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# The constitution of the nature of mathematics in the lecturing practices of three university mathematics teachers

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*The study reported in this paper investigates how notions of the nature of mathematical knowledge and mathematical objects are articulated in the discursive practices of three university mathematics teachers at a Swedish university. The data consists of video recorded lectures, and the analyses were informed by classifications presented by Lerman (1990) and Davis and Hersh (1981). The results indicate that different epistemological and ontological positions are indeed constituted through the discourse. Although the discourse is generally highly objectified, the ways in which mathematical objects are introduced differ. Mostly the discourse was within an absolutist paradigm, but there were also examples of how the socio-historical nature of mathematical knowledge is emphasized.*

**Keywords:** Tertiary mathematics, university teaching, epistemology of mathematics, ontology of mathematics, mathematical discourse.

## INTRODUCTION

Recent years have seen an increasing research interest in university mathematics teaching, including a growing number of studies focusing on so-called traditional mathematics teaching (e.g., Gücler, 2013; Viirman, 2014b; Weber, 2004). One aspect of university mathematics teaching, which has so far received less attention, is how notions of the character of mathematics are constituted through the teaching practices in mathematics lectures. The present paper is an attempt at addressing this issue. There are many ways of characterizing mathematics (see, e.g., Devlin's (2000) four faces of mathematics), but in this short paper I have chosen to focus on the classic philosophical questions of epistemology and ontology, that is, on the nature of mathematical knowledge and mathematical

objects, and how these notions are manifested in the teaching discourse.

## THEORETICAL PERSPECTIVES

The study takes its theoretical starting point in the commognitive framework of Sfard (2008). In this theory, mathematics is viewed as a discursive activity, with mathematical discourse constituted by its use of *words*, *visual mediators*, *narratives* (sequences of utterances speaking of objects, relations between and/or processes upon objects, and subject to endorsement or rejection within the discourse) and *routines* (repetitive patterns characteristic of the discourse). In the present paper the focus will be mainly on narratives and substantiation routines, that is, routines aimed at deciding whether to endorse previously constructed narratives. Mathematical discourse is also characterized by its high degree of objectification, where words in the discourse are viewed as signifying independently existing objects (ibid, p. 300).

Concerning the questions of the epistemology and ontology of mathematics, these have been central topics within the philosophy of mathematics for hundreds of years, and a more thorough discussion is far beyond the scope of this paper. For the purposes of this study the distinctions made by Lerman (1990) and Davis and Hersh (1981) will suffice. Lerman, in a study investigating the relation between views of the nature of mathematics and teaching practice, distinguishes between absolutist and fallibilist epistemologies. The absolutist sees mathematical knowledge as certain, absolute and timeless, and views the history of mathematics as “a demonstration of the errors and mistakes along the way to certain knowledge” (Lerman, 1990, p. 54–55). Fallibilism (inspired by Wittgenstein and Lakatos) on the other hand, “sees the growth of mathematical

knowledge as a process of conjectures, proofs and refutations, and accepts the uncertainty of mathematical knowledge as part of the nature of mathematics” (ibid, p. 54). A similar stance is taken by Davis and Hersh, (1981), who also consider mathematical ontology. They distinguish between Platonism, where mathematical objects are seen as real, having an objective existence outside of human experience, and an alternative position, later denoted as humanism (Hersh, 1997), where mathematical objects are seen as human creations, but still objective in the sense of being external to the consciousness of any single individual. Instead they belong to the social, non-material culture of mankind.

Thus, the focus of this paper is on how the epistemology and ontology of mathematics are expressed in the teaching discourse. The question that the paper aims at answering is the following: How are notions of the nature of mathematical knowledge, and of mathematical objects, articulated through the discursive practices of the teachers?

## PREVIOUS RESEARCH

Traditionally, questions about the nature of mathematics in relation to teaching practices have been handled in the context of research on teacher beliefs. That is, the focus has been on what beliefs about the nature of mathematics teachers possess, and how these beliefs might impact on their teaching practice (for overviews of the early achievements in this field, see, e.g., Pajares, 1992; Thompson, 1992). The study by Ernest (1989) can serve as an example. Ernest identifies three categories for characterizing individuals’ views of the nature of mathematics: Instrumentalist, Platonist, and Problem-Solving. However, Ernest’s study, like most similar studies, focuses mostly on pre-service or practicing teachers in elementary or secondary school, although there are some examples of studies of university teachers’ beliefs about mathematics (e.g., Mura, 1993; Speer, 2008). Also, studying teacher beliefs shifts the focus from the teaching to the teacher, which runs contrary to the aim of the present study. Furthermore, the very notion of teacher beliefs as a topic of research has been the subject of much criticism, from a methodological standpoint – beliefs being notoriously hard to define and gain access to (Speer, 2008; Skott, 2009) as well as on a purely conceptual level (Skott, 2013). Skott suggests that we move away from the objectified construct of beliefs, focusing instead on “patterns of participation” in social

practices, that is, on the processes said to give rise to beliefs (Skott, 2013, p. 549). This approach is similar to Sfard’s (2008) commognitive framework, with its focus on discursive practice as a patterned activity.

