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# Convergence to equilibrium for Fokker-Planck equation with a general force field

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## Abstract

The aim of this paper is to study the convergence of the solution of the Fokker-Planck equation to the associated stationary state when time goes to infinity. The force field which we consider here is of a general structure, that is it may not derive from a potential. The proof is based on an adequate splitting  $L = B + A$  of the Fokker-Planck operator  $L$  and the use of Krein-Rutmann theory.

KEYWORDS: Fokker-Planck equation, Semigroup,  $m$ -dissipative, linear operators, Krein-Rutman theorem, asymptotic convergence.

## Contents

<b>1</b>	<b>Introduction and main results</b>	<b>2</b>
1.1	Hypotheses . . . . .	3
1.2	Main results . . . . .	4
<b>2</b>	<b>Definitions and notations</b>	<b>4</b>

<b>3 Preliminary results</b>	<b>6</b>
<b>4 Existence of a stationary solution</b>	<b>18</b>
<b>5 Exponential stability</b>	<b>24</b>

# 1 Introduction and main results

In this paper we study the Fokker-Planck equation

$$\begin{cases} \partial_t f = \Delta f + \operatorname{div}(\mathbf{E}f) \\ f(0, x) = f_0. \end{cases} \quad (1.1) \quad \boxed{\text{eq:1.1}}$$

We assume that  $(1 + |x|^2)^{k/p} f_0 \in L^p(\mathbb{R}^d)$  for some  $k \geq 0$  and  $\mathbf{E} : \mathbb{R}^d \mapsto \mathbb{R}^d$ . Unlike previous works, we consider a general vector field  $\mathbf{E}$ , which does not derive necessarily from a potential. Under suitable assumptions on the vector field we establish that  $f(t) \rightarrow M(f_0)G$  with exponential rate as time  $t$  goes to infinity. The function  $G$  stands for the positive stationary solution of [\(1.1\)](#) with total mass equal to 1, and the mass of a function is defined throughout this paper by

$$M(g) := \int_{\mathbb{R}^d} g(x) dx, \quad \text{for all integrable function } g$$

We thus generalise similar results obtained by M.P. Gualdani, S. Mischler and C. Mouhot in [\[6\]](#), where the force field  $\mathbf{E}$  is assumed to be of the form  $\mathbf{E} = \nabla\varphi + U$ , with  $\varphi : \mathbb{R}^d \mapsto \mathbb{R}$  is of class  $\mathcal{C}^2$ , and  $U : \mathbb{R}^d \mapsto \mathbb{R}^d$  is of class  $\mathcal{C}^1$  are such that,  $\operatorname{div}(Ue^{-\varphi}) = 0$ . In that case, it is noticed that  $\mu := e^{-\varphi}$  is a stationary solution, and a Poincaré inequality

$$\int_{\mathbb{R}^d} |u - M(u)|^2 e^{-\varphi} dx \leq C \int_{\mathbb{R}^d} |\nabla u|^2 e^{-\varphi} dx$$

holds. For instance one may choose a function  $\varphi$  of the form

$$\varphi(x) = \frac{1}{\gamma} |x|^\gamma + c_0,$$

where  $\gamma \geq 1$  and  $c_0 \in \mathbb{R}$  is such that

$$\int_{\mathbb{R}^d} e^{-\varphi(x)} dx = 1,$$

that is  $\mu = e^{-\varphi}$  is a probability measure. Thus under some appropriate conditions on the function  $\varphi$  they prove that the solution  $f$  of (1.1) converges to  $M(f_0) \mu$ , when  $t$  goes to infinity, where we denote

$$M(f_0) := \int_{\mathbb{R}^d} f_0(x) d(x) = \int_{\mathbb{R}^d} f_0(x) dx. \quad (1.2) \quad \boxed{\text{eq:1.2}}$$

## 1.1 Hypotheses

Throughout this paper, we assume that the following assumption (**H0**) holds:

(**H0**) The vector field  $\mathbf{E} \in W_{\text{loc}}^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$ .

Furthermore in the statement of our results we may require some of the following assumptions:

(**H1**) For some constants  $\alpha, \alpha_2 > 0 \geq 0$ , and  $\beta, \beta_2 \in \mathbb{R}$  and exponents

$$1 < \gamma \leq \gamma_2 \leq 2$$

and for all  $x \in \mathbb{R}^d$ , the following holds

$$\alpha|x|^\gamma - \beta \leq x \cdot \mathbf{E} \leq \alpha_2|x|^{\gamma_2} + \beta_2. \quad (1.3) \quad \boxed{\text{eq:1.3}}$$

(**H2**) There exist  $\beta_0 \in \mathbb{R}$  and some  $k > 0$  such that the following holds

$$-\text{div}(\mathbf{E}) + k \frac{x}{1+|x|^2} \cdot \mathbf{E} \geq \beta_0. \quad (1.4) \quad \boxed{\text{eq:1.4}}$$

(**H3**) There exists  $\omega^* > 0$  and  $R > 0$  large enough, such that for all  $k > 0$ ,  $p \in [2, +\infty)$ , and for all  $x \in \mathbb{R}^d$  satisfying  $|x| > R$ , we have

$$-\frac{kd + k(k+d-2)|x|^2}{(1+|x|^2)^2} - \frac{p}{p'} \text{div}(\mathbf{E}) + k \frac{x}{1+|x|^2} \cdot \mathbf{E} \geq \omega^*. \quad (1.5) \quad \boxed{\text{eq:1.5}}$$

An example of such vector fields is given by

$$\mathbf{E} := \mathbf{E}_0 + \mathbf{E}_1,$$

where

$$\mathbf{E}_0 := \frac{\nabla \langle x \rangle^\gamma}{\gamma} \quad \text{and} \quad \mathbf{E}_1 \in \mathcal{C}^1(\mathbb{R}^d; \mathbb{R}^d), \quad \mathbf{E}_1 \rightarrow 0 \quad \text{when} \quad |x| \rightarrow +\infty.$$

## 1.2 Main results

We shall denote by  $L_k^2$  the space

$$L_k^2 := L_k^2(\mathbb{R}^d) := \left\{ f \in L^2(\mathbb{R}^d) ; \int_{\mathbb{R}^d} |f(x)|^2 \langle x \rangle^k dx < \infty \right\}$$

and the linear operator  $(L, D(L))$  by

$$Lu := \operatorname{div}(\nabla u + \mathbf{E}u), \quad \text{for } u \in D(L) := \{u \in L_k^2 ; Lu \in L_k^2\}. \quad (1.6) \quad \boxed{\text{eq:1.6}}$$

We shall see that the linear operator  $L$  generates a  $C_0$ -semigroup on  $L_k^2$ . Now we are in a position to state our first result regarding the stationary solution of [\(1.1\)](#).

thm:GS

**Theorem 1.1.** *Let  $(L, D(L))$  be defined by [\(1.6\)](#). Assume that hypotheses **(H1)** and **(H2)** hold with  $p = 2$  in **(H2)**. Then the equation [\(1.1\)](#) has a stationary solution. More precisely there exists a unique function  $G \in L_k^2$ , strictly positive, such that*

$$LG = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} G(x) dx = 1. \quad (1.7) \quad \boxed{\text{eq:1.8}}$$

The following is our main result regarding the evolution equation.

thm:Cv-GS

**Theorem 1.2.** *Assume that hypotheses **(H1)**, **(H2)** and **(H3)** hold and let  $f_0 \in L_k^2$  the initial datum of [\(1.1\)](#) be given. Then there exists a unique function  $f \in C([0, \infty) ; L_k^2)$  solution to [\(1.1\)](#). Moreover, there exist a real number  $\omega > 0$  and a constant  $C > 0$  such that*

$$\forall t \geq 0 \quad \|f(t) - M(f_0)G\|_{L_k^2} \leq C \exp(-\omega t) \|f_0 - M(f_0)G\|_{L_k^2}. \quad (1.8) \quad \boxed{\text{eq:1.9}}$$

## 2 Definitions and notations

We recall in this section some definitions and notations. We denote

$$\langle x \rangle := (1 + |x|^2)^{1/2} \quad \text{and} \quad \langle x \rangle^k := (1 + |x|^2)^{\frac{k}{2}}.$$

We define the Lebesgue spaces  $L_k^p(\mathbb{R}^d)$  and the Sobolev spaces  $H_k^1(\mathbb{R}^d)$  as follows:

$$L_k^p(\mathbb{R}^d) := \left\{ f \in L^p(\mathbb{R}^d) ; \int_{\mathbb{R}^d} |f(x)|^p \langle x \rangle^k dx < \infty \right\},$$

for all  $p, k \in \mathbb{R}$  such that  $k > 0$  and  $p \geq 1$ . We endow these spaces with their natural norms. The scalar product of  $L_k^2$  is defined by

$$(f|g)_k := \int_{\mathbb{R}^d} f(x) \overline{g(x)} \langle x \rangle^k dx,$$

for all  $f$  and  $g \in L_k^2$ . We set

$$H_k^1(\mathbb{R}^d) := \left\{ f \in L_k^2(\mathbb{R}^d) ; \nabla f \in (L_k^2(\mathbb{R}^d))^d \right\}.$$

It is clear that the space  $C_c^\infty(\mathbb{R}^d)$  is dense in  $L_k^p$  for  $p < \infty$ , as well as in  $H_k^1$ .

We recall that  $R_L(\lambda) := (\lambda - L)^{-1}$  denotes the resolvent operator of  $L$  for some given  $\lambda$  such that the operator  $(\lambda - L)$  has a bounded inverse and  $S_L(t) := \exp(tL)$  denotes the semigroup generated by  $L$ .

