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Theory of semiotic mediation in teaching and learning linear algebra: In search of a viewpoint in the use of ICT

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This paper presents a research project on the description of a linear algebra course within the perspective of the Theory of Semiotic Mediation (TSM), in order to construct mathematical meanings prioritizing artefact–sign (i.e., ICT tools) and social context relationships. I will first describe the project and, following a theoretical framework of the TSM, discuss a priori epistemological analysis of GeoGebra’s potentiality to be used as an artefact to bring out key linear algebra concepts in future didactic interventions. I also present the results of a pilot study elaborating the potentiality of GeoGebra for students’ construction of the mathematical meaning of free variables. Future steps of the project are also outlined.

Keywords: Theory of semiotic mediation, teaching and learning linear algebra, ICT.

LEARNING LINEAR ALGEBRA: EPISTEMOLOGICAL ISSUES

Because of the strength of generalization, linear algebra is a powerful theory used to frame problems belonging to quite different contexts, but at the same time, it may be difficult to construct in itself. From a didactic point of view, merely taking a theoretical approach, i.e., introducing *vector space* concept to students, that is, only giving axioms, means asking students to enter a meaningless game of symbols, because the historical genesis of linear algebra indicates numerous lengthy steps to form an evolution of the idea of vector spaces. The notion of a *vector* begins with Aristotle who represented ‘force’ in geometric terms (Chong, 1985). However, after approximately 2000 years, mathematical representations of vectors were used as points and directed line segments by Gauss and Hamilton in terms of Cartesian geometry.

Researchers used *matrices* to represent these geometrical ideas, as well as linear equations, and, by practical discernment, they obtained certain extensions to elimination techniques, n-tuples, determinants and transformations. However, all these developments were operational, sometimes having different theoretical elements, and so used different mathematical languages. There was, however, a missing *unified-general* approach covering and connecting all of them. In 1888, Giuseppe Peano defined the vector space concept with an axiomatic system, as *a set* satisfying certain axioms (Dorier, 2000). This was a formal definition, and opened a door to non-geometric vector spaces, such as polynomials and square matrices. Since we, as linear algebra lecturers, use all of these concepts together, one can conclude that the use of different notations, depictions and axiomatic language is essential for the teaching of linear algebra. From a didactical point of linear algebra education, the axiomatic-formal system of the course reveals a learning obstacle in teachers’ hands; the *obstacle of formalism* (Dorier, Robert, Robinet, & Rogalski, 2000). This obstacle is associated with the specific *terminology* of linear algebra and appears when students are faced with the mathematical *symbol language* triangle, composed of equations, matrices and vectors. Students waver ‘under an avalanche of new words, symbols, definitions and theorems’ and therefore, for many students, ‘linear algebra is no more than a catalogue of very abstract notions’ (ibid. p. 95). In summary, the formalism obstacle can be considered as students’ failure to grasp linear algebra’s symbols and their associated-corresponding *mathematical meanings*. From the viewpoint of *semiotic registers* (Duval, 2006) of such mathematical meanings, learning linear algebra needs *conversion* between different registers; ‘graphical’, ‘tabular’ and ‘symbolic languages’ of linear

algebra or, in other words, students should have ‘cognitive flexibility’ to overcome the obstacle of formalism (Dorier & Artigue, 2001, p. 270).

The construction of mathematical meanings cannot be easily achieved through a direct use of ICT, needing a careful *didactic design* of tasks to exploit the use of *artefacts* (Mariotti, 2012). I will try to overcome the obstacle of formalism using such didactic designs enriched with ICT tools, in particular GeoGebra (5.0 version) (its potentiality will be presented in detail), for students’ construction of mathematical meanings of linear algebra concepts. Taking a semiotic approach, I will focus on the *Theory of Semiotic Mediation* (TSM) (Bartolini Bussi & Mariotti, 2008); both the design of the tasks and analyses of the processes.