Unfortunately, studies looking at how notions of the nature of mathematics, as described for instance in the literature cited above, are expressed through teaching practice are rare. The previously mentioned study by Lerman (1990) could be said to be one, although the principal focus of the empirical analysis in his (mainly theoretical) paper is on what student teachers’ interpretations of a teaching episode tells us about their views of mathematics. Another example, which also happens to concern university teaching, is the study by Österholm (2010), investigating “what types of epistemologies are conveyed through properties of mathematical discourse in two lectures” (p. 241). Despite Österholm’s paper being framed in the language of belief research, the analyses in fact focus solely on the epistemological character of mathematics as conveyed through the discourse. To this end, Österholm considers the types of statements and the type of argumentations used by the teachers. Of particular relevance for the present study is the distinction between use-statements, related to procedural knowledge; and object-statements, related to conceptual knowledge. A dominance of object-statements can be seen in the calculus lecture, indicating a focus on conceptual knowledge.

## METHOD

The analyses presented in this paper are based on data collected for my doctoral thesis (Viirman, 2014a). This data consists of video recordings of first-semester mathematics lectures by seven teachers at three different Swedish universities, approximately two hours of video for each teacher. The teachers were selected among those volunteering to participate in the study, aiming for variety both in teaching experience and in topics taught. In this study, however, I am using data from lectures by three of the teachers (denoted A1, A3 and A4 in what follows, in accordance with other publications arising from this data, e.g., Viirman, 2014a; 2014b). These were chosen since the analysis conducted for the doctoral thesis indicated that they were the richest and most varied with regard to the aims of the present study. All three teachers work at the same university, one of the largest and most well-established in Sweden, and are experienced

teachers, having taught university mathematics for more than 10 years. Teacher A1 is female, while the other two are male. All three lectures were given in courses aimed at engineering students, and were taught in a traditional style, with the lecturer talking and writing on the board. The number of students ranged from about 50 (teacher A1) to about 150 (teachers A3 and A4). Teacher A1 taught an introductory course, preparatory for calculus; teacher A3 taught linear algebra; while teacher A4 taught single-variable calculus. The topic in all three lectures was various aspects of the function concept. For more detail on the process of data collection, see (Viirman, 2014a; 2014b).

As part of the work on the thesis, the video recorded lectures were transcribed verbatim, speech as well as the writing on the board. For the present study, the transcribed lectures given by teachers A1, A3 and A4 were then analysed, first separately and then in comparison, looking specifically at how mathematical objects and the nature of mathematical knowledge were expressed through the teachers' discourse, focusing on word use and narratives. Regarding the teachers' use of definitions, and how new mathematical objects were introduced I looked, for instance, at how notions of agency were expressed in the discourse – whether these new objects were spoken of as originating outside of the discourse or within it. Concerning means of substantiation I looked, for instance, for utterances suggesting change in such means over time. Throughout I used the categorisations of Lerman (1990) and Davis and Hersh (1981) to guide my analyses. I want to stress that this is not a study of teachers' beliefs. I make no claims as to whether the ways in which mathematical objects and mathematical knowledge are articulated in the discursive activity in the lectures have any bearing whatsoever on the views the teachers might be holding regarding these matters.

## RESULTS

Considering first the question of mathematical ontology, the discourse documented in this study is typically mathematical in that it is generally highly objectified. The mathematical objects are spoken of as being independently existing [1]:

Teacher A4 So, it's about continuity, and that is a property that functions can have.

Teacher A3 There is a transformation that we call, I don't know, id for identity, that takes every vector to itself.

In fact, functions are so much like physical objects that they can be moved around:

Teacher A1 That is a function; it is the function  $x^2$  that I move one step to the right and two steps upwards.

Teacher A4 What happens to this function when  $x$  is bigger than one? (...) It goes down, yes, and then it will wander here, and get bigger and bigger and bigger.

However, looking at how new mathematical objects are introduced, narratives are framed in different ways, suggesting different ontological positions. Consider the following example:

Teacher A1 for us to know what we are talking about, we have to begin by saying exactly what we mean by a function. I think that most of you already have a feeling for what it is that a function is, but maybe you haven't seen exactly a definition. Because, you know, it is like this in mathematics that all words we use, we have to say exactly what we mean by them so that we are totally agreed, if I say that all functions have a certain property, then all have to agree with me what objects we are talking about. We have to agree about what we mean by the word function.