To prove Theorems [1.1](#) and [1.2](#), our approach is based on the decomposition of the operator  $L$  as follows: for an appropriately chosen bounded operator  $B$ , we shall split  $L$  in the form  $L = B + A$ , where the operator  $A$  is so that there is some  $\tau_0 \in \mathbb{R}$  such that for all  $\tau > \tau_0$ , the linear operator  $A - \tau I$  is  $m$ -dissipative, (See J. Scher and S. Mischler [\[8\]](#)). With the above assumptions we shall describe the appropriate splitting for  $L$ , and thereby for  $S_L(t)$ . Indeed the mild solution of

$$\frac{df}{dt} - Lf = \frac{df}{dt} - Af - Bf = 0,$$

is given by

$$f(t) = S_L(t)f_0 \text{ and also by } f(t) = S_A(t)f_0 + \int_0^t S_A(t-\tau)BS_L(\tau)f_0 d\tau.$$

Where  $S_A(t) = \exp(tA)$ . This means that we can write the semigroup  $S_L(t)$  split as follows:

$$S_L(t) = S_A(t) + (S_A * BS_L)(t), \tag{2.1} \quad \boxed{\text{eq:Rel-Conv}}$$

where the convolution  $*$  is defined by

$$(S_A * (BS_L))(t) := \int_0^t S_A(t-\tau)BS_L(\tau)f_0 d\tau.$$

Analogously we may consider the equation

$$\frac{dv}{dt} - Av = 0 \quad \text{with initial data } v(0, x) = v_0(x),$$

and since we have  $Av := Lv - Bv$ , we conclude that

$$\frac{dv}{dt} - Lv = -Bv \quad \text{with} \quad v(0, x) = v_0(x).$$

This implies that

$$S_A(t)v_0 = S_L(t)v_0 - \int_0^t S_L(t - \tau)BS_A(\tau)v_0 d\tau.$$

and then one sees again that [\(2.1\)](#) can also be written as follows:

$$S_L(t) = S_A(t) + (S_L * BS_A)(t). \tag{2.2}$$

[eq:Rel-Conv-bis](#)

The remainder of this paper is organized as follows. In section 3, we state preliminary results which are used in the sequel. In section 4, existence of a stationary solution will be investigated. And in the last section 5, we explore the stability issue for the evolution equation [\(1.1\)](#) and prove our second main result, Theorem [1.2](#).

### 3 Preliminary results

In this section we discuss the existence of a solution for the evolution equation [\(1.1\)](#) in some Banach spaces. To this end we study some properties of the linear operator  $(L, D(L))$ .

[thm:Exist-SG](#)

**Proposition 3.1.** *Assume that [\(H2\)](#) holds with  $p = 2$ . Consider  $(L, D(L))$  defined by [\(1.6\)](#). Then there exists  $\lambda_0 := \lambda_0(2) \in \mathbb{R}$  such that for all  $\lambda \geq \lambda_0$  and for  $f \in L_k^2$  we have*

$$((\lambda - L)f|f)_k \geq (\lambda - \lambda_0)\|f\|_{L_k^2}.$$

That is, the linear operator  $(L, D(L))$  is the generator of a  $C_0$ -semigroup of contraction on  $L_k^2$ . Moreover if  $f_0 \in L_k^2$  is the initial data of [\(1.1\)](#), there exists  $f \in C([0, T]; L_k^2)$  solution of [\(1.1\)](#) associated to  $f_0$  and  $f(t) := S_L(t)f_0$ .

*Proof.* We aim to show that there exists  $\lambda_0 \in \mathbb{R}$  such that for any  $\lambda \geq \lambda_0$ , the operator  $L - \lambda I$  is  $m$ -dissipative, more precisely

$$((\lambda - L)f|f)_k \geq (\lambda - \lambda_0)\|f\|_{L_k^2}.$$

For  $f_0 \in L_k^2$ , consider the following problem: find a function  $f \in D(L)$  such that

$$-Lf + \lambda f = f_0. \quad (3.1) \quad \boxed{\text{eq:2.1}}$$

We begin by showing that for an appropriate choice of  $\lambda$ , the operator  $-L + \lambda I$  is coercive. Indeed for,  $\varphi \in C_c^\infty$ , we have

$$(-L\varphi|\varphi)_k = - \int_{\mathbb{R}^d} \operatorname{div}(\nabla\varphi(x) + \mathbf{E}(x)\varphi(x))\varphi(x) \langle x \rangle^k dx.$$

Thus integrating by parts we obtain

$$(-L\varphi|\varphi)_k = \int_{\mathbb{R}^d} (\nabla\varphi(x) + \mathbf{E}(x)\varphi(x)) \cdot \nabla(\varphi(x) \langle x \rangle^k) dx.$$

Now computing  $\nabla(\varphi(x) \langle x \rangle^k) = \nabla\varphi(x) \langle x \rangle^k + \varphi(x) \nabla \langle x \rangle^k$ , and using this expression in the identity above, we get

$$(-L\varphi|\varphi)_k = I_1 + I_2 + I_3 + I_4,$$

where for convenience we denote

$$I_1 := \int_{\mathbb{R}^d} |\nabla\varphi(x)|^2 \langle x \rangle^k dx, \quad I_2 := \int_{\mathbb{R}^d} \varphi(x) \nabla\varphi(x) \cdot \mathbf{E}(x) \langle x \rangle^k dx,$$

$$I_3 := \int_{\mathbb{R}^d} \varphi(x) \nabla\varphi(x) \cdot \nabla \langle x \rangle^k dx, \quad \text{and} \quad I_4 := \int_{\mathbb{R}^d} \varphi(x)^2 \mathbf{E}(x) \cdot \nabla \langle x \rangle^k dx.$$

We rewrite  $I_2$  as

$$I_2 = \frac{1}{2} \int_{\mathbb{R}^d} \nabla(\varphi^2) \cdot \mathbf{E} \langle x \rangle^k dx,$$

and then we integrate by parts to obtain

$$I_2 = -\frac{1}{2} \int_{\mathbb{R}^d} \varphi^2 \operatorname{div}(\mathbf{E} \langle x \rangle^k) dx.$$

Since  $\operatorname{div}(\mathbf{E} \langle x \rangle^k) = \operatorname{div}(\mathbf{E}) \langle x \rangle^k + \mathbf{E} \cdot \nabla \langle x \rangle^k$ , then  $I_2$  becomes:

$$I_2 = -\frac{1}{2} \int_{\mathbb{R}^d} \varphi^2 \operatorname{div}(\mathbf{E}) \langle x \rangle^k dx - \frac{1}{2} \int_{\mathbb{R}^d} \varphi^2 \mathbf{E} \cdot \nabla \langle x \rangle^k dx.$$

Summing  $I_2$  and  $I_4$  we obtain

$$I_2 + I_4 = -\frac{1}{2} \int_{\mathbb{R}^d} \varphi^2 \operatorname{div}(\mathbf{E}) \langle x \rangle^k dx + \frac{1}{2} \int_{\mathbb{R}^d} \varphi^2 \mathbf{E} \cdot \nabla \langle x \rangle^k dx.$$

The term  $I_3$  can be also rewritten as

$$I_3 = \frac{1}{2} \int_{\mathbb{R}^d} \nabla(\varphi^2) \cdot \nabla \langle x \rangle^k dx,$$

which, integrated by parts, yields

$$I_3 = -\frac{1}{2} \int_{\mathbb{R}^d} \varphi^2 \Delta \langle x \rangle^k dx.$$

Summing together  $I_1$ ,  $I_2$ ,  $I_3$  and  $I_4$  we obtain

$$(-L\varphi|\varphi)_k = \int_{\mathbb{R}^d} |\nabla\varphi|^2 \langle x \rangle^k dx + \frac{1}{2} \int_{\mathbb{R}^d} \varphi^2 \frac{\Psi_k(x)}{\langle x \rangle^k} \langle x \rangle^k dx, \quad (3.2) \quad \boxed{\text{eq:2.2}}$$

where we have set

$$\Psi_k(x) := -\Delta \langle x \rangle^k - \operatorname{div}(\mathbf{E}) + \mathbf{E} \cdot \nabla \langle x \rangle^k. \quad (3.3) \quad \boxed{\text{eq:2.3}}$$

Using the expressions

$$\nabla \langle x \rangle^k := k \frac{x}{1 + |x|^2} \langle x \rangle^k, \quad (3.4) \quad \boxed{\text{eq:nabla-x}}$$

and

$$\Delta \langle x \rangle^k := (kd + k(k + d - 2)|x|^2) \frac{1}{(1 + |x|^2)^2} \langle x \rangle^k, \quad (3.5) \quad \boxed{\text{eq:laplacian-x}}$$

we obtain

$$\frac{\Psi_k(x)}{\langle x \rangle^k} = -\frac{kd + k(k + d - 2)|x|^2}{(1 + |x|^2)^2} - \operatorname{div}(\mathbf{E}) + \frac{kx}{1 + |x|^2} \cdot \mathbf{E}.$$

Thus using hypothesis **(H2)** with  $p = 2$  we find

$$\frac{\Psi_k(x)}{\langle x \rangle^k} \geq \beta_0 - \frac{kd + k(k + d - 2)|x|^2}{(1 + |x|^2)^2}.$$

now setting

$$\lambda_0 := \max_{x \in \mathbb{R}^d} \left[ \frac{kd}{1 + |x|^2} + \frac{k(k - 2)|x|^2}{(1 + |x|^2)^2} - \beta_0 \right]. \quad (3.6) \quad \boxed{\text{eq:2.4}}$$

One obtains the following inequality

$$(-L\varphi|\varphi)_k \geq \int_{\mathbb{R}^d} |\nabla\varphi|^2 \langle x \rangle^k dx - \lambda_0 \int_{\mathbb{R}^d} \varphi^2 \langle x \rangle^k dx \geq 0. \quad (3.7) \quad \boxed{\text{eq:2.5}}$$

This inequality, together with a density argument, implies that  $\lambda - L$  is coercive for all  $\lambda > \lambda_0$ . This means that for all  $\lambda > \lambda_0$  we have

$$((\lambda - L)\varphi|\varphi) \geq \int_{\mathbb{R}^d} |\nabla\varphi|^2 \langle x \rangle^k dx + (\lambda - \lambda_0) \int_{\mathbb{R}^d} \varphi^2 \langle x \rangle^k dx.$$

Then an elementary application of Lax-Milgram lemma in the space  $H_k^1(\mathbb{R}^d)$  implies that for any  $\lambda > \lambda_0$  and any  $f_0 \in L_k^2$ , equation (3.1) has a unique solution  $f \in H_k^1$ .

