \left(\sum_{l=1}^j \delta_l^H - \sum_{l=1}^j \delta_l^L \right) [a_{j+1}^L - a_j^L] \\
&> \sum_{j=1}^{|J|-1} \left(\sum_{l=1}^j \delta_l^H - \sum_{l=1}^j \delta_l^L \right) [a_{j+1}^H - a_j^H] \\
&= \sum_{j \in J} (\delta_j^L - \delta_j^H) a_j^H
\end{aligned}$$

Hence, whenever one incentive constraint is binding, the other is slack. \square

Proof of Proposition 5. We prove the statement only for the case $j_S^L = J$. The other case is analogous but using the preferences of the type H . Let δ be a communication equilibrium inducing some revelation of information, so that $|\text{supp}(\delta)| > 1$. Because preferences are single-peaked, $a_j^L < a_{j+1}^L$ for all $j \in J$. A.1 implies (4.1). Then,

$$\sum_{j \in J} (\delta_j^L - \delta_j^H) a_j^L = \sum_{j=1}^{|J|-1} \left(\sum_{l=1}^j \delta_l^H - \sum_{l=1}^j \delta_l^L \right) [a_{j+1}^L - a_j^L] < 0$$

Hence, any communication equilibrium is non-revealing. \square

Proof of Proposition 6. Let δ be a communication device solving the problem (3.2) for p . Then, $pa^H(\delta) + (1-p)a^L(\delta) = \text{cav } \hat{a}(p)$ (recall that (3.2) has no information incentive constraints). If δ is NR, there is nothing to prove. Hence, assume that δ induces some revelation of information, so that $|\text{supp}(\delta)| > 1$. Since $p \in [p_{j_S^L}^-, p_{j_S^H}^+]$, $\delta_j^H = \delta_j^L = 0$ for any action $j < j_S^L$ or $j > j_S^H$. Thus we have that

$$\begin{aligned}
\sum_{j \in J} (\delta_j^L - \delta_j^H) a_j^L &= \sum_{j=1}^{|J|-1} \left(\sum_{l=1}^j \delta_l^H - \sum_{l=1}^j \delta_l^L \right) [a_{j+1}^L - a_j^L] \\
&= \sum_{j=j_S^L}^{j_S^H} \left(\sum_{l=1}^j \delta_l^H - \sum_{l=1}^j \delta_l^L \right) [a_{j+1}^L - a_j^L] > 0
\end{aligned}$$

since $j_R^H > j_R^L$ implies (4.1) and $a_{j+1}^L < a_j^L$ for all $j > j_S^L$. Similarly,

$$\begin{aligned}
\sum_{j \in J} (\delta_j^L - \delta_j^H) a_j^H &= \sum_{j=1}^{|J|-1} \left(\sum_{l=1}^j \delta_l^H - \sum_{l=1}^j \delta_l^L \right) [a_{j+1}^H - a_j^H] \\
&= \sum_{j=j_S^L}^{j_S^H} \left(\sum_{l=1}^j \delta_l^H - \sum_{l=1}^j \delta_l^L \right) [a_{j+1}^H - a_j^H] < 0
\end{aligned}$$

because $a_{j+1}^H > a_j^H$ for all $j < j_S^H$. We conclude that δ is incentive compatible for the sender, and thus it is a communication equilibrium achieving $\text{cav } \hat{a}(p)$. \square

Proof of Proposition 7. We start proving the first assertion. Fix $j < j_S$ and $p \in [p_j^-, p_j^+]$. Consider the communication device $\hat{\delta}$ defined by

$$\begin{aligned}\hat{\delta}_j^H &= \frac{p_j^- \hat{\rho}_j^-}{p}, & \hat{\delta}_{j+1}^H &= \frac{p_j^+ \hat{\rho}_j^+}{p}, & \hat{\delta}_{j_R}^H &= \frac{1 - \hat{\rho}_j^- - \hat{\rho}_j^+}{p} \\ \hat{\delta}_j^L &= \frac{(1 - p_j^-) \hat{\rho}_j^-}{1 - p}, & \hat{\delta}_{j+1}^L &= \frac{(1 - p_j^+) \hat{\rho}_j^+}{1 - p}, & \hat{\delta}_{j_R}^L &= 0\end{aligned}$$

This communication device induces posterior beliefs $p_j = p_j^-$, $p_{j+1} = p_j^+$ and $p_{j_R} = 1$, with probabilities $\hat{\rho}_j^-$, $\hat{\rho}_j^+$ and $1 - \hat{\rho}_j^+ - \hat{\rho}_j^-$, respectively. Clearly, $\hat{\delta}$ is incentive compatible for the receiver. By the definition of $(\hat{\rho}_j^-, \hat{\rho}_j^+)$, the incentive constraint asserting that type L should not gain by reporting H is binding at $\hat{\delta}$. Then, proposition 3 implies that the constraint asserting that type H should not gain by reporting L is slack. Therefore, $\hat{\delta}$ is a communication equilibrium of $\Gamma(p)$.

