

APPENDIX: results of the intersection between a 3D cone and a vertical plane passing by the cone apex.

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This section details the computation of a plane-cone intersection, knowing that the plane always contains the cone apex. The purpose is to compute the half-apex angle δ of the resulting intersection.

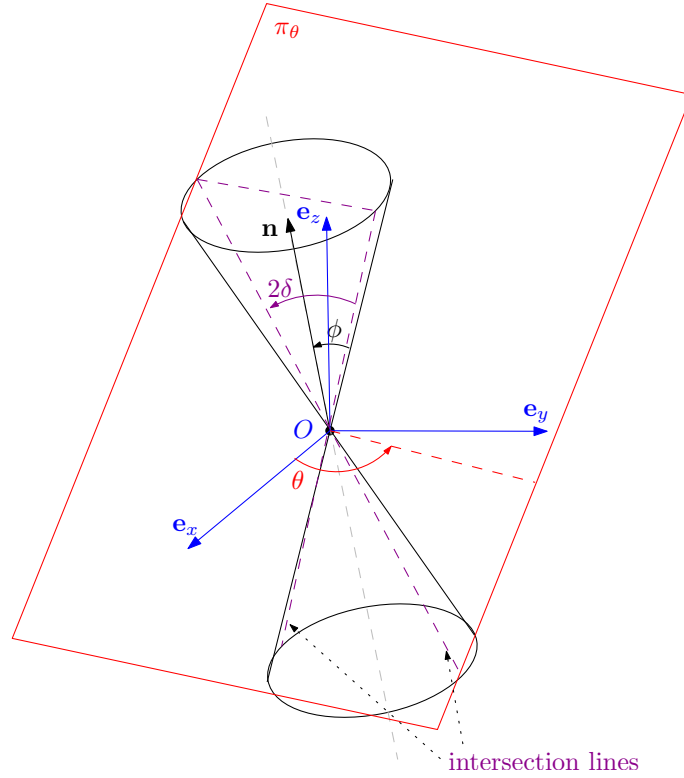


Figure 1: Illustration of a 3D cone of apex O , orientation \mathbf{n} and half-apex angle ϕ so that $\tan(\phi) = \mu$. The case of an intersection with a vertical plane π_θ resulting in a 2D cone (formed by the two ‘intersection lines’) is presented.

Let us denote the origin by $O = (0, 0, 0)^T$. Considering a cone¹ of apex O , orientation \mathbf{n} and tangent coefficient μ (see Figure 1). Its equation can be written as:

$$(y n_z - z n_y)^2 + (z n_x - x n_z)^2 + (x n_y - y n_x)^2 = \mu^2 (n_x x + n_y y + n_z z)^2 \quad (1)$$

¹Here we consider the classical shape of a cone corresponding to two symmetric half-cones with respect to the apex.

Which can be rewritten as:

$$Az^2 + B(x, y)z + C(x, y) = 0 \quad (2)$$

$$\begin{cases} A = n_x^2 + n_y^2 - n_z^2\mu^2 \\ B(x, y) = 2(1 + \mu^2)n_z(n_x x + n_y y) \\ C(x, y) = n_x^2\mu^2 x^2 + n_y^2\mu^2 y^2 - n_z^2 x^2 - n_z^2 y^2 + 2n_x n_y (1 + \mu^2)xy - n_z^2(x^2 + y^2) \end{cases}$$

The equation of a vertical plane π_θ defined by $(O; \cos(\theta)\mathbf{e}_x + \sin(\theta)\mathbf{e}_y; \mathbf{e}_z)$ is:

$$x \sin(\theta) - y \cos(\theta) = 0 \quad (3)$$

We define:

$$\psi = \frac{\pi}{2} - \text{atan2}(n_z, n_x \cos(\theta) + n_y \sin(\theta))$$

A cone is *vertical* when $-\arctan(\mu) \leq \psi \leq \arctan(\mu)$ or $\pi - \arctan(\mu) \leq \psi \leq \pi + \arctan(\mu)$.

1 Case $A \neq 0$

1.1 Case $\theta \neq \pm \frac{\pi}{2}$

For now, we assume that $A \neq 0$ and $\theta \neq \pm \frac{\pi}{2}$. A solution to the system of (1)-(3) exists if and only if $\Delta \geq 0$, where:

$$\Delta = (n_x^2 + n_z^2)\mu^2 - n_y^2 - (n_x^2 + n_y^2 n_z^2 \mu^2) \tan(\theta)^2 + 2n_x n_y (1 + \mu^2) \tan(\theta)$$

If $\Delta > 0$, the intersection results in a 2D cone. Then, the half-cone selection can be performed considering the direction of \mathbf{n} . If $\Delta = 0$, the intersection gives a line which is not relevant for our application. $\Delta < 0$ yields an intersection reduced to O , which is also not relevant.