RESEARCH QUESTIONS AND THEORETICAL FRAMEWORK

This study is a one-year, post-doctoral research project, planned to start in February 2015, focusing on the following research questions:

- *How should a linear algebra course be designed within a didactic-semiotic perspective?*
- *Does this approach overcome the formalism obstacle of linear algebra students?*

Taking into account the TSM, I intend to address both research questions. This is because the TSM is a Vygotskian-rooted approach in math education proposed by Bartolini Bussi and Mariotti (2008). It relates to the *semiosis* feature of math, which bases the teacher’s actions in a *social context* and on the hypothesis that the *production of signs* can be elaborated on when the teacher intentionally uses an artefact to accomplish a math task within a communication-oriented process. By use of specific artefacts in the mediation process, the TSM aims to construct math meanings in students’ mental schemes; in other words, to transform *personal meanings* into *math meanings*, while they solve the proposed tasks as *mediator*. In this process, the main focus is on the *emergence of signs* that foster students’ possible math learning. Within this aim, the TSM is constructed on two key elements: *the notion of the semiotic potential of an artefact* and *the notion of a didactic cycle*. The semiotic potential of an artefact is associated with its ‘... *evocative power*, stressing the distinction between

meanings emerging from the activity with the artefact and the math meanings evoked by such activity’ (Mariotti, 2013, p. 442). In other words, it is related to the potential for math meanings to emerge whilst students solve a mathematical task. The notion of a *didactic cycle* is about the design of the teaching-learning process, especially describing semiotic processes: (i) *activities with artefact* (students work in pairs or in small groups), (ii) *individual production of signs* and (iii) *collective production of signs* (Bartolini Bussi & Mariotti, 2008, pp. 754–755). Iteration of such didactic-semiotic (environment) processes aims to foster the evolution of personal signs-meanings to (desired) math signs-meanings. This is because the (desired) evolution of the signs is ‘artefact signs’, ‘pivot (hinge) signs’ and culturally accepted ‘mathematical signs’ (ibid, pp. 756–757). Therefore, as an important component of this polysemy of the artefact, the teacher has a central role; she should orchestrate mediation with specific social activities to exploit the semiotic potential of the artefact (Mariotti, 2013), i.e., her role should be surrounded by an interacting triangle of *use of artefact*, *personal meanings*, and *math meanings* in a socially-communicative environment.

Why I select TSM in this project to overcome obstacle of formalism

Harel (2000) proposes three teaching principles for learning linear algebra (p. 180):

- the *concreteness principle* associated with results stemming from the use of the *axiomatic language* of linear algebra and students’ pedagogical needs, in particular, the transition from geometric to abstract features. For instance, the concept of a polynomial as a vector is not concrete for students if they cannot comprehend the mathematical notion of linear independency.
- the *necessity principle* that refers to instructional activities which should form a *problematic environment* for the construction of mathematical concepts, with students seeing an ‘intellectual need’ (ibid, p. 185).
- the *generalization principle* that reflects students’ generalization of driven concepts after the problem-solving process; in particular, with the help of the *intellectual need* character of the proposed learning environment.