Notice that

$$a(\hat{\delta}) = \hat{\rho}_j^- \hat{a}(p_j^-) + \hat{\rho}_j^+ \hat{a}(p_j^+) + (1 - \hat{\rho}_j^+ - \hat{\rho}_j^-) a_J^H$$

Then, we have to show that $\hat{\delta}$ is an optimal solution of the primal problem for p .

Now, let $\hat{\gamma}$ be a solution of the dual for p . By lemma 3, $\gamma^* \equiv \hat{\gamma}(H | L) > 0$. Because the incentive constraint of the type H is slack, complementary slackness implies that $\hat{\gamma}(L | H) = 0$. Therefore, the virtual utilities of the sender are

$$\alpha_i^H(p, \gamma^*) = a_i^H - \frac{\gamma^*}{p} a_i^L, \quad \alpha_i^L(p, \gamma^*) = a_i^L + \frac{\gamma^*}{1-p} a_i^L$$

Consider the (p, γ^*) -virtual persuasion game. By inducing posterior beliefs $p_j = p_j^-$, $p_{j+1} = p_j^+$ and $p_J = 1$, with probabilities $\hat{\rho}_j^-$, $\hat{\rho}_j^+$ and $1 - \hat{\rho}_j^+ - \hat{\rho}_j^-$, respectively, the sender guarantees an expected virtual payoff equal to

$$\begin{aligned}& \hat{\rho}_j^- \hat{\alpha}(p_j^-; p, \gamma^*) + \hat{\rho}_j^+ \hat{\alpha}(p_j^+; p, \gamma^*) + (1 - \hat{\rho}_j^+ - \hat{\rho}_j^-) \alpha_J^H(p, \gamma^*) \\ &= \hat{\rho}_j^- \left[p_j^- \alpha_j^H(p, \gamma^*) + (1 - p_j^-) \alpha_j^L(p, \gamma^*) \right] + \hat{\rho}_j^+ \left[p_j^+ \alpha_{j+1}^H(p, \gamma^*) + (1 - p_j^+) \alpha_{j+1}^L(p, \gamma^*) \right] \\ & \quad + (1 - \hat{\rho}_j^+ - \hat{\rho}_j^-) \alpha_J^H(p, \gamma^*) \\ &= \hat{\rho}_j^- \hat{a}(p_j^-) + \hat{\rho}_j^+ \hat{a}(p_j^+) + (1 - \hat{\rho}_j^+ - \hat{\rho}_j^-) a_J^H \\ & \quad + \gamma^* \left[\hat{\rho}_j^- \frac{p - p_j^-}{p(1-p)} a_j^L + \hat{\rho}_j^+ \frac{p - p_j^+}{p(1-p)} a_{j+1}^L - \frac{1 - \hat{\rho}_j^+ - \hat{\rho}_j^-}{p} a_J^L \right] \\ &= \hat{\rho}_j^- \hat{a}(p_j^-) + \hat{\rho}_j^+ \hat{a}(p_j^+) + (1 - \hat{\rho}_j^+ - \hat{\rho}_j^-) a_J^H \\ & \quad + \gamma^* \sum_{i \in J} [\hat{\delta}_i^L - \hat{\delta}_i^H] a_i^L \\ &= \hat{\rho}_j^- \hat{a}(p_j^-) + \hat{\rho}_j^+ \hat{a}(p_j^+) + (1 - \hat{\rho}_j^+ - \hat{\rho}_j^-) a_J^H\end{aligned}$$

Then, the sender guarantees $a(\hat{\delta})$ in both the primal problem and the dual problem. Thus, by the optimality property of weak duality

$$a^*(p) = \hat{\rho}_j^- \hat{a}(p_j^-) + \hat{\rho}_j^+ \hat{a}(p_j^+) + (1 - \hat{\rho}_j^+ - \hat{\rho}_j^-) a_J^H$$

Therefore, $\hat{\delta}$ satisfies the complementary slackness condition (3.8) and the optimality condition (3.9) for the dual variables $\hat{\gamma}$. We conclude that $\hat{\delta}$ is an optimal solution of the primal problem for p .