Now, for  $\mu > 0$  given and  $f_0 \in L_k^2$  the equation

$$[I + \mu(\lambda_0 - L)]f = f_0 \quad f \in D(L) \tag{3.8} \quad \boxed{\text{eq:2.6}}$$

has a unique solution, since it is equivalent to

$$\left(\frac{1}{\mu} + \lambda_0\right)f - Lf = \frac{1}{\mu}f_0$$

and as we have  $\lambda := \lambda_0 + \frac{1}{\mu} > \lambda_0$ . Multiplying (3.8) by  $f$  yields

$$\|f\|_{L_k^2}^2 \leq (f_0|f)_k,$$

from which we deduce that the unique solution of (3.8) satisfies

$$\|f\|_{L_k^2}^2 \leq \|f_0\|_{L_k^2}^2,$$

that is the operator  $(L - \lambda_0 I)$  is  $m$ -dissipative. This means that the operator  $(L - \lambda I)$  generates a  $C_0$ -semigroup of contraction on  $L_k^2$ :

$$\|e^{-t(\lambda_0 - L)} f_0\|_{L_k^2} \leq \|f_0\|_{L_k^2}.$$

From this it is classical to deduce that the solution of (1.1) is given by

$$f(t) := e^{tL} f_0 = S_L(t) f_0$$

and that  $f \in C([0, +\infty); L_k^2)$ . Moreover we have

$$\|e^{tL} f_0\|_{L_k^2} \leq e^{\lambda_0 t} \|f_0\|_{L_k^2}.$$

This completes the proof of the Proposition [3.1](#). □

It is useful to note that for any  $f_0 \in L_k^2$  with  $f_0 \geq 0$ , when the hypothesis (H3) is satisfied and  $\lambda > -\omega^*/2$  the solution of

$$\lambda f - Lf = f_0, \quad f \in D(L), \quad (3.9) \quad \text{eq:Weak-Max}$$

satisfies  $f \geq 0$ .

**Lemma 3.2. (Weak maximum principle).** *Assume that the hypotheses (H0) and (H3) are satisfied. We have*

$$(Lf|f^-) \geq \int_{\mathbb{R}^d} |\nabla f^-(x)|^2 \langle x \rangle^k dx + \frac{\omega^*}{2} \int_{\mathbb{R}^d} |f^-(x)|^2 \langle x \rangle^k dx, \quad (3.10) \quad \text{eq:Coercive-1}$$

where  $\omega^*$  is given by (1.5). If  $f_0 \in L_k^2$  and  $f_0 \geq 0$ , then for any  $\lambda > -\omega^*/2$  the solution of (3.9) exists and satisfies  $f \geq 0$ .

*Proof.* Indeed, to see this, we note that for  $f \in D(L)$  we clearly have  $f^- \in H_k^1$  and thus

$$(Lf|f^-)_k = - \int_{\mathbb{R}^d} \nabla f(x) \cdot \nabla f^-(x) \langle x \rangle^k dx - \int_{\mathbb{R}^d} f(x) \mathbf{E}(x) \cdot \nabla (f^-(x) \langle x \rangle^k) dx.$$

Now, since we have

$$f = f^+ - f^-, \quad \nabla f = \nabla f^+ - \nabla f^- \quad \text{and} \quad f^+ \nabla f^- = 0,$$

in a first step this allows us to see that

$$\begin{aligned} (Lf|f^-)_k &= \int_{\mathbb{R}^d} |\nabla f^-|^2 \langle x \rangle^k dx + \int_{\mathbb{R}^d} f^-(x) \nabla f^-(x) \cdot \nabla \langle x \rangle^k dx \\ &\quad + \int_{\mathbb{R}^d} f^-(x) \mathbf{E} \cdot \nabla f^-(x) \langle x \rangle^k dx + \int_{\mathbb{R}^d} |f^-(x)|^2 \mathbf{E} \cdot \nabla \langle x \rangle^k dx. \end{aligned}$$

Now integrating by parts in the second and the third terms of the identity above we obtain

$$\begin{aligned} (Lf|f^-)_k &= \int_{\mathbb{R}^d} |\nabla f^-|^2 \langle x \rangle^k dx - \frac{1}{2} \int_{\mathbb{R}^d} |f^-|^2 \Delta \langle x \rangle^k dx \\ &\quad - \frac{1}{2} \int_{\mathbb{R}^d} |f^-|^2 \operatorname{div}(\mathbf{E}) \langle x \rangle^k dx + \frac{1}{2} \int_{\mathbb{R}^d} |f^-|^2 \mathbf{E} \cdot \nabla \langle x \rangle^k dx. \end{aligned}$$

Using the expressions of  $\nabla \langle x \rangle^k$  and that of  $\Delta \langle x \rangle^k$  on the one hand, and the fact that  $\omega^*$  satisfies (1.5), we conclude that

$$(Lf|f^-) \geq \int_{\mathbb{R}^d} |\nabla f^-(x)|^2 \langle x \rangle^k dx + \frac{\omega^*}{2} \int_{\mathbb{R}^d} |f^-(x)|^2 \langle x \rangle^k dx,$$

which is precisely (3.10).

Also, proceeding as above one can see that for any  $f \in H_k^1$  we have

$$(-Lf|f) \geq \int_{\mathbb{R}^d} |\nabla f(x)|^2 \langle x \rangle^k dx + \frac{\omega^*}{2} \int_{\mathbb{R}^d} |f(x)|^2 \langle x \rangle^k dx. \quad (3.11) \quad \text{eq:Coercive-H3}$$

Therefore, thanks to the Lax-Milgram theorem equation (3.9) has a unique solution when  $\lambda > -\omega^*/2$ . Moreover, when  $f_0 \geq 0$ , multiplying (3.9) by  $f^-$  and integrating by parts we have

$$0 \leq (f_0|f^-) = -\lambda \|f^-\|_{L_k^2}^2 - (Lf|f^-)_{L_k^2}.$$

Using (3.10) it follows that

$$0 \leq - \int_{\mathbb{R}^d} |\nabla f^-(x)|^2 \langle x \rangle^k dx - \left( \lambda + \frac{\omega^*}{2} \right) \int_{\mathbb{R}^d} |f^-(x)|^2 \langle x \rangle^k dx,$$

which, since  $\lambda + (\omega^*/2) > 0$  implies  $f^- \equiv 0$ , that is  $f \geq 0$ .  $\square$

Our next useful result is the fact that the semigroup  $S_L(t)$  is positivity preserving.

lem:Positif

**Lemma 3.3.** *Let  $f_0 \in D(L)$  be nonnegative and assume that the hypotheses (H0) and (H3) hold. Then  $S_L(t)f_0 \geq 0$ , that is the solution  $f$  of (1.1) associated to the initial data  $f_0$  is nonnegative.*

*Proof.* Assume that  $f_0 \in D(L)$  with  $f_0 \geq 0$ , and consider the equation

$$\partial_t f = Lf, \quad f(0) = f_0. \quad (3.12) \quad \text{eq:2.7}$$

We aim to show that  $f(t) \geq 0$  for all  $t \geq 0$ . To this end we consider  $f^-$  the negative part of  $f$ . It is clear that since  $f \in C^1([0, \infty); L_k^2)$ , we have  $f \in D(L)$  and  $f^- \in H_k^1$ . Therefore we may multiply (3.12) by  $-f^-$ , to get

$$- \int_{\mathbb{R}^d} \partial_t f f^- \langle x \rangle^k dx = -(Lf|f^-)_k \leq -\frac{\omega^*}{2} \|f^-\|_{L_k^2}^2,$$

thanks to [\(3.10\)](#). Since  $f^- \partial_t f = -\partial_t((f^-)^2)/2$  we see that

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^d} |f^-(x)|^2 \langle x \rangle^k dx \leq -\frac{\omega^*}{2} \|f^-\|_{L_k^2}^2,$$

and using Gronwall's lemma we conclude that

$$\int_{\mathbb{R}^d} |f^-(x)|^2 \langle x \rangle^k dx \leq \exp(-\omega^* t) \int_{\mathbb{R}^d} |f_0^-(x)|^2 \langle x \rangle^k dx,$$

from which, since  $f_0^- \equiv 0$ , we infer that  $f^- \equiv 0$ , that is  $f \geq 0$ . □

lem:SG-Lp

**Lemma 3.4.** *Assume that the hypotheses **(H0)**–**(H3)** hold for some  $p \in [2, \infty)$ . Then there exists  $\lambda_0(p) \in \mathbb{R}$  such that for any  $\lambda \geq \lambda_0(p)$ , the semigroup  $S_L(t)$  generated by  $(L, D(L))$  is also a  $C_0$ -semigroup on  $L_k^p$ .*

*Proof.* Let  $f_0 \in L_k^p$  and assume that  $f_0 \geq 0$ . We aim to show that for  $\lambda > \lambda_0(p)$  the equation [\(1.1\)](#) has a unique solution  $\varphi \in L_k^p$  such that  $L\varphi \in L_k^p$ . To this end we consider the following problem: find  $\varphi \in L_k^p$  such that,

$$-L\varphi + \lambda\varphi = f_0. \tag{3.13} \span style="float: right; border: 1px solid black; padding: 2px;">eq:2.9$$

We begin by observing that for  $f_0 \in C_c^\infty$  and  $f_0 \geq 0$ , the above equation has a unique solution  $\varphi \in D(L)$  provided  $\lambda > \lambda_0(p)$ , where  $\lambda_0(p)$  will be precised later. We are going to show that when  $\lambda > \lambda_0(p)$ , where  $\lambda_0(p)$  is large enough, we have

$$\|\varphi\|_{L_k^p} \leq C \|f_0\|_{L_k^p}$$

for a constant  $C$  independant of  $f_0$ . Then a standard density argument shows that for any  $f_0 \in L_k^p$  and  $f_0 \geq 0$  equation [\(3.13\)](#) has a unique solution  $\varphi \in L_k^p$  such that  $L\varphi \in L_k^p$  and  $\varphi \geq 0$ .

Let  $\zeta_0 \in C_c^\infty([0, \infty))$  such that

$$\zeta_0(s) = \begin{cases} 1 & \text{if } 0 \leq s \leq 1 \\ 0 & \text{if } s \geq 2, \end{cases}$$

where  $0 \leq \zeta_0 \leq 1$  and  $-2 \leq \zeta_0'(s) \leq 0$ . For any integer  $n \geq 1$  we define

$$\zeta_n(x) := \zeta_0\left(\frac{|x|}{n}\right). \tag{3.14} \span style="float: right; border: 1px solid black; padding: 2px;">eq:Def-zeta-n$$

Since  $f_0 \in L^2$ , we know that for  $\lambda > \lambda_0(2)$ , where  $\lambda_0(2)$  is given by Proposition 3.1, there exists a unique solution  $\varphi \in D(L)$  of (3.13). Thus we may multiply the latter equation by  $\varphi^{p-1}\zeta_n$  and integrate by parts to obtain:

$$\begin{aligned} \lambda \int \varphi^p \zeta_n \langle x \rangle^k dx + \int \nabla \varphi \cdot \nabla (\varphi^{p-1} \zeta_n \langle x \rangle^k) dx + \int \varphi \mathbf{E} \cdot \nabla (\varphi^{p-1} \zeta_n \langle x \rangle^k) dx \\ = \int f_0 \varphi^{p-1} \zeta_n \langle x \rangle^k dx \end{aligned} \quad (3.15)$$

In order to make the proof more clear, we are going to treat the second and third integrals of the first line of the above equality separately, and show the Lemma in several steps.

**Step 1.** The second term in the first line of the identity (3.15) can be written as

$$\int \nabla \varphi \cdot \nabla (\varphi^{p-1} \zeta_n \langle x \rangle^k) dx = A_1 + A_2 + A_3, \quad (3.16) \quad \boxed{\text{eq:Identity-2}}$$

where we have set

$$A_1 := (p-1) \int_{\mathbb{R}^d} |\nabla \varphi|^2 \varphi^{p-2} \zeta_n \langle x \rangle^k dx, \quad A_2 := \int_{\mathbb{R}^d} \varphi^{p-1} \nabla \varphi \cdot \nabla \zeta_n \langle x \rangle^k dx, \quad (3.17) \quad \boxed{\text{eq:Def-A1A2}}$$

and

$$A_3 := \int_{\mathbb{R}^d} \varphi^{p-1} \nabla \varphi \cdot \zeta_n \nabla \langle x \rangle^k dx. \quad (3.18) \quad \boxed{\text{eq:Def-A3}}$$

Regarding  $A_2$ , writing  $\varphi^{p-1} \nabla \varphi$  as  $\nabla(\varphi^p)/p$  and integrating by parts we have

$$A_2 = -\frac{1}{p} \int_{\mathbb{R}^d} \varphi^p (\Delta \zeta_n) \langle x \rangle^k dx - \frac{1}{p} \int_{\mathbb{R}^d} \varphi^p \nabla \zeta_n \cdot \nabla \langle x \rangle^k dx. \quad (3.19) \quad \boxed{\text{eq:A2}}$$

Using the expressions

$$\nabla \zeta_n(x) := \zeta'_0 \left( \frac{|x|}{n} \right) \frac{x}{n|x|} \quad \text{and} \quad \nabla \langle x \rangle^k = \frac{kx}{1+|x|^2}$$

in (3.19) we obtain finally

$$A_2 = -\frac{1}{p} \int_{\mathbb{R}^d} \varphi^p (\Delta \zeta_n) \langle x \rangle^k dx - \frac{1}{p} \int_{\mathbb{R}^d} \varphi^p \zeta'_0 \left( \frac{|x|}{n} \right) \frac{k|x|}{n(1+|x|^2)} \langle x \rangle^k dx. \quad (3.20) \quad \boxed{\text{eq:A2-bis}}$$

Analogously the term  $A_3$  can be also rewritten and one may check that

$$\begin{aligned}
A_3 &= \frac{1}{p} \int_{\mathbb{R}^d} \nabla(\varphi^p) \cdot \zeta_n \nabla \langle x \rangle^k dx \\
&= -\frac{1}{p} \int_{\mathbb{R}^d} \varphi^p \nabla \zeta_n \cdot \nabla \langle x \rangle^k dx - \frac{1}{p} \int_{\mathbb{R}^d} \varphi^p \zeta_n \Delta \langle x \rangle^k dx \\
&= -\frac{1}{p} \int_{\mathbb{R}^d} \varphi^p \zeta'_0 \left( \frac{|x|}{n} \right) \frac{k|x|}{n(1+|x|^2)} \langle x \rangle^k dx \\
&\quad - \frac{1}{p} \int_{\mathbb{R}^d} \varphi^p \zeta_n \frac{kd + k(k+d-2)|x|^2}{(1+|x|^2)^2} \langle x \rangle^k dx. \tag{3.21} \quad \boxed{\text{eq:A3}}
\end{aligned}$$

Summing the equalities [\(3.20\)](#) and [\(3.21\)](#) we obtain

$$\begin{aligned}
A_2 + A_3 &= -\frac{1}{p} \int_{\mathbb{R}^d} \varphi^p (\Delta \zeta_n) \langle x \rangle^k dx - \frac{2}{p} \int_{\mathbb{R}^d} \varphi^p \zeta'_0 \left( \frac{|x|}{n} \right) \frac{k|x|}{n(1+|x|^2)} \langle x \rangle^k dx \\
&\quad - \frac{1}{p} \int_{\mathbb{R}^d} \varphi^p \zeta_n \frac{kd + k(k+d-2)|x|^2}{(1+|x|^2)^2} \langle x \rangle^k dx. \tag{3.22} \quad \boxed{\text{eq:A2A3}}
\end{aligned}$$

The facts that

$$0 \leq \zeta_n(x) \leq 1, \quad -2 \leq \zeta'_0(s) \leq 0, \quad |\zeta''_0(s)| \leq C,$$

and

$$-\Delta \zeta_n = \frac{-1}{n^2} \Delta \zeta_0 \left( \frac{|x|}{n} \right) \geq \frac{-C}{n^2} 1_{[n \leq |x| \leq 2n]},$$

allow us to conclude first that

$$-\int_{\mathbb{R}^d} \varphi^p \zeta'_0 \left( \frac{|x|}{n} \right) \frac{k|x|}{n(1+|x|^2)} \langle x \rangle^k dx \geq 0,$$

and then from [\(3.22\)](#) we infer that, since there exists a constant  $C > 0$  such that

$$\frac{kd + k(k+d-2)|x|^2}{1+|x|^2} \leq C,$$

we finally have

$$A_2 + A_3 \geq -C \int_{[n \leq |x| \leq 2n]} \varphi^p \frac{\langle x \rangle^k}{n^2} dx - C \int_{\mathbb{R}^d} \varphi^p \frac{\langle x \rangle^k}{1+|x|^2} dx. \tag{3.23} \quad \boxed{\text{eq:Minor-A2A3}}$$

**Step 2.** The third term in the first line of the identity [\(3.15\)](#) <sup>eq:Identity-1</sup> can be written as

$$\int \varphi \mathbf{E} \cdot \nabla(\varphi^{p-1} \zeta_n \langle x \rangle^k) dx = A_4 + A_5 + A_6, \quad (3.24)$$

where we have set

$$A_4 := (p-1) \int_{\mathbb{R}^d} \varphi^{p-1} \nabla \varphi \cdot \zeta_n \mathbf{E}(x) \langle x \rangle^k dx, \quad A_5 := \int_{\mathbb{R}^d} \varphi^p \nabla \zeta_n \cdot \mathbf{E}(x) \langle x \rangle^k dx \quad (3.25) \quad \boxed{\text{eq:A4A5}}$$

and

$$A_6 := \int_{\mathbb{R}^d} \varphi^p \mathbf{E}(x) \cdot \zeta_n \nabla \langle x \rangle^k dx. \quad (3.26) \quad \boxed{\text{eq:A6}}$$

Proceeding as above  $A_4$  can be rewritten as

$$\begin{aligned} A_4 &= \frac{p-1}{p} \int_{\mathbb{R}^d} \nabla(\varphi^p) \cdot \zeta_n(x) \mathbf{E}(x) \langle x \rangle^k dx \\ &= -\frac{p-1}{p} \int_{\mathbb{R}^d} \varphi^p \zeta_n \operatorname{div}(\mathbf{E}(x)) \langle x \rangle^k dx - \frac{(p-1)}{p} \int_{\mathbb{R}^d} \varphi^p \nabla \zeta_n \cdot \mathbf{E}(x) \langle x \rangle^k dx \\ &\quad - \frac{p-1}{p} \int_{\mathbb{R}^d} \varphi^p \zeta_n \mathbf{E}(x) \cdot \nabla \langle x \rangle^k dx. \end{aligned}$$

Summing  $A_4, A_5, A_6$  we get

$$\begin{aligned} A_4 + A_5 + A_6 &= -\frac{1}{p'} \int_{\mathbb{R}^d} \varphi^p \zeta_n \operatorname{div}(\mathbf{E}(x)) \langle x \rangle^k dx \\ &\quad + \frac{1}{p} \int_{\mathbb{R}^d} \varphi^p \nabla \zeta_n \cdot \mathbf{E}(x) \langle x \rangle^k dx \\ &\quad + \frac{1}{p} \int_{\mathbb{R}^d} \varphi^p \zeta_n \mathbf{E}(x) \cdot \nabla \langle x \rangle^k dx. \end{aligned} \quad (3.27)$$

Since  $0 \leq -\zeta'_0 \leq 2$ , using the assumption [\(1.3\)](#) <sup>eq:1.3</sup> one checks that for  $n$  large enough so that  $x \cdot \mathbf{E}(x) \geq 0$  for  $|x| \geq n$ ,

$$\nabla \zeta_n(x) \cdot \mathbf{E}(x) = \zeta'_0 \left( \frac{|x|}{n} \right) \frac{x}{n|x|} \cdot \mathbf{E} \geq \zeta'_0 \left( \frac{|x|}{n} \right) \frac{\alpha_2 |x|^{\gamma_2} + \beta_2}{n|x|} \mathbf{1}_{[n \leq |x| \leq 2n]}$$

and so for  $n$  large enough we have

$$\nabla \zeta_n(x) \cdot \mathbf{E}(x) \geq -2 \frac{\alpha_2 |x|^{\gamma_2} + \beta_2}{|x|^2} \mathbf{1}_{[n \leq |x| \leq 2n]}. \quad (3.28) \quad \boxed{\text{eq:E-gradzeta}}$$

Then using hypothesis **(H2)** and the inequality [\(3.28\)](#) we obtain

$$A_4 + A_5 + A_6 \geq \frac{\beta_0}{p} \int_{\mathbb{R}^d} \varphi^p \zeta_n \langle x \rangle^k dx - \frac{2}{p} \int_{\mathbb{R}^d} \varphi^p \frac{\alpha_2 |x|^{\gamma_2} + \beta_2}{|x|^2} \langle x \rangle^k dx.$$

Thus, setting

$$\frac{\Psi_{k,p}}{\langle x \rangle^k} := \frac{1}{p} \left[ \frac{C}{1 + |x|^2} + \frac{2k}{1 + |x|^2} + \frac{C}{1 + |x|^2} + 2 \frac{\alpha_2 |x|^{\gamma_2} + \beta_2}{|x|^2} \right],$$

and using [\(3.23\)](#), we have that

$$A_2 + A_3 + A_4 + A_5 + A_6 \geq -\frac{1}{p} \int_{\mathbb{R}^d} \varphi^p \frac{\Psi_{k,p}}{\langle x \rangle^k} \langle x \rangle^k dx. \quad (3.29) \quad \boxed{\text{eq:A2-A6}}$$

**Step 3.** Now if we define

$$\lambda_0(p) := \max_{x \in \mathbb{R}^d} \left[ \frac{\Psi_{k,\zeta}}{\langle x \rangle^k} - \frac{\beta_0}{p} \right]$$

we obtain, thanks to [\(3.29\)](#), [\(3.17\)](#), [\(3.16\)](#) and [\(3.15\)](#), that

$$\begin{aligned} \int_{\mathbb{R}^d} f_0 \varphi^{p-1} \zeta_n \langle x \rangle^k dx &= \int_{\mathbb{R}^d} (\lambda - L) \varphi \varphi^{p-1} \zeta_n \langle x \rangle^k dx \\ &\geq (\lambda - \lambda_0(p)) \int_{\mathbb{R}^d} \varphi^p \zeta_n \langle x \rangle^k dx + (p-1) \int_{\mathbb{R}^d} |\nabla \varphi|^2 \varphi^{p-2} \zeta_n \langle x \rangle^k dx. \end{aligned}$$

We may fix  $\lambda$  such that  $\lambda - \lambda_0(p) \geq 1$  and upon using Young's inequality, that is the fact that  $ab \leq \varepsilon a^p/p + b^{p'}/p'$  for  $a, b \geq 0$ , and choosing  $a := f_0$  and  $b := \varphi^{p-1}$ , we conclude that we have

$$p(p-1) \int_{\mathbb{R}^d} |\nabla \varphi|^2 \varphi^{p-2} \zeta_n \langle x \rangle^k dx + \int_{\mathbb{R}^d} \varphi^p \zeta_n \langle x \rangle^k dx \leq \int_{\mathbb{R}^d} |f_0|^p \zeta_n \langle x \rangle^k dx, \quad (3.30)$$

It is clear now that letting  $n$  tend to  $\infty$ , we deduce that  $\varphi$ , the solution of [\(3.13\)](#), belongs to  $L_k^p$  and

$$\|\varphi\|_{L_k^p} \leq \|f_0\|_{L_k^p},$$

and that moreover

$$\int_{\mathbb{R}^d} |\nabla \varphi|^2 \varphi^{p-2} \zeta_n \langle x \rangle^k dx < \infty.$$

To finish the proof of the Lemma, when  $f_0 \geq 0$  belongs only to  $L_k^p$  we consider a sequence  $f_{0n} \in C_c^\infty$  such that  $f_{0n} \geq 0$  and  $f_{0n} \rightarrow f_0$  in  $L_k^p$  and we conclude by verifying easily that the corresponding solutions  $\varphi_n$  converge to  $\varphi$  as  $n \rightarrow \infty$ .

Indeed we have also  $\|\varphi\|_{L_k^p} \leq \|f_0\|_{L_k^p}$  and  $L\varphi \in L_k^p$ , which means that the operator  $L - \lambda I$  is  $m$ -dissipative on  $L_k^p$ .  $\square$

Next we prove the following Nash type inequality which is going to be useful later.

lem:Nash-k

**Lemma 3.5.** *Let  $f \in L_{k/2}^1(\mathbb{R}^d) \cap H_k^1(\mathbb{R}^d)$ , assume that  $k > 0$  when  $d \geq 2$  and  $k \geq 2$  when  $d = 1$ . Then there exists a constant  $C > 0$  such that the following inequality holds*

$$\|f\|_{L_k^{2+\frac{4}{d}}}^2 \leq C \|f\|_{L_{k/2}^1}^{\frac{4}{d}} \cdot \|\nabla f\|_{L_k^2}^2. \quad (3.31) \quad \text{eq:2.11}$$

*Proof.* Let  $f \in L_{k/2}^1(\mathbb{R}^d) \cap H_k^1(\mathbb{R}^d)$ , we write

$$\int_{\mathbb{R}^d} |f(x)|^2 \langle x \rangle^k dx = \int_{\mathbb{R}^d} |f(x) \langle x \rangle^{\frac{k}{2}}|^2 dx.$$

Therefore

$$\|f\|_{L_k^{2+\frac{4}{d}}}^2 = \|f \langle \cdot \rangle^{\frac{k}{2}}\|_{L^2}^{2+\frac{4}{d}}.$$

Let us set  $\varphi(x) := f(x) \langle x \rangle^{\frac{k}{2}}$ , then by the Nash's classical inequality (J. Nash [\[10\]](#)) we have

$$\|\varphi\|_{L^2}^{2+\frac{4}{d}} \leq C \|\varphi\|_{L^1}^{\frac{4}{d}} \|\nabla \varphi\|_{L^2}^2,$$

since  $f \langle \cdot \rangle^{\frac{k}{2}} \in H^1$ . With simple calculations we can see that

$$\nabla \varphi = \langle x \rangle^{\frac{k}{2}} \nabla f + \frac{k}{2} f(x) \langle x \rangle^{\frac{k}{2}-2} x.$$

Thus we have

$$\begin{aligned} \|\nabla \varphi\|_{L^2}^2 &= \int_{\mathbb{R}^d} |\nabla f|^2 \langle x \rangle^k dx + \frac{k^2}{4} \int_{\mathbb{R}^d} |f|^2 |x|^2 \langle x \rangle^{k-4} dx \\ &\quad + \frac{k}{2} \int_{\mathbb{R}^d} 2f(x) \nabla f(x) \cdot x \langle x \rangle^{k-2} dx \end{aligned}$$

Integrating by parts the third integral on the right hand side above, we obtain

$$\|\nabla\varphi\|_{L^2}^2 = \int_{\mathbb{R}^d} |\nabla f|^2 \langle x \rangle^k dx - \frac{k}{2} \int_{\mathbb{R}^d} f^2 \operatorname{div}(\langle x \rangle^{k-2} x) dx + \frac{k^2}{4} \int_{\mathbb{R}^d} |f|^2 |x|^2 \langle x \rangle^{k-4} dx.$$

Since we have

$$\operatorname{div}(\langle x \rangle^{k-2} x) = (d + (d + k - 2)|x|^2) \langle x \rangle^{k-4}.$$

We get

$$\|\nabla\varphi\|_{L^2}^2 = \|\nabla f\|_{L_k^2}^2 + \int_{\mathbb{R}^d} |f|^2 \left[ \frac{k^2}{4} |x|^2 - \frac{k}{2} (d + (d + k - 2)|x|^2) \right] \langle x \rangle^{k-4} dx.$$