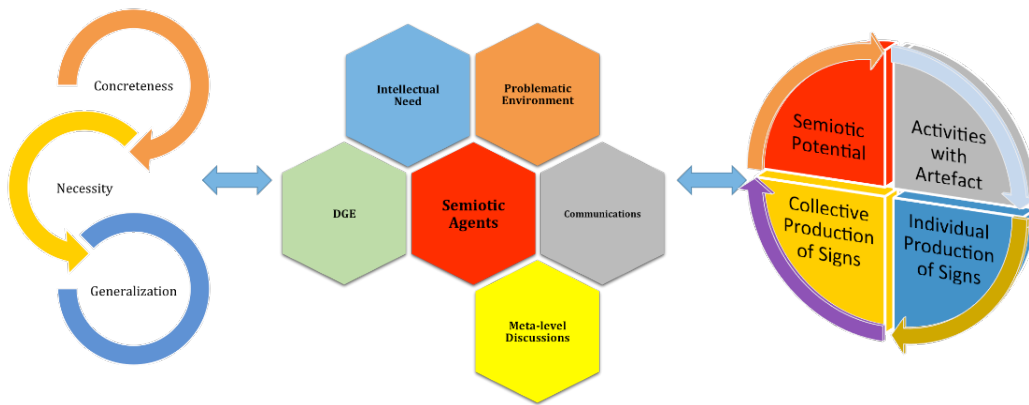


Figure 1: Key epistemological concepts in relation to the TSM

In summary, one can conclude that construction of math meanings of the geometric features of linear algebra through ICT tools can provide dialectic relationships among abstract features. The use of ICT didactic designs should provide a problematic environment (necessity) (Turgut, 2013); preparatory insights (concreteness) for non-geometric linear algebra (Harel, 2000), and math discussions for comprehending abstract concepts. In other words, as Dorier and Artigue (2001) emphasize, the lecturer should create an environment for students, which provides an opportunity to reflect on ‘*meta-level discussions*’ of problems, referring to the generalization principle. This can also be an opportunity to *unify* linear algebraic concepts (p. 271).

Several attempts have been done within the semiotic perspective to analyse and construct math meanings of linear algebra with ICT tools. Sierpiska, Trgalová, Hillel and Dreyfus (1999) designed a research program with Cabri–geometry II of a geometric model of vector space in order to overcome the obstacle of formalism. Lengthy teaching experiments (within different conceptual perspectives) reveal that students are able to grasp math meanings, in particular specific concepts, such as linear combination (ibid, p. 129). Hillel and Dreyfus (2005) investigate how the conditions of *communication* influence the construction of math meanings, whilst students are attempting to solve tasks regarding the projection of vectors and an approximation with Maple CAS. Episodes were constructed on different semiotic systems, in particular *agents* for communication; the students themselves, an observer, the computer and Maple, classroom teacher, classroom notes and text. At the end of the sessions, the agents (communicative semiotic environment) were able to contribute to students’ construction of new meanings of linear algebra

concepts. The researchers also emphasize the Maple role as a ‘silent mediator’ that stems from students’ wait for-know-decide-use process of the commands. In summary, this was a glimpse of the use of dynamic geometry environments (DGE) as an artefact in the teaching/ learning linear algebra.

To sum up, all these together imply a puzzle including several keywords. I postulate and summarize the following key concepts to overcome the obstacle of formalism that, all together, fit with the TSM (Figure 1).

The keywords in Figure 1 have dialectic relationships with the TSM’s core elements. For instance, intellectual need and problematic environment are related, but the dialectic relationship between them also implies the notion of the semiotic potential of the artefact and task designs. Consequently, the TSM underpins this process, because the TSM is a powerful framework for both the design of didactic cycles and analysis of the signs. Another fact is that the communication-meta level discussions process corresponds to Individual and Collective Productions of Signs and so on.

Within the framework of Figure 1, I hypothesize that GeoGebra may be a powerful artefact to use in task designs because one main potentiality of the use of GeoGebra is its involvement of a 3D interface, and hypothetically, this might help students to *unify* the different representations of linear algebra. However, it needs a grain analysis to describe the key points that we want to emerge from its use.