Assume that $a_j^L = a_j^L$ and consider $p \in (p_j^-, p_j^+]$. Then, $\hat{\rho}_j^+ = 0$ and $\hat{\rho}_j^- < 1$. Hence, by lemma 1, $a_j^H > a_j^H$. Therefore, we have

$$\begin{aligned} a^*(p) &= \hat{\rho}_j^- \hat{a}(p_j^-) + (1 - \hat{\rho}_j^-) a_j^H \\ &> \hat{\rho}_j^- \hat{a}(p_j^-) + (1 - \hat{\rho}_j^-) a_j^H \\ &= p a_j^H + (1 - p) a_j^L \\ &= \hat{a}(p) \end{aligned}$$

Assume now that $a_j^L > a_j^L$ and consider $p \in (p_j^-, p_j^+)$, so that, $0 < \hat{\rho}_j^+ < 1$. Observe that

$$\begin{aligned} a^*(p) &= \hat{\rho}_j^- \hat{a}(p_j^-) + \hat{\rho}_j^+ \hat{a}(p_j^+) + (1 - \hat{\rho}_j^+ - \hat{\rho}_j^-) a_j^H \\ &= \hat{a}(p) + \hat{\rho}_j^+ \left[p_j^+ (a_{j+1}^H - a_j^H) + (1 - p_j^+) (a_{j+1}^L - a_j^L) \right] + (1 - \hat{\rho}_j^+ - \hat{\rho}_j^-) (a_j^H - a_j^H) \\ &= \hat{a}(p) + \hat{\rho}_j^+ \left[p_j^+ (a_{j+1}^H - a_j^H) + (1 - p_j^+) (a_{j+1}^L - a_j^L) + \left(\frac{p - p_j^-}{\hat{\rho}_j^+(1 - p_j^-)} - \frac{p_j^+ - p_j^-}{1 - p_j^-} \right) (a_j^H - a_j^H) \right] \\ &= \hat{a}(p) + \hat{\rho}_j^+ \left[p_j^+ (a_{j+1}^H - a_j^H) + (1 - p_j^+) (a_{j+1}^L - a_j^L) + \frac{p_j^+ - p}{1 - p} \frac{a_{j+1}^L - a_j^L}{a_j^L - a_j^L} (a_j^H - a_j^H) \right] \\ &> \hat{a}(p) + \hat{\rho}_j^+ \left[p_j^+ (a_{j+1}^H - a_j^H) + (1 - p_j^+) (a_{j+1}^L - a_j^L) + \frac{p_j^+ - p}{1 - p} \frac{a_{j+1}^L - a_j^L}{a_j^L - a_j^L} (a_j^L - a_j^L) \right] \\ &= \hat{a}(p) + \hat{\rho}_j^+ \left[p_j^+ (a_{j+1}^H - a_j^H) + (1 - p_j^+) (a_{j+1}^L - a_j^L) - \frac{p_j^+ - p}{1 - p} (a_{j+1}^L - a_j^L) \right] \\ &> \hat{a}(p) + \hat{\rho}_j^+ \left[p_j^+ (a_{j+1}^L - a_j^L) + (1 - p_j^+) (a_{j+1}^L - a_j^L) - \frac{p_j^+ - p}{1 - p} (a_{j+1}^L - a_j^L) \right] \\ &= \hat{a}(p) + \hat{\rho}_j^+ \frac{1 - p_j^+}{1 - p} (a_{j+1}^L - a_j^L) \\ &> \hat{a}(p) \end{aligned}$$

We proceed in a similar fashion for (ii), this time using the communication device

$$\begin{aligned} \tilde{\delta}_{j-1}^H &= \frac{p_j^- \tilde{\rho}_j^-}{p}, & \tilde{\delta}_j^H &= \frac{p_j^+ \tilde{\rho}_j^+}{p}, & \tilde{\delta}_j^H &= 0 \\ \tilde{\delta}_{j-1}^L &= \frac{(1 - p_j^-) \tilde{\rho}_j^-}{1 - p}, & \tilde{\delta}_j^L &= \frac{(1 - p_j^+) \tilde{\rho}_j^+}{1 - p}, & \tilde{\delta}_j^L &= \frac{1 - \tilde{\rho}_j^- - \tilde{\rho}_j^+}{1 - p} \end{aligned}$$

□

7. References

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