Note that

$$\left[ \frac{k^2}{4} |x|^2 - \frac{k}{2} (d + (d + k - 2)|x|^2) \right] = -\frac{k}{2} \left[ d + \left( d + \frac{k}{2} - 2 \right) |x|^2 \right].$$

Therefore when  $d \geq 2$  we have

$$d + \frac{k}{2} - 2 > 0$$

and

$$\|\nabla\varphi\|_{L^2}^2 \leq \|\nabla f\|_{L_k^2}^2$$

Otherwise if  $d = 1$ , we assume that  $k \geq 2$  and thus, in this case also

$$\|\nabla\varphi\|_{L^2}^2 \leq \|\nabla f\|_{L_k^2}^2$$

Replacing  $\varphi$  by  $f \langle x \rangle^{\frac{k}{2}}$ , we obtain the lemma.  $\square$

## 4 Existence of a stationary solution

In this section we are interested in the existence and uniqueness of a stationary solution. To find this solution we want to use the Krein-Rutmann's theorem revisited by J. Scher and S. Mischler [8]. For this we need some notions of Banach lattices, which we are going to recall. Let us consider the  $L_k^2$  space equipped with its natural partial order  $\geq$ . We set

$$(L_k^2)_+ := \{f \in L_k^2; f \geq 0\} \tag{4.1}$$

thm:Kat

**Lemma 4.1** (Kato's inequality). *For all  $f \in D(L)$  we have*

$$L|f| \geq \operatorname{sgn}(f)Lf, \quad (4.2) \quad \text{eq:Kato}$$

*in the sense of distributions.*

For more details on kato's inequality, one can refer to W. Arendt [1], T. Kato [5] R.Nagel and H. Ulig [9] and B. Simon [13]. For more convenience we give the following proof.

*Proof.* Let  $f \in D(L)$ , by definition of  $D(L)$  we have  $f \in L_k^2$  and  $Lf \in L_k^2$ . This implies in particular that  $f \in H_k^1$ , that is  $\nabla f \in L_k^2$  and we have also  $\Delta f \in L_{\text{loc}}^2$ . Now consider the function

$$j_\varepsilon(s) := (\varepsilon^2 + s^2)^{1/2} - \varepsilon$$

for  $s \in \mathbb{R}$ . A simple calculation gives

$$j'_\varepsilon(s) = \frac{s}{\sqrt{\varepsilon^2 + s^2}} \quad \text{and} \quad j''_\varepsilon(s) = \frac{\varepsilon^2}{(\varepsilon^2 + s^2)^{3/2}}.$$

Note that  $j''_\varepsilon(s) \geq 0$ . For  $f \in D(L)$  we compute  $\nabla j_\varepsilon(f)$  and  $\Delta j_\varepsilon(f)$  that is

$$\nabla j_\varepsilon(f) = j'_\varepsilon(f)\nabla f \quad (4.3)$$

and

$$\Delta j_\varepsilon(f) = j''_\varepsilon(f)|\nabla f|^2 + j'_\varepsilon(f)\Delta f. \quad (4.4)$$

Then for all  $\varphi \in C_c^\infty$  such that  $\varphi \geq 0$ , since  $j''_\varepsilon(f) \geq 0$  we have

$$\langle \Delta j_\varepsilon(f), \varphi \rangle \geq \langle j'_\varepsilon(f)\Delta f, \varphi \rangle. \quad (4.5)$$

Since  $j_\varepsilon(f) \rightarrow |f|$  in  $L_k^2$  and  $\Delta \varphi \in L^\infty \cap L^1$ , we have in particular

$$\Delta j_\varepsilon(f) \rightarrow \Delta|f| \quad \text{in} \quad \mathcal{D}'(\mathbb{R}^d).$$

Therefore

$$\langle \Delta j_\varepsilon(f), \varphi \rangle \rightarrow \langle \Delta|f|, \varphi \rangle.$$

On the other hand using the definition of  $L$  we obtain the following inequality

$$\langle Lj_\varepsilon(f), \varphi \rangle \geq \langle j'_\varepsilon(f)\Delta f + \operatorname{div}(j_\varepsilon(f)\mathbf{E}), \varphi \rangle.$$

The right hand side term of the above inequality can be rewritten as follows:

$$\langle j'_\varepsilon(f)\Delta f + \operatorname{div}(j_\varepsilon(f)\mathbf{E}), \varphi \rangle = \langle j'_\varepsilon(f)\Delta f + j'_\varepsilon(f)\nabla f \cdot \mathbf{E} + j_\varepsilon(f)\operatorname{div}(\mathbf{E}), \varphi \rangle.$$

Due to the fact that  $j'_\varepsilon(f) \rightarrow \operatorname{sgn}(f)$  a.e. on  $\mathbb{R}^d$ , one sees that

$$\langle j'_\varepsilon(f)\Delta f, \varphi \rangle = \int_{\mathbb{R}^d} j'_\varepsilon(f)\Delta f \varphi dx \rightarrow \int_{\mathbb{R}^d} \operatorname{sgn}(f)\Delta f \varphi dx.$$

On the other hand, since  $\mathbf{E}$  and  $\operatorname{div}(\mathbf{E})$  belong to  $L^\infty_{\operatorname{loc}}(\mathbb{R}^d)$ , and using the fact that  $\nabla f \cdot \mathbf{E} \in L^2_{\operatorname{loc}}$ , as  $\varepsilon \rightarrow 0$ , we have

$$\langle j'_\varepsilon(f)\nabla f \cdot \mathbf{E}, \varphi \rangle = \int_{\mathbb{R}^d} j'_\varepsilon(f)\nabla f \cdot \mathbf{E} dx \rightarrow \int_{\mathbb{R}^d} \operatorname{sgn}(f)\nabla f \cdot \mathbf{E} \varphi dx.$$

So finally we infer that

$$\langle j'_\varepsilon(f)\Delta f + j'_\varepsilon(f)\nabla f \cdot \mathbf{E}, \varphi \rangle \rightarrow \langle \operatorname{sgn}(f)\Delta f + \operatorname{sgn}(f)\nabla f \cdot \mathbf{E}, \varphi \rangle,$$

when  $\varepsilon$  goes to 0. Analogously we have

$$\langle j_\varepsilon(f)\operatorname{div}(\mathbf{E}), \varphi \rangle = \int_{\mathbb{R}^d} j_\varepsilon(f)\operatorname{div}(\mathbf{E})\varphi dx \rightarrow \int_{\mathbb{R}^d} |f|\operatorname{div}(\mathbf{E})\varphi dx,$$

when  $\varepsilon$  goes to 0, since  $|j_\varepsilon(f)| \leq |f| \in L^2_{\operatorname{loc}}$ . Now we remind that  $|s| = \operatorname{sgn}(s) \cdot s$ . Therefore we have

$$\langle |f|\operatorname{div}(\mathbf{E}), \varphi \rangle = \langle \operatorname{sgn}(f) f \operatorname{div}(\mathbf{E}), \varphi \rangle,$$

and thus we obtain

$$\langle L|f|, \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \langle Lj_\varepsilon(f), \varphi \rangle \geq \langle \operatorname{sgn}(f)\Delta f + \nabla|f| \cdot \mathbf{E} + |f|\operatorname{div}(\mathbf{E}), \varphi \rangle.$$

However one may check that

$$\begin{aligned} \nabla|f| \cdot \mathbf{E} + |f|\operatorname{div}(\mathbf{E}) &= \operatorname{sgn}(f)\nabla f \cdot \mathbf{E} + \operatorname{sgn}(f)f\operatorname{div}(\mathbf{E}) \\ &= \operatorname{sgn}(f)[\nabla f \cdot \mathbf{E} + f\operatorname{div}(\mathbf{E})], \end{aligned}$$

from which we conclude that

$$\operatorname{sgn}(f)\Delta f + \nabla|f| \cdot \mathbf{E} + |f|\operatorname{div}(\mathbf{E}) = \operatorname{sgn}(f)Lf,$$

and finally that for all  $\varphi \in \mathcal{D}(\mathbb{R}^d)$  such that  $\varphi \geq 0$  we have

$$\langle L|f|, \varphi \rangle \geq \langle \operatorname{sgn}(f)Lf, \varphi \rangle,$$

which is precisely the Kato's inequality [\(4.2\)](#). □

The Kato's inequality will stay true if we replace  $|f|$  by the positive part of  $f$ , that is  $f^+ := (|f| + f)/2$ . In this case we have

$$Lf^+ \geq \frac{(1 + \operatorname{sgn}(f))}{2} Lf = \mathbf{1}_{\{f>0\}} \cdot Lf.$$

**Remark 4.2.** *It is well known that if  $L$  satisfies Kato's inequality then this is equivalent to say that the semigroup which is generated by  $L$  is a positivity preserving semigroup, in the sense that if  $f_0 \geq 0$  then  $S_L(t)f_0 \geq 0$ . See for instance B. Simon [13], R. Nagel and H. Uhlig [9] [theorem 4.1 page 121]) or W. Arendt ([1] [theorem 1.6 page 159]).*

**Remark 4.3.** *The Kato's inequality implies also the weak maximum principle: in other words, if  $f \in D(L)$  and  $Lf \leq 0$ , then  $f \geq 0$ .*

thm:Str-MP

**Lemma 4.4** (Strong maximum principle). *Let  $f \in D(L) \cap W^{2,\infty}(\mathbb{R}^d)$ , then the linear operator  $L$  satisfies a strong maximum principle. i.e*

$$(f \geq 0, \quad f \not\equiv 0 \quad \text{and} \quad Lf \leq 0) \quad \Rightarrow \quad f > 0. \quad (4.6)$$