Semiotic potential of some tools of GeoGebra: A priori analysis

The following *priori* epistemological analysis of GeoGebra will describe our goals and answer the following questions; what key linear algebra concept can

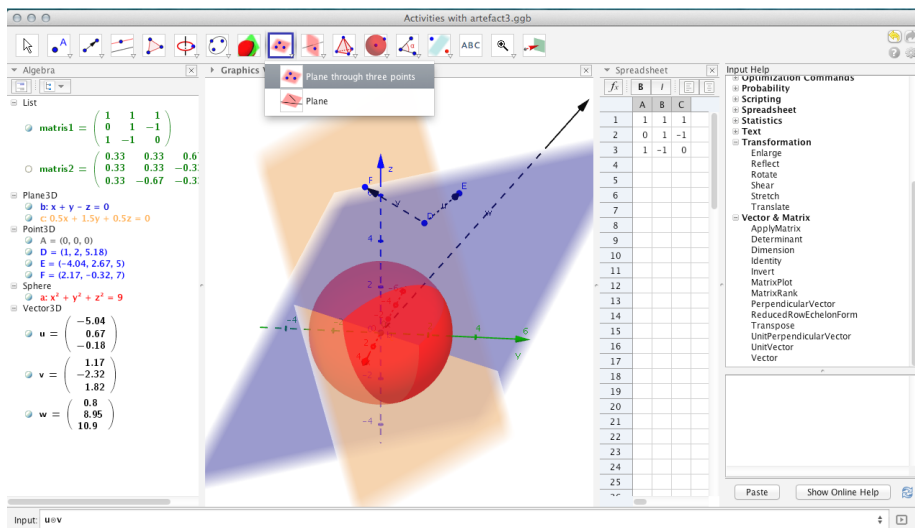


Figure 2: GeoGebra interface with different windows

emerge from the artefact, and what can we do with GeoGebra? As a consequence, this process leads us to discuss the *semiotic potential of the artefact* that is the core element of the TSM. The analyses of semiotic potential has a *dialectic* relationship between explanation of goals and attempting to see what happens when we present the designed tasks to students, and then see which of their *utilization schemes* are useful to transform into mathematical meanings. To sum up, the process of outlining the semiotic potential of an artefact needs a deep analysis encompassing an epistemological and didactic-cognitive perspective that will give us a framework to assist in the design of the second phase of the TSM. Let us first describe this artefact's tools in relation to linear algebra.

In the GeoGebra interface (Figure 2), all the windows can be viewed together; the CAS, Graphics, 3D Graphics, Spreadsheet and Algebra. Using a spreadsheet table, such as Excel, one can form matrices with any rows or columns. They immediately appear in the Algebra window. With the help of the Input Help column (right side), other forms can also be accessed. The key element, a vector, can be composed by the Input line or in the Spreadsheet window in the form of co-ordinates and vectors (also with manipulations, cross product or forming a line) that appear simultaneously in the Graphics window. The Graphics window enables the plotting of a 3D view of the lines, planes and surfaces. At the same time, the tools of this window provide certain 3D applications, but the important one is the plane through three points. In addition, the Transformation part in the Input Help window provides a matrix transformations application in the plane, but not in 3D. As a limitation, one

can see several tools in the windows, but the tools may be limited for the purpose of the tasks to prevent possible cognitive loads on the students. The viewing of these windows all together can create an environment for the possible *conversions* of different semiotic registers by students that may help them unify linear algebraic concepts. Therefore, I decided to consider the following four different semiotic registers (in sense of Duval, 2006): *algebraic register* (AR), *2D graphics register* (2DGR), *3D graphics register* (3DGR), and *spreadsheet register* (SR). Conversion of these registers might help students to shift different representations of linear algebraic concepts. Besides this powerful feature of the artefact, 2DGR has a *slider* component that forms an environment as a *dynamic variation*. This process is also applicable in other registers by *moving*, or the controlled movement of the mouse. These may be important in enabling students to evolve meanings of particular notions, because *variation* in 2DGR and 3DGR, provided by the slider and AR, may be key elements in the construction of associated meanings in the design of cycles for our future didactic interventions. I limit myself to the emergence of 'free variables in \mathbb{R}^3 ' in the system of linear equations and associated geometric invariants with the following pilot study, which aforementioned registers can evoke students' learning.

PILOT STUDY AS A TEACHING EXPERIMENT

In this part, I attempt to elaborate the semiotic potential of GeoGebra, in particular, the use of AR, 2DGR (slider tool) and 3DGR, in the construction of a link between the system of linear equations, augmented matrices and intersection of planes. In other words,

I focus on the question, ‘will the use of those semiotic registers evoke construction of mathematical meaning of free variables?’ For this purpose, I designed a task involving manipulation on different registers (inspired by the problem in Anton, 1981, p. 54). The educational goals of the experiment are:

- conversions among 2DGR (slider), 3DGR and AR,
- evolution of personal meanings to mathematical meanings of a free variable,
- fostering construction of the mathematical link among the concepts.

Using three registers, I prepared a task, as described in Figure 3. The participants of the experiment were two sophomore level undergraduate mathematics education students, and the experiment was implemented at the beginning of a linear algebra course, following the topic of solving the system of linear equations. A number of students volunteered to participate, but only two were selected according to their mathematical background and communication skills. They had performed moderately on former courses, and had taken only three mathematical courses; general mathematics, abstract mathematics and geometry. The fact is that they knew the equations of a plane, and the augmented matrices corresponding to the system of linear equations. The students were unable to use GeoGebra since, prior to the experiment, I had introduced the main tools of the software by removing any unnecessary tools, i.e., the only tools in the 3DGR were *move* and *rotate*. Thereafter, they practised dragging and shifting in the windows.

Task: Move the sliders a and b , and explore and explain what is happening in the GeoGebra interface systematically.

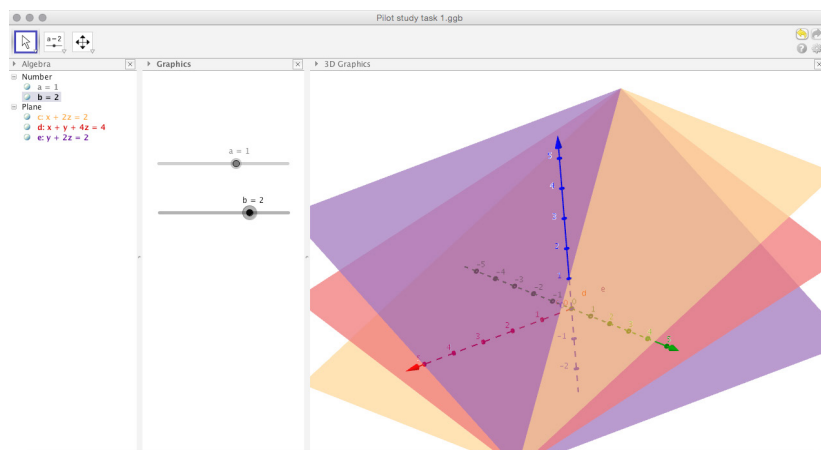


Figure 3: GeoGebra interface of the proposed task

The expected situations in this experiment were; (i) the students’ analyses using two sliders for a and b (2DGR), (ii) at the same time analysing the variations of the equations in AR, (iii) and the positions and manipulations of the plane in the 3DGR. Overall, the task was prepared to analyse the system of linear equations in terms of the variations of a and b , (for instance when $a=0$, $b=2$, the planes coincide). I estimated that, after they analysed such variations, they would focus on the system in AR, and thereafter, they would build an augmented matrix in order to attempt to solve the system. As a next step, they would compare their results within dynamic variations in 2DGR and 3DGR, also associated with the *intellectual needs* of the task. In the end, they would first construct a mathematical meaning of *two free variables*, as well as its corresponding meanings on the planes’ movements and intersections (c , d , e in AR, with yellow, red and purple planes in 3DGR).