Here, even though the functions are explicitly spoken of as objects, it is still clear that regarding the properties of these objects it is up to the participants in the mathematical activity to decide what they are. But all participants have to agree in order to be able to use them meaningfully. This way of talking is very much in accordance with the humanist philosophy, as formulated by Davis and Hersh (1981). Later in the same lecture, we find the following example. The teacher is discussing the unit circle, an example drawn on the board.

Teacher A1 We would perhaps want this to be a function (...) but then when we insert something which isn't one or minus one

then we get- there are two y-values that fit, so the function doesn't give us exactly one, it gives more than one, and then it isn't a function.

Hence, although we, the participants in the mathematical activity, have ourselves constructed this object called function, we are not free to do whatever we want with it. Once we have decided what to mean by the word, it comes equipped with properties that are not ours to decide over. It is not enough that we want something to be a function; it has to agree with what we decided that a function should be. Again this corresponds to the humanist position, where mathematical objects, although human creations, still have objective properties.

Yet another example from the same lecture: having introduced trigonometric functions through right triangles the teacher notes that these functions are only defined on the interval  $(0, \pi/2)$ , and continues:

Teacher A1 We would want these functions sine and cosine to be defined for all real numbers, we would like to exchange this little piece for the whole of  $R$  (...) then we have to figure out how to do this, and before we do this we have to say how we are going to measure angles.

Again, it is up to us as participants to decide how to define the sine and cosine functions outside of the interval  $(0, \pi/2)$ . But, it can't be done any way we want. It has to agree with what has already been decided, that is, with how the functions are defined for acute angles.

On the other hand, the introduction of mathematical objects can be done through narratives framed in a very different manner:

Teacher A3 A function from  $R^m$  to  $R^n$  is called a linear transformation if it is linear, that is, if it satisfies two conditions:

The teacher then goes on to describe the two linearity conditions. After giving, as an example, a characterization of all linear transformations from  $R$  to  $R$ , he concludes:

Teacher A3 this was a very small example, and an example that has completely negli-

ble importance for the rest of the course, but anyway it says something about that linear transformations are very rare. In any case they are very important.

As presented here, there are some things called functions, and among these a small number satisfy certain conditions, and these are called linear transformations. Indeed, these are spoken of as rare, almost as if talking about rare birds or flowers. There is nothing in the way these objects are discussed suggesting that they are anything other than objectively existing, independently of our working with them. No reasons are given for the linearity conditions looking the way they do, for instance. Looking at the example given at the start of the section about the identity transformation, the transformation is there, we as practitioners of mathematics just give it a name. Similarly, concerning the relationship between matrices and linear transformations:

Teacher A3 Every linear transformation defines a matrix which we will call the standard matrix of the linear transformation.

Again, the transformation defines the matrix; we only have to give it a name. This way of speaking about mathematical objects fits with the Platonist position.

Considering epistemology, that is, the questions of mathematical knowledge and how it is obtained, much of the discourse documented in this study is consistent with an absolutist paradigm. Claims are mainly justified using traditionally mathematical means of substantiation, for instance through mathematical proof or by reference to already established mathematical facts (for more detail on the teachers' substantiation routines, see Viirman, 2014b). Some examples:

Teacher A4 It isn't obvious that it is like this, this is not a property that all functions have, but for us it was a consequence of what? Two things. The basic properties of limits (...) and the standard limits.

Teacher A3 Why is it linear? It is linear precisely because matrix multiplication works like that.

The mathematical substantiations are mostly done through algebraic and numerical reasoning, as in the



following example, where the aim is calculating the range of the function  $f(x) = (x - 1)^2 + 2$ :

Teacher A1 In this case, there is an even square here, plus 2. Even squares can be zero or bigger. (...) Hence it can be 2 plus something positive, so it can be all numbers that are greater or equal to 2.

Or here, part of the process of calculating the value of  $\sin(\pi/4)$  from a diagram of half a square:

Teacher A1  $[x_0^2 + x_0^2 = 1^2 \Rightarrow x_0^2 = 1/2 \Rightarrow x_0 = 1/\sqrt{2}]$

There are also examples of using geometrical reasoning to substantiate claims:

Teacher A3 That these two vectors are perpendicular, this you can see immediately geometrically (...) If we project something which is perpendicular to  $v$  on  $v$ , then we just get zero, right? [Draws a vector pointing straight upwards, draws another vector pointing straight to the right and marks it by "v", marks the angle between the two vectors as right.] Here is  $v$  [points to the vector  $v$ ], here is something which is perpendicular [points to the vertical vector], if we project it down here [points to the base of the vertical vector] then we get sort of nothing.

In a similar fashion, teacher A3 shows the linearity of rotations in the plane by geometrical arguments, drawing vectors and their images under the transformation and showing why the linearity conditions are satisfied.

There are also examples of substantiations emphasizing the cumulative nature of mathematical knowledge, referring back to previously known facts, as in the excerpt quoted above, as well as in the following examples:

Teacher A4 If someone were to twist your arm and say: How do you know that? Then it is precisely from the good old rules of limits.

Teacher A4 And now it's time to reconnect this part of the brain that you have neglect-

ed for some time, namely complex numbers. Because the fundamental theorem of algebra says that a third degree polynomial has three roots, in general complex.

This is still consistent with an absolutist paradigm. There are however some examples of substantiations consistent with a fallibilist epistemology, where the historical development of mathematics is explicitly used in the teaching, emphasising the socio-historical nature of mathematical knowledge. Some examples:

Teacher A4 In fact, this is how one often defines continuity during the 18<sup>th</sup> century, this means that the graph of this function if we are to draw it [Draws a coordinate system, and draws a connected curve] it hangs together like this; I can draw it without lifting the chalk from the blackboard.

Teacher A4 This is the beginning of a line of work that is quite important within analysis, and which gained momentum during the latter half of the 19<sup>th</sup> century, when the deal was to find really pathological functions, that test our understanding of the function concept and what we can assume (...) A lot of great mathematicians spent time on this, and some people thought that it was totally nuts, such things don't exist, they are totally insignificant, can't be used for anything, and that turned out to be totally wrong.

Teacher A4 I just read an article by a German mathematician from the late 19<sup>th</sup> century (...) and he spent half a page in his article explaining what this meant, it wasn't established at that time, the terminology was vague, and it isn't totally trivial what it means.

This way of using historical examples indicates how mathematical definitions are subject to change over time, and how even what is to be counted as mathematics is the subject of disagreement and controversy.

## DISCUSSION

The results of the study show how different philosophical positions on the nature of mathematical knowl-

edge and mathematical objects are articulated in the teaching discourse. The study is thus an example of how notions traditionally studied within the context of research on teachers' beliefs can indeed be studied purely on the level of discourse.

One prominent characteristic of the discourse of all three lectures documented in this study was the use of a highly objectified language, something Österholm (2010) also notes in his study. Some of the differences that could be seen, for instance, regarding the type of substantiations used for claims made, can probably be explained by differences in the courses taught. For instance, in an introductory course such as the one taught by teacher A1 one does not expect substantiation through previously established facts to be very prominent.

Still, there are other differences, more directly related to notions of the nature of mathematics, which are not obviously explained by differences in topic or level of the courses taught. For instance, the results show clearly how two different, and indeed contrasting, positions concerning mathematical ontology can be seen in the teaching discourse: a humanist position, emphasizing how the definition is something we as participants in the mathematical activity have agreed upon, and a more depersonalized Platonist position where mathematics is presented as something that is discovered, appearing fully formed.

Concerning the epistemology of mathematics, although the discourse documented in the study is generally consistent with an absolutist paradigm of mathematical knowledge, such statements may be interpreted differently depending on other aspects of the discourse. For instance, statements suggesting an absolutist position may be interpreted differently in the light of whether emphasis is generally placed on the man-made character of mathematical objects or, on the other hand, on a more Platonist way of talking about mathematical objects. Similarly, an emphasis on the socio-historical development of mathematics (humanism position) makes an absolutist interpretation of the mathematical discourse less likely. One might point out here, however, that the use of historical examples does not in itself necessarily suggest a fallibilist epistemology. As Lerman (1990) indicates, “[w]hilst it may be generally accepted that, in its external history, it is influenced by cultural determinants and social factors, the prevailing view is that the mathe-

mathematical knowledge that results is self-justificatory in terms of its truth” (p. 54). It might be possible to interpret some of the statements quoted above in this way. For instance, on the topic of the classic theorems on continuous functions (the maximum value theorem, the mean value theorem etc.) when the teacher notes that “it wasn’t established at that time, the terminology was vague”, this could be taken to mean that mathematicians have now established the true state of affairs. Although such an interpretation might be less likely, one would have to gather more data, both in the form of further recordings of his teaching, and data on how students interpret the use of the history of mathematics in teaching, to be able to draw more certain conclusions.

Indeed, data on the students’ interpretations of the teaching practices would be useful, to establish whether the notions of mathematics constituted through the teachers’ discursive practices actually have any effect on student learning. It has been claimed by many within the mathematics education community that students’ beliefs about mathematics affect their learning (e.g., Pajares, 1992). Even if one accepts Skott’s (2013) critique of the concept of belief, increased knowledge of how notions of the nature of mathematics are constituted through teaching practices, and of how students interpret these practices, would be useful to gain further insight into the relationship between teaching and learning mathematics.

More generally, although the present study is small, the conclusion that notions traditionally considered as belonging to the field of belief research can actually be studied as features of discourse could be of importance to any researcher interested in such aspects of mathematical teaching and learning.

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## ENDNOTE

1. All excerpts have been translated from Swedish by the author. Text within [square brackets] indicates writing on the board.