*Proof.* Let  $f \in D(L)$  be such that  $f \not\equiv 0$  and  $Lf \leq 0$  on  $\mathbb{R}^d$ . By the weak maximum principle, which is a consequence of Kato's inequality Lemma 4.1, we know that  $f \geq 0$ , and actually for any  $R > 0$  we have

$$M(R) := \inf_{|x| \leq R} f(x) \geq 0.$$

If there exists  $R_0 > 0$  and  $x_0 \in B(0, R_0)$  such that  $f(x_0) = 0$ , then  $M(R_0) = 0$ , and consequently for any  $R > R_0$  we have also  $M(R) = 0$ . Therefore, according to Hopf maximum principle (see for instance Theorem 5, chapter 2, section 3 of the classical book of M.H. Protter & H.F. Weinberger [12]) we have  $f \equiv 0$  in  $\overline{B}(0, R)$ , for all  $R$ , and this contradicts the fact that  $f \not\equiv 0$  on  $\mathbb{R}^d$ .  $\square$

thm:M-Cons

**Lemma 4.5** (Mass conservation for the semigroup). *Let  $(L, D(L))$  be defined by (1.6). Then we have the following identity.*

$$L^*1 = 0, \quad \text{in } \mathcal{D}(\mathbb{R}^d), \quad (4.7)$$

where  $L^*$  is the formal adjoint:

$$L^*\varphi = \Delta\varphi - \mathbf{E} \cdot \nabla\varphi, \quad \forall \varphi \in C_c^\infty(\mathbb{R}^d).$$

□

thm:adj

**Proposition 4.6.** Let  $(L, D(L))$  be defined by [\(1.6\)](#) and [leg:1.6](#). Assume that hypothesis **(H1)** holds. Then there exists  $b \in \mathbb{R}$  and a function  $\psi \in D(L^*)$ ,  $\psi > 0$ , such that

$$L^*\psi \geq b\psi, \quad \text{in } \mathcal{D}'(\mathbb{R}^d). \quad (4.8) \quad \text{eq:3.9}$$

*Proof.* Let  $\alpha_0 > 0$  and consider the function

$$\psi(x) := \langle x \rangle^{-\alpha_0} = (1 + |x|^2)^{-\alpha_0/2}.$$

One checks that if  $\alpha_0$  is large enough, then  $\psi \in D(L^*)$ . Using the expressions [\(3.4\)](#) and [\(3.5\)](#), where  $k$  is replaced with  $-\alpha_0$ , one checks that

$$\frac{L^*\psi}{\psi} = \frac{\alpha_0(\alpha_0 + 2 - d)|x|^2 - \alpha_0 d}{(1 + |x|^2)^2} + \frac{\alpha_0 x \cdot \mathbf{E}}{1 + |x|^2}.$$

Now using the fact that according to [\(1.3\)](#) and [leg:1.3](#) the function  $x \mapsto x \cdot \mathbf{E}$  has a growth more than  $\alpha|x|^\gamma + \beta$  with  $\gamma \leq 2$  it is clear that

$$\frac{L^*\psi}{\psi} \geq b := \inf_{x \in \mathbb{R}^d} \left[ \frac{\alpha_0(\alpha_0 + 2 - d)|x|^2 - \alpha_0 d}{(1 + |x|^2)^2} + \frac{\alpha_0(\alpha|x|^\gamma + \beta)}{1 + |x|^2} \right] > -\infty.$$

Thus we have proved the Proposition. □

We will set

$$\begin{aligned} \omega(L) &:= \inf \{ b \in \mathbb{R}, L - bI \text{ is } m\text{-dissipative} \}, \\ s(L) &:= \sup \{ \operatorname{Re}(z), z \in \Sigma(L) \} \end{aligned}$$

and  $\Sigma(L)$  is the spectrum of  $L$ .

Now we will recall a result of S. Mischler and J. Scher [\[8\]](#), [JM](#), Theorem 4.3, page 39] which reads :

thm:SM-JC

**Theorem 4.7.** We consider an operator  $L$ , which is a generator of a semi-group  $S_L(t)$  on a Banach lattice of functions  $\mathbf{X}$  and we assume that:

**(1)** we have  $L = A + B$ , where  $B$  is a bounded linear operator and  $A$  is such that there exists  $\tau \in \mathbb{R}$ , such that  $A - \tau I$  is  $m$ -dissipative.

(2) There exist  $b \in \mathbb{R}$  and  $\psi \in D(L^*) \cap \mathbf{X}_+ \setminus \{0\}$ , such that we have

$$L^*\psi \geq b\psi.$$

(3)  $S_L$  is a positivity preserving semigroup.

(4)  $L$  satisfies the strong maximum principle.

Then we have  $s(L) = \omega(L)$ , and denoting this common value by  $\lambda$ , there exists  $G \in D(L)$  such that

$$G > 0, \quad LG = \lambda G \quad \text{with} \quad \lambda := s(L).$$

The originality of this theorem is the fact that, it establishes a spectral theory result like Krein-Rutmann's in a non compact framework. It allows us to circumvent the lack of compactness of the linear operator. It keeps the philosophy of Krein-Rutman theorem while weakening its assumptions.

*Proof.* (of Theorem [1.1](#))  
Let  $(L, D(L))$  be defined by [\(1.6\)](#). From Theorem [3.1](#) applied to the operator  $L$  we have the existence of a real number  $\lambda$ , such that  $L - \lambda I$  be  $m$ -dissipative, and generates a semigroup on the space  $\mathbf{X} := L_k^2$ . We are going to verify that the conditions of Theorem [4.7](#) are satisfied.

(1) We know that the operator  $L$  can be split in the following way: for all  $f \in D(L)$ ,

$$Lf := Bf + Af,$$

where  $B$  is a bounded operator, and defined as follows there exists  $M > 0$  and  $n \geq 1$ , such that for  $\zeta_n$  given by [\(3.14\)](#)

$$Bf := M\zeta_n f \quad \text{and} \quad f \in D(L).$$

And the linear operator  $A$  is such that there exists a real number  $\tau$ , such that  $A - \tau$  is  $m$ -dissipative and

$$Af := Lf - Bf, \quad \text{for all} \quad f \in D(L).$$

The Proposition [4.6](#) applied to  $L$  leads to the following:

(2) There exists  $b_0 \in \mathbb{R}$ , such that for some  $b > b_0$ , we have a function  $\psi > 0$ ,  $\psi \in D(L^*)$ , such that

$$L^*\psi \geq b\psi.$$

An application of Lemma [4.1](#) to the operator  $L$  allows us to deduce that  $L$  satisfies Kato's inequality. This means that:

(3) For all  $f \in D(L)$  we have  $L|f| \geq \text{sgn}(f)Lf$ . Therefore the semigroup generated by  $(L, D(L))$  is positivity preserving semigroup.

Using Lemma [4.4](#) one can assert that the linear operator  $L$  satisfies the strong maximum principle. i.e

(4) If  $(f \not\equiv 0, f \geq 0$  and  $Lf \leq 0)$ , then  $f > 0$  in  $\mathbb{R}^d$ .

Then applying Theorem [4.7](#) we conclude that , there exists  $G > 0$ , such that

$$LG = 0.$$

This completes the proof of the Theorem [1.1](#). □

## 5 Exponential stability

In this section we want to prove Theorem [1.2](#). And this proof will be based on the decomposition of the operator  $L = A + B$ , with a regular bounded operator  $B$  and a linear operator  $A$ , such that  $A - \tau I$  is  $m$ -dissipative. Before starting the proof of Theorem [1.2](#), we state the following results which will be useful for the sequel.

[thm:Contract](#)

**Proposition 5.1.** *Let  $f_0 \in D(L)$  be the initial data of [\(1.1\)](#). Assume that hypotheses **(H0)**—**(H3)** hold for  $p = 2$ . For  $M > 0$  and  $n \geq 1$ , let  $\zeta_n$  be given by [\(3.14\)](#), set*

$$Bf := M\zeta_n f \quad \text{and} \quad \text{for all } f \in D(L).$$

$$Af := Lf - Bf, \quad \text{for all } f \in D(L) \tag{5.1} \quad \text{eq:4.1}$$

Then we may fix  $n \geq 1$  and  $M > 0$  large enough so that there exists  $\omega_0 > 0$  satisfying the following property: for all  $f_0 \in L_k^2$  we have

$$\|S_A(t)f_0\|_{L_k^2} \leq e^{-\omega_0 t} \|f_0\|_{L_k^2}. \tag{5.2} \quad \text{eq:4.2}$$

*Proof.* Let  $f_0 \in D(L)$ . We consider the following equation

$$\begin{cases} \partial_t f - Af = 0 \\ f(0, x) = f_0(x) \end{cases} \quad (5.3) \quad \boxed{\text{eq:4.3}}$$

There exists  $\tau \in \mathbb{R}$  such that  $A - \tau I$  is  $m$ -dissipative, according to the definition of the linear operator  $A$ . Then there exists  $f \in C^1([0, T]; L_k^2)$  solution of (5.3). To simplify notations, let us set  $f(t, x) := S_A(t)f_0(x)$ . As the function  $f \in C^1([0, T]; L_k^2)$  it makes sense to write

$$\frac{d}{dt} \int_{\mathbb{R}^d} |f(t, x)|^2 \langle x \rangle^k dx = \int_{\mathbb{R}^d} (Af(t, x))f(t, x) \langle x \rangle^k dx$$

Using (5.1), we have

$$\int_{\mathbb{R}^d} (Af)f \langle x \rangle^k dx = \int_{\mathbb{R}^d} Lf f \langle x \rangle^k dx - M \int_{\mathbb{R}^d} f^2 \zeta_n \langle x \rangle^k dx.$$