Procedure and data analysis

The experiment was implemented as a teacher-researcher, involving Buse and Deniz (pseudonyms), working as a pair on a computer screen. The data consisted of video-recorded interviews, screen recorder software and students’ productions, with the teaching episode ending after 40 minutes. Within a semiotic lens, the data was analysed with respect to the TSM’s frame of categories of signs (Bartolini Bussi & Mariotti, p. 756); emergence of *artefact signs*, *pivot-hinge signs* and *math signs*. Artefact signs are a production driven as a result of an immediate use of the artefact characterizing the proposed task; math signs refer to math meanings, such as definitions, a proposition or a math proof. Pivot signs refer to hy-

brid terms in natural language, such as ‘object’ or ‘thing’, associated with math terminology (ibid.).

Summary of the results

After the teacher had introduced the task, the students tried to make certain interpretations by moving the sliders and analyzing the different registers together, and, thereafter, they focused on the equations in the AR. In this way, they comprehended what was differentiating in the plane equations whilst they dragged the sliders, since they realized that a system of linear equations existed with the key values being $a=0$ and $b=2$. They also formed the planes’ equations with respect to a and b , and Buse pointed out that the solution of the system must be related to these values. In this process, the teacher was orienting the students to focus on the relationship between the a and b values as well as the solution of the system of linear equations. The following excerpt is drawn from the discussion from which the signs evolved: from artefact type to mathematical signs.

- 45 Teacher: You said at the moment, values of $a=0$ and $b=2$ must be in relation to a solution of this [indicating the system on the paper sheet] system, how you are sure about that?
- 46 Buse: Because changing the values here [indicating the 2DGR], shows us different type intersections here [indicating 3DGR], in fact, sliders are affecting the equations [meaning the AR]. These intersections must be similar to the position of the lines that we discussed.
- 47 Deniz: Exactly, look [moving the sliders] if $a=0$ and $b=2$, all the planes coincide. Oh yes, it is already obvious here [referring AR], in all the equations $z=1$, as in the case of coinciding lines, therefore, I think, the solution must be infinite here.

Thereafter, the students also explain other cases using their knowledge, stemming from an analysis of the system of linear equations in R^2 : when $a \neq 0$, $b \neq 2$, the intersection of the planes is ‘single point’, ‘exact solution’ and ‘consistent system’. Similarly, if $a = 0$, $b \neq 2$, they mention ‘parallel planes’, ‘no solution’, and ‘inconsistent system’. Thereafter, the teacher asks how to find the relationship between the augmented matrix and the interpretations expressed by the students. They use the Gauss-Jordan elimination method. The following excerpt is drawn from this discussion.

- 71 Buse: Look [showing the solution her pair], I put $a=0$ and $b=2$ to see what will happen, I could only calculate $z=1$. I can not find x or y .
- 72 Deniz: We can not find, look, the second and third rows are completely zero.
- 73 Teacher: What does it mean? How can you relate this fact with your initial interpretation?
- 74 Deniz: Any real number can satisfy this system if we put it instead of x or y . They are independent from the plane equations, since, this is consistent with the picture here [indicating 3DGR], where there are two variables that we cannot find, but the solution, I mean, the intersection is a plane: a two-dimensional thing. But I am not sure whether this hypothesis is valid for other cases.
- 75 Teacher: Let’s analyse other cases then.
- 76 Deniz: [She is moving the sliders, checking her hypothesis] If we take $a=1$ and $b=2$, there is again an infinite solution, [looking at the matrix form] and we can not find y either x , but $x=y$, since we have one free variable, the intersection is a line; a one dimensional thing.

As a consequence, Buse points out the exact solution, ‘no free variable’, and therefore, ‘single point and 0-dimensional thing’. To sum up, I observed a semiotic chain (Bartolini Bussi & Mariotti, 2008) in the use of different semiotic registers, from artefact signs to math signs of *free variables* such as: ‘chancing value’ (item 46); ‘intersection of the planes’ (item 46); ‘solution types’ (item 47); ‘independence from equation’ (item 74); ‘variable’ (item 74); ‘dimension’ (items 74–75); and ‘free variable’ (item 76).

CONCLUSIONS

Even if linear algebra does not consist of only geometrical features, as Harel (2000) states, it can be a powerful ‘corridor to the more abstract algebraic concepts’ (p. 185). In this project, I aim to construct math meanings of geometrical features in linear algebra concepts within the semiotic potential of an artefact; with GeoGebra, and particularly in this work, I focus on the notion of a free variable. Through this experiment, I conclude that the emergence of the notion of a free variable has a strong link with student recognition of geometry and to relating geometric

objects' invariants with their infinitive characteristics, such as variables. The use of different registers helped students to articulate the different cases. In the experiment, I realized that the main feature was the students' use of the software. Because they did not know how to use it as an instrument, they were not able to master the use of the sliders when they changed windows. The project will continue with underpinning teaching experiments and case studies to analyze the semiotic potential of the mentioned software in terms of, 'what key linear algebra concepts can emerge' through its use, and I will also point out students' utilization schemes in sense of Rabardel (1995). The description of such utilization schemes may also be a basis for describing possible meanings that may emerge during the designed activities.

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REFERENCES

- Anton, H. (1981). *Elementary Linear Algebra* (3rd ed.). New York, NY: John Wiley & Sons.
- Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: Artifacts and signs after a Vygotskian perspective. In L. English, M. Bartolini Bussi, G. Jones, R. Lesh, & D. Tirosch (Eds.), *Handbook of International Research in Mathematics Education* (2nd ed.) (pp. 746–783). Mahwah, NJ: Lawrence Erlbaum.
- Chong, C. T. (1985). *Some remarks on the history of linear algebra*. Given talk at the Workshop on Linear Algebra and Its Applications, 9–10 September 1985, National University of Singapore.
- Dorier, J.-L. (2000). Epistemological analysis of the genesis of the vector spaces. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 3–81). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Dorier, J.-L., & Artigue, M. (2001). Research into the teaching and learning of linear algebra. In D. Holton (Ed.), *The teaching and learning of mathematics at university level* (pp. 255–273). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Dorier, J.-L., Robert, A., Robinet, J., & Rogalski, M. (2000). The obstacle of formalism in linear algebra. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 85–124). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103–131.
- Harel, G. (2000). Three principles of learning and teaching mathematics. In J.-L. Dorier (Ed.), *On the teaching of linear algebra* (pp. 177–189). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Hillel, J., & Dreyfus, T. (2005). What's a best fit? Construction of meaning in a linear algebra session. In J. Kilpatrick, C. Hoyles, & O. Skovsmose (Eds.), *Meaning in mathematics education* (pp. 181–204). New York, NY: Springer.
- Mariotti, M. A. (2012). ICT as opportunities for teaching–learning in a mathematics classroom: The semiotic potential of artefacts. In T. Y. Tso (Ed.), *Proceedings of the 36th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 25–35). Taipei, Taiwan: PME.
- Mariotti, M. A. (2013). Introducing students to geometric theorems: how the teacher can exploit the semiotic potential of a DGS. *ZDM*, 45, 441–452.
- Sierpinska, A., Trgalová, J., Hillel, J., & Dreyfus, T. (1999). Teaching and learning linear algebra with Cabri. In O. Zaslavsky (Ed.), *Proceedings of the 23rd Conference of the International Group for the Psychology of the Mathematics Education* (pp. 119–134). Haifa, Israel: Israel Institute of Technology.
- Rabardel, P. (1995). *Les homes et les technologies—Approche cognitive des instruments contemporains*. Paris, France: Armand Colin.
- Turgut, M. (2013). Applications *Mathematica* into teaching of linear algebra: The case of least – squares. In E. Faggiano & A. Montone (Eds.), *Proceedings of the 11th International Conference on Technology in Mathematics Teaching-ICTMT11* (pp. 286–291). Bari, Italy: Università degli Studi di Bari Aldo Moro.