The equations of the lines forming the 2D cone are:

$$\begin{cases} y = \tan(\theta)x \\ z = K^\pm x \end{cases}, \quad K^\pm = \frac{n_x n_z (1 + \mu^2) + n_y n_z (1 + \mu^2) \tan(\theta) \pm \sqrt{\Delta}}{A}$$

Thus, one can compute the half-apex angle δ of the 2D cone:

$$\cos(2\delta) = \varepsilon \frac{1 + \tan(\theta)^2 + K^+ K^-}{\sqrt{(1 + \tan(\theta)^2 + K^{+2})(1 + \tan(\theta)^2 + K^{-2})}}$$

where ε is equal to -1 when the cone is *vertical* and $A < 0$. It is equal to 1 otherwise.

1.2 Case $\theta = \pm \frac{\pi}{2}$

Now we consider the case $\theta = \pm \frac{\pi}{2}$. The intersection type is about:

$$\Lambda = (n_y^2 + n_z^2)\mu^2 - n_x^2$$

The equations of the lines forming the 2D cone become:

$$\begin{cases} x = 0 \\ z = G^\pm y \end{cases}, \quad G^\pm = \frac{n_y n_z (1 + \mu^2) \pm \sqrt{\Lambda}}{A}$$

The half-apex angle δ of the 2D cone:

$$\cos(2\delta) = \varepsilon \frac{1 + G^+ G^-}{\sqrt{(1 + G^{+2})(1 + G^{-2})}}$$

where ε is equal to -1 when the cone is *vertical* and $A < 0$. It is equal to 1 otherwise.

2 Case $A = 0$

Now we consider that $A = 0$, which is equivalent to $(1 + \mu^2)n_z^2 = 1$. In such case, one of the intersection lines is parametrized by $(0, 0, z)^T$. The following equations detail the expression of the second line, in order to compute δ .

2.1 Case $\theta \neq \pm \frac{\pi}{2}$

If $\theta \neq \pm \frac{\pi}{2}$, we have:

$$B(x) = 2(1 + \mu^2)n_z(n_x + n_y \tan(\theta))x$$

If $n_x + n_y \tan(\theta) = 0$, $B(x) = 0$. From (2), we have $C(x) = 0$. Thus, the second line is perpendicular to \mathbf{e}_z , and:

$$\delta = \frac{\pi}{4} \text{ rad}$$

Otherwise, if $n_x + n_y \tan(\theta) \neq 0$, since $n_z \neq 0$, we have:

$$\forall x \neq 0, B(x) \neq 0$$

Thus, we can write the equations of the second intersection line (restricted to the \mathbf{n} direction by the absolute values):

$$\begin{cases} y = \tan(\theta)x \\ z = Hx \end{cases}$$

$$H = -\frac{(n_x^2 + n_y^2 \tan^2(\theta))\mu^2 - n_x^2 \tan^2(\theta) - n_y^2 + 2n_x n_y (1 + \mu^2) \tan(\theta) - n_z^2 (1 + \tan^2(\theta))}{2(1 + \mu^2)|n_z||n_x + n_y \tan(\theta)|}$$

The half-apex angle of the 2D cone formed by the intersection lines can be computed as follows:

$$\cos(2\delta) = \frac{H}{\sqrt{(1 + \tan^2(\theta) + H^2)}}$$

2.2 Case $\theta = \pm \frac{\pi}{2}$

Now if $\theta = \pm \frac{\pi}{2}$, $x = 0$ so we have:

$$\begin{cases} B(y) = 2(1 + \mu^2)n_z n_y y \\ C(y) = n_y^2 \mu^2 y^2 - n_x^2 y^2 - n_z^2 y^2 \end{cases}$$

If $n_y = 0$, $B(y) = 0$. Thus $C(y) = 0$ which results in $y = 0$. So the second line is included in the first one $(0, 0, z)^T$, it is not relevant to compute δ .

Otherwise, if $n_y \neq 0$, the second line (restricted to the \mathbf{n} direction by the absolute values) equations are:

$$\begin{cases} x = 0 \\ z = Ly \end{cases}, \quad L = -\frac{n_y^2(1 + \mu^2) - 1}{2(1 + \mu^2)|n_y||n_z|}$$

The half-apex angle becomes:

$$\cos(2\delta) = \frac{L}{\sqrt{(1 + L^2)}}$$