Simple computations give us the following formula

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} f^2 \langle x \rangle^k dx &= -2 \int_{\mathbb{R}^d} |\nabla f|^2 \langle x \rangle^k dx + \int_{\mathbb{R}^d} f^2 \Delta \langle x \rangle^k dx \\ &\quad + \int_{\mathbb{R}^d} f^2 \operatorname{div}(\mathbf{E}) \langle x \rangle^k dx - \int_{\mathbb{R}^d} f^2 \mathbf{E} \cdot \nabla \langle x \rangle^k dx \\ &\quad - M \int_{\mathbb{R}^d} f^2 \zeta_n \langle x \rangle^k dx. \end{aligned}$$

Using the fact that  $\int_{\mathbb{R}^d} |\nabla f|^2 \langle x \rangle^k dx$  and  $M \int_{\mathbb{R}^d} f^2 \zeta_n \langle x \rangle^k dx$  are positive, one can deduce that

$$\frac{d}{dt} \int_{\mathbb{R}^d} f^2 \langle x \rangle^k dx \leq \int_{\mathbb{R}^d} f^2 \Delta \langle x \rangle^k dx + \int_{\mathbb{R}^d} f^2 \operatorname{div}(\mathbf{E}) \langle x \rangle^k dx - \int_{\mathbb{R}^d} f^2 \mathbf{E} \cdot \nabla \langle x \rangle^k dx.$$

Using the expressions (3.4) and (3.5), we can write

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} f^2 \langle x \rangle^k dx &\leq \int_{\mathbb{R}^d} f(x)^2 \frac{kd + k(k + d - 2)|x|^2}{(1 + |x|^2)^2} \langle x \rangle^k dx \\ &\quad + \int_{\mathbb{R}^d} f^2 \operatorname{div}(\mathbf{E}) \langle x \rangle^k dx - \int_{\mathbb{R}^d} f(x)^2 \frac{kx \cdot \mathbf{E}}{1 + |x|^2} \langle x \rangle^k dx. \end{aligned}$$

Then hypothesis **(H3)** allows us to write that

$$\frac{d}{dt} \int_{\mathbb{R}^d} f^2 \langle x \rangle^k dx \leq -2\omega^* \int_{\mathbb{R}^d} f^2 \langle x \rangle^k dx.$$

Integrating in time between 0 and  $t$ , or using Gronwall's lemma, we find that

$$\int_{\mathbb{R}^d} f^2 \langle x \rangle^k dx \leq \|f_0\|_{L_k^2}^2 e^{-2\omega^* t}.$$

Thus

$$\|S_A(t)f_0\|_{L_k^2}^2 \leq \|f_0\|_{L_k^2}^2 e^{-2\omega^* t}.$$

This completes the proof of the proposition.  $\square$

For the reader's convenience we give the proof which follows S. Mischler-J. Scher <sup>JM</sup>[8].

*Proof.* (of Theorem <sup>thm:Cv-GS</sup>1.2). By iterating the formula (<sup>eq:1.2</sup>1.2), one has

$$S_L(t) = S_A(t) + (S_A * BS_A)(t) + (S_L * BS_A * BS_A)(t). \quad (5.4) \quad \boxed{\text{sdp}}$$

Since  $S_L(t)$  is a  $\mathcal{C}_0$ -semi group, there exists  $\omega_1 \in \mathbb{R}$  and  $C_0 \geq 1$ , such that  $\|S_L(t)\| \leq C_1 e^{\omega_1 t}$ . Now we choose  $a_1$  a real number, such that  $a_1 > \omega_1$ . Thus using the inverse Laplace transform formula we obtain the following representation:

$$(S_L * BS_A * BS_A)(t) := \frac{1}{2i\pi} \int_{a_1 - i\infty}^{a_1 + i\infty} e^{zt} R_L(z) (BR_A(z))^2 dz.$$

Since 0 is a simple eigenvalue of  $L$  one can define the projection operator  $\Pi$  on the space generated by  $G$ , which is the eigenfunction associated to 0. The projection operator  $\Pi$  is defined as follows: for all  $f \in D(L)$

$$\Pi f = M(f)G, \quad \text{where} \quad M(f) := \int_{\mathbb{R}^d} f(x) dx.$$

Using Proposition <sup>thm:Contract</sup>5.1 we can easily establish that

$$\|(S_A * BS_A)(t)\| \leq C^2 \|B\| t e^{-\omega^* t}. \quad (5.5) \quad \boxed{\text{eq:sdp1}}$$

It has been shown in <sup>JM</sup>[8, Theorem 2.1] that if there exist two linear operators  $A$  and  $B$  such that  $L = A + B$ , where  $A$  and  $B$  are given as in Proposition

[thm:Contract](#) [5.1](#) and such that [\(5.5\)](#) holds, then a spectral gap exists, that is we can find  $a^* > 0$  such that the spectrum

$$\Sigma(L) \subset \{z \in \mathbb{C} \mid \operatorname{Re}(z) < -a^*\} \cup \{0\}.$$

Now we choose  $a$  such that  $0 < a < a^*$  and we define

$$J_a(t) := \frac{1}{2i\pi} \int_{-a-i\infty}^{-a+i\infty} e^{zt} R_L(z) (BR_A(z))^2 dz.$$

Using [\(5.4\)](#) we have

$$\begin{aligned} S_L(t)(I - \Pi) &= S_A(t)(I - \Pi) + (S_A * BS_A)(t)(I - \Pi) \\ &\quad + \frac{1}{2i\pi} \int_{-a-i\infty}^{-a+i\infty} e^{zt} R_L(z) (BR_A(z))^2 (I - \Pi) dz, \end{aligned}$$

Choose  $\omega_2 \in (a, a^*)$ , and consider  $z = -a + is$ , for  $s \in \mathbb{R}$ . Since

$$(zI - A)R_A(z) = I,$$

then

$$R_A = \frac{1}{z} (I - AR_A(z)) = \frac{1}{-a + is} (I - AR_A(-a + is)).$$

Thus we have

$$\begin{aligned} \|J_a(t)\| &\leq C \int_{-\infty}^{+\infty} |e^{zt}| \|R_L(-a + is)\| \|BR_A(-a + is)\|^2 ds \\ &\leq C e^{-at} \int_{-\infty}^{+\infty} \|R_L(-a + is)\| \|BR_A(-a + is)\|^2 ds \\ &\leq C e^{-at} \int_{-\infty}^{+\infty} \frac{1}{(a^2 + s^2)} \|R_L(-a + is)\| \|B(I - AR_A(-a + is))\|^2 ds. \end{aligned}$$

We know that since the operator  $B(I - AR_A(-a + is))$  is bounded uniformly in  $s$ , then there exists a constant  $C > 0$  such that

$$\forall s \in \mathbb{R}, \quad \|B(I - AR_A(-a + is))\| \leq C.$$

Since  $L$  generates a  $\mathcal{C}_0$ -semigroup. One has

$$\|R_L(-a + is)\| \leq \frac{C_0}{\omega_2 - a}.$$

Consequently we obtain

$$\|J_a(t)\| \leq C e^{-a t} \frac{C_0}{-a + \omega_2} \int_{-\infty}^{+\infty} \frac{1}{(a^2 + s^2)} ds,$$

as  $t \rightarrow +\infty$ , therefore we have

$$\|J_a(t)\| \leq \frac{C_0}{\omega_2 - a} e^{-at}. \quad (5.6) \quad \boxed{\text{eq:ja}}$$

Using the identity [\(5.4\)](#) <sup>lsdp</sup> we have

$$\begin{aligned} S_L(t)(I - \Pi)f_0 &= S_A(t)(I - \Pi)f_0 + (S_A * BS_A)(t)(I - \Pi)f_0 \\ &\quad + (S_L * BS_A * BS_A)(t)(I - \Pi)f_0. \end{aligned}$$

Thus we have

$$\begin{aligned} \|S_L(t)(I - \Pi)f_0\|_{L_k^2} &\leq \|S_A(t)(I - \Pi)f_0\|_{L_k^2} + \|(S_A * BS_A)(t)(I - \Pi)f_0\|_{L_k^2} \\ &\quad + \|(S_L * BS_A * BS_A)(t)(I - \Pi)f_0\|_{L_k^2}. \end{aligned}$$

Since we know that

$$\|(S_L * BS_A * BS_A)(t)(I - \Pi)f_0\|_{L_k^2} \leq \|J_a(t)\| \cdot \|(I - \Pi)f_0\|_{L_k^2}$$

and that thanks to [\(5.6\)](#) <sup>eq:ja</sup> we have

$$\|(S_L * BS_A * BS_A)(t)(I - \Pi)f_0\|_{L_k^2} \leq \frac{C_0}{\omega_2 - a} e^{-at} \cdot \|(I - \Pi)f_0\|_{L_k^2}.$$

On the other hand we have prove that

$$\|(S_A * BS_A)(t)(I - \Pi)f_0\|_{L_k^2} \leq C \|B\| t e^{-\omega^* t} \|(I - \Pi)f_0\|_{L_k^2}.$$

We may use Proposition [5.1](#) <sup>thm:Contract</sup> to obtain that

$$\|S_A(t)(I - \Pi)f_0\|_{L_k^2} \leq C e^{-\omega^* t} \|(I - \Pi)f_0\|_{L_k^2}.$$

Now we choose  $0 < \omega < \min(\omega^*, a)$  and then we have

$$\|S_L(t)(I - \Pi)f_0\|_{L_k^2} \leq C(\omega, t) e^{-\omega t} \|(I - \Pi)f_0\|_{L_k^2}.$$

This completes the proof of Theorem [1.2](#) <sup>thm:Cv-GS</sup>. □

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