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# First steps in re-inventing Euler's method: A case for coordinating methodologies

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*In this report, we highlight the epistemic actions and concomitant discursive shifts of four students as they reinvent the fundamental idea and technique in Euler's method. We use this case to further the theoretical and methodological coordination of the Abstraction in Context (AiC) approach, with its associated model commonly used for the analysis of processes of constructing knowledge by individuals, and small groups and the Documenting Collective Activity (DCA) approach, with its methodology commonly used for identifying normative ways of reasoning with groups of students. In this report, we show students' first steps towards re-inventing Euler's method and explicate the theoretical and methodological commonalities of AiC and DCA.*

**Keywords:** Documenting collective activity, abstraction in context, networking theories, Euler's method.

## INTRODUCTION

Research at the undergraduate level is moving beyond the documentation of student difficulties towards the design, implementation, and analysis of innovative learning environments where students reinvent important mathematical ideas and methods. For example, in differential equations, research has documented that students are able to reinvent, given appropriate task sequences and learning environments, Euler's method, bifurcation diagrams, and even an analytic approach for solving systems of linear differential equations (e.g., Rasmussen, 2007). Such reinventions are, from our perspective, both individual and collective accomplishments. Methodological approaches for analysing such accomplishments, however, are sorely needed. In this report, we highlight the epistemic actions and concomitant argumentation of four students as they reinvent the fundamental idea and

technique in Euler's method. We use this case to further the theoretical and methodological coordination of the Abstraction in Context (AiC) approach and the Documenting Collective Activity (DCA) approach (see Hershkowitz et al., 2014; Tabach et al., 2014 for initial attempts at coordinating these two approaches). The two approaches have various theoretical and methodological commonalities that we will refer to as environmental and underlying ones; the analysis in the present paper led to the discovery of additional commonalities that we will refer to as environmental and underlying internal ones. We explicate these commonalities to set the stage for the analysis of student reinvention, but first begin with a brief summary of the AiC and DCA approaches.

## ABSTRACTION IN CONTEXT AND THE RBC+C MODEL

Abstraction in Context (AiC) is a theoretical framework for investigating processes of constructing and consolidating abstract mathematical knowledge (Hershkowitz et al., 2001). Abstraction is defined as an activity of vertically reorganizing previous mathematical constructs within mathematics and by mathematical means, interweaving them into a single process of mathematical thinking so as to lead to a construct that is new to the learner. According to AiC, the genesis of an abstraction passes through three stages (ibid): (i) the arising of the need for a new construct, (ii) the emergence of the new construct, and (iii) the consolidation of that construct. AiC includes a theoretical/methodological model, according to which the description and analysis of the emergence of a new construct and its consolidation relies on a limited number of epistemic actions: Recognizing, Building-with, and Constructing (RBC).

These epistemic actions are often observable as they are expressed by learners verbally, graphically, or otherwise. Recognizing takes place when the learner recognizes a specific previous knowledge construct as relevant to the problem currently at hand. Building-with is an action comprising the combination of recognized constructs in order to achieve a localized goal, such as the actualization of a strategy or the solution of a problem. The model suggests Constructing as the central epistemic action of mathematical abstraction. Constructing consists of assembling and interweaving previous constructs by vertical mathematization to produce a new construct. It refers to the first time the new construct is expressed by the learner.

Recognizing actions are nested within building-with actions, and recognizing and building-with actions are nested within constructing actions. Moreover, constructing actions are at times nested within more holistic constructing actions. Therefore the model is called the nested epistemic actions model of abstraction in context, or simply the RBC+C model. The second "C" stands for Consolidation. The consolidation of a new construct is evidenced by students' ability to progressively recognize its relevance more readily and to use it more flexibly in further activity.

### **DOCUMENTING COLLECTIVE ACTIVITY OVERVIEW**

The methodological approach of documenting collective activity (DCA) is theoretically grounded in the emergent perspective (Cobb & Yackel, 1996), a basic premise of which is that mathematical learning is a constructive process that occurs while participating in and contributing to the collective activity of the classroom. The collective activity of a class refers to the normative ways of reasoning that develop as students work together to solve problems, explain their thinking, represent their ideas, etc. These normative ways of reasoning can be used to describe the mathematical activity of a group and may or may not be appropriate descriptions of the characteristics of each individual student in the group. A mathematical idea or way of reasoning becomes normative when there is empirical evidence that it functions in the classroom as if it is shared. The empirical approach makes use of Toulmin's model of argumentation, the core of which consists of Data, Claim, and Warrant. Typically, the data consist of facts or procedures that lead to the conclusion that is made. To further improve the strength

of the argument, speakers often provide more clarification that connects the data to the claim, which serves as a warrant. It is not uncommon, however, for rebuttals or qualifiers to arise once a claim, data, and warrant have been presented. Backing provides further support for the core of the argument.

The following three criteria are used to determine when a way of reasoning becomes normative: 1) When the backing and/or warrants for particular claim are initially present but then drop off, 2) When certain parts of an argument (the warrant, claim, data, or backing) shift position within subsequent arguments, or 3) When a particular idea is repeatedly used as either data or warrant for different claims across multiple days. See Rasmussen and Stephan (2008) for an illustration of the first two criteria.

### **ENVIRONMENTAL COMMONALITIES**

The use of both methodologies, AiC and DCA, requires very explicit classroom norms. First, they require classrooms in which students are routinely explaining their thinking, listening to and indicating agreement or disagreement with each other's reasoning, etc. If such norms are not in place, then evidence is unlikely to be found of challenges, rebuttals, and negotiations that lead to ideas where knowledge is constructed and starts functioning as if shared by the whole class. We call such classrooms "inquiry classrooms." Second, they require the intentional use of tasks that were purposefully designed to offer students opportunities for constructing new knowledge by engaging them in problem solving and reflective activities allowing for vertical mathematization.

Both methodologies focus on the ways in which mathematical progress is achieved and spreads in the classroom. RBC+C focuses on individuals or small groups working in the classroom and DCA focuses on group discussions. In this sense, the two methodologies complement each other in analyzing a sequence of lessons including individual and group work and in tracing how knowledge is constructed and becomes normative along this sequence.

### **UNDERLYING COMMONALITIES**

Other characteristics of a classroom culture in which DCA and RBC+C methodologies might be enacted together are that the tasks are designed to afford inquiry

and the emergence of new constructs by vertical mathematization from previous constructs; such learning materials allows for interweaving collaborative work in both small-group work and whole-class discussions, where the teacher adopts a role that encourages inquiry in the above sense.

Another underlying characteristic relates to the centrality of the shared knowledge. AiC defined shared knowledge as “a common basis of knowledge which allows the students in the group to continue together the construction of further knowledge in the same topic” (Hershkowitz et al., 2007, p. 42). This definition relates to cognitive aspects. We find its counterpart in sociological terms, in the phrase “function as if shared” used by the DCA approach. What is common between the two constructs is the point that each operationalizes when particular ideas or ways of reasoning are, from a researcher’s viewpoint, beyond justification for participants. At the collective level, ideas or ways of reasoning that function as if shared have the status of accepted mathematical truths for the group. At the individual level, consolidation results in individuals accepting something as a mathematical truth.

### FIRST STEPS TO REINVENTING EULER'S METHOD

We begin with the following excerpt, used also in Stephan and Rasmussen (2002) and in Tabach and colleagues (2014) but for different purposes. It is a discussion between Liz, Joe, Deb and Jeff, four students in a class of 29 STEM, first year undergraduate students, working on the following problem during group work on the first lesson:

*Consider the following rate of change equation, where  $P(t)$  is the number of rabbits at time  $t$  (in years):  $dP/dt = 3P(t)$  or in shorthand notation  $dP/dt = 3P$ . Suppose that at time  $t = 0$  we have 10 rabbits (think of this as scaled, so we might actually have 1000 or 10,000 rabbits). Figure out a way to use this rate of change equation to approximate the future number of rabbits at  $t = 0.5$  and  $t = 1$ .*

Prior to this task students received no instruction on Euler’s method, but the class did develop graphical depictions of what the exact solution should more or less look like (e.g., not linear but increasing at an increasing rate). The excerpt includes a DCA analysis and an RBC analysis. The DCA analysis classifies the shaded parts according to Toulmin’s model as data

[D], claim [C], warrant [W], backing [B], or qualifier [Q]. For example, D2 is the Data used for Claim 2. We indicate at the end of a turn if one of the three criteria has been met. The RBC analysis is based on an a priori analysis of the activity that yielded the following knowledge elements intended to be constructed: Csy – establishing connection between  $P$  and  $dP/dt$  (if you know  $P$  you can find  $dP/dt$ ); Cpit – population iteration (given  $P$  and  $dP/dt$  at a moment in time allows one to find  $P$  at a later time); and Crit – rate of change iteration (applying Csy at that later time one can find the corresponding  $dP/dt$ ); and finally Cit: Cpit and Crit can be combined into a repeating loop. We conjecture that in previous courses students constructed  $dP/dt$  as a ratio (Crat) and hence they can recognize and build-with this construct. To keep things transparent we omit mentioning previous constructs to which our analysis does not explicitly refer. RBC actions were italicized in students’ talk and coded in the third column as recognising (R), building-with (B), constructing (C) or consolidating (CC). This side-by-side analysis was done to facilitate coordination between the RBC+C and DCA methodologies. This coordination is then helpful for analysing student’s re-invention of Euler’s method.

- 1 Liz *I would plug in the population of rabbits for  $P$  to determine the rate of change initially. What is the rate of change when time equals zero [W1]. So if we had a graph, its kind of like what we were just talking about, we are trying to determine the rate of change when this time is equal to zero [B1].* R B
- 2 Joe *Oh ok. This is where 10 rabbits at zero [D1].* R
- 3 Liz *What do you think?*
- 4 Deb *Oh ok, so I get the rate of change at time initially the rate of change would be 3 [sic] [C1]. Did I multiply it right?* R B Csy
- 5 Liz *And then I guess the simple ...*
- 6 Joe *How did you do that?*
- 7 Liz *Okay, well this [D2] [differential equation] is the change in the population over the change in time [C2].* Rrat
- 8 Joe *Right.*
- 9 Liz *Okay, and this 3 I’m taking as being the constant or whatever you call the growth rate. And this  $P$  of  $t$  is the population at any given point of time  $t$ , but this is just short hand notation for it . So I*

- thought, if we know the population is ten when our time equals zero [D1 & D2 elaborated], can we plug in the  $P(t)$  population at time zero and find out what initially the rate of change is [W1]? B Csy
- 10 Joe It would be  $10 = 3 \dots$
- 11 Liz Times 10 Csy
- 12 Jeff Okay I see so it would be 30 [C1]. Csy
- 13 Liz 30, I mean does that,
- 14 Jeff Yeah that does make sense.
- 15 Joe Well, wouldn't  $10 = 3P(t)$ ? [C3] At time zero we have 10 rabbits [D3]. (Note that his claim is incorrect) R B
- 16 Liz Well 10 is actually the population [D4] so I'm taking that that has to actually be the population at time  $t$ . I don't think it's telling us how the population is changing which would be  $dP/dt$  [C4]. CCsy
- 17 Liz So if we have that [initial rate of change is 30] [D5], the question is how can we use that to help us figure out the population after a half unit elapsed? [32 sec pause] (identifies a need to construct Cpit) Rsy
- 18 Jeff How would we work time into the equation?
- 19 Liz If we think of it right now as our time equals zero, we could say... B
- 20 Deb We have the 30 [D5]. Rsy
- 21 Liz We have the 30 to work [D5] with, so couldn't we say we don't [5 second pause] Bsy
- 22 Deb You said the population is 10 right [D5]? B
- 23 Liz um hm.
- 24 Deb So three times ten would give us our rate of change [D5]. Say 0.5 years passes, this is our rate of change. Then we'll take that 0.5 times the rate of change [W5] which will give us what, the new amount of rabbits plus the old amount of rabbits. [C5] [Criterion 2 met for Csy, see turn 12 where this was C1] Cpit
- 25 Liz So the old amount of rabbits is ten [D6] R
- 26 Deb Am I making sense?
- 27 Jeff I think so, so that would be 25 [C5], is that what you're saying? Cpit
- 28 Liz Okay I think I get what you're saying. So we're at time zero and we have 10 rabbits, and the rate of change is 30 [D6] so its going to grow at a rate of 30 rabbits per year [C6]? [Criterion 2 met for Csy, similar to turn 24 by Deb] Cpit
- 29 Deb Right. So we'll have 30 more rabbits. [D7]
- 30 Liz But we only want to go a half a year.
- 31 Deb So it'll be 0.5 times 30, [W7] which is 15 [C7]. [Criterion 2 met for part of Cpit (namely that 30 is also the change over one year), claim C6 is now D7] CCpit
- 32 Liz And so we're really not figuring out the rate of change we figuring... Well this is the rate of change and we're using the rate of change to figure out the number of rabbits we are going to increase by in half a year [B5]. Cpit
- 33 Deb Well the new population...
- 34 Joe Well if  $t$  is 0 [D8] then we have 0 [C8]. But you said when  $t$  is zero we have 10 [Rebuttal to Argument 1]. (note that his assertion is incorrect) Rsy
- 35 Liz I think it just means initially we have 10 [Rebuttal to C8]. R
- 36 Joe Well according to this when  $t$  is zero [D8] we would have zero rabbits. Or the rate of change would be 0 [C8]. B
- 37 Liz Well actually we're going to multiply it by a half a year [B5, continuation of turn 32]. Cpit
- 38 Deb This is what I did. First I looked at the fact that this is a rate of change equation. So this is telling me how many rabbits are being produced every year [W10]. So If I know 3 times the original population is produced every year, then I have 3 times 10 is produced every year [Criterion 3 met for Csy]. But I want to know how many is produced in 0.5 years [D10]. So I know how many rabbits are produced per year, so if I multiply that by 0.5 then I'll know how many more rabbits have been produced. So I take that new number that I get and add it to the old population [C10] CCpit
- 39 Deb Uh huh, so then I find the one with my new rate of change [W11], so I just take that population and put it in for  $p$  [D11]. And that is 3 times whatever that is [C11]. Bsy
- 40 Liz Do you get what Deb is saying?
- 41 Jeff Yeah you get 25 and then you get 55 (sic) [W11]. Bsy
- 42 Deb I think we should make a chart like he did. [showing her paper to Jeff] But

- this would be your equation. This would be your 0.5, and then rabbits per year, and that will be your new amount of rabbits that's been added, then you add that to your old amount of rabbits, and you'll get your new population [B11].* CCpit
- 43 Jeff *I think you can go  $dp/dt=30$ , actually your  $dt$  will be 0.5, and then you do it again for the next one [C12]. [Criterion 1 met for Csy]* Bsy
- 44 Liz *What do you have right there?*
- 45 Deb *You take your old rate of change which we already know is 30 rabbits per year, and how much time that has passed equals 0.5. So 0.5 times 30 will get me how many new rabbits I have [D13]. So I take the new amount of rabbits I have and add it to the old amount of rabbits I have and that will give me the new population. And once I know the new population I know the new rate of change because I know the rate of change is right here. [C13]* CCpit Crit
- 46 Liz *And the reason for putting in the new population would be what? (identifying a need to build Cpit which Deb has already constructed)*
- 47 Deb *Because now my population is larger and I know the population changes at a constant of 3 times whatever that population is [W13].* CCrit
- 48 Liz *Okay, so basically, I get you up into the point where you say you want to put in, what I understand is that we found our rate of change initially at time zero and that we are using that to find out what our population is after half a year. If we are expected to grow by 30 rabbits in a year then, in a half a year we grow by 15 rabbits. So we'll have 15, [D14].* CCpit
- 49 Deb *No no*
- 50 Liz *I mean 25 because 15 plus 10 is 25 [D14].* CCpit
- 51 Jeff *Then we have to do it again [C14]. [turns 48–51 repeat with specific values Argument 13]* Crit
- 52 Liz *Then you start over again [C14], so its kind of like our new initial population, so we could label it time equals zero if we wanted to [B14].* Crit

Since space constraints prohibit a complete accounting of the individual constructions and normative ways of reasoning evidenced in this episode we only highlight individual constructing actions associated with Cpit, the method for computing the next population value. By recognizing and building-with previous constructs (e.g., turns 1, 4, 7, 9, 20) we see Deb first construct Cpit in turn 24, followed by Jeff in turn 27 and Liz in turns 28+32+37. Per the DCA methodology we see that “knowing P means you can find  $dP/dt$ ” (Csy) functions as if shared at the collective level per Criterion 2 (in turns 24 and 31 this idea was Data whereas in turn 4 it was a Claim). Even in this brief analysis we see how the coordination of the RBC+C and DCA traces well the individual and collective processes in mathematical progress.

The epistemic actions and concomitant discursive shifts resulted in these students reinventing the core idea of Euler's method, namely Cit. One way to express this core idea is in the following algorithm:  $P_{next} = P_{now} + ((dP/dt)|_{now}) * 0.5$ . Indeed, this particular formulation of Euler's method would be a viable extension of students' natural language. In particular, in turns 25–32, three of the four students essentially co-create the first step of the iterative process and then in turn 45 Deb succinctly provides a verbal summary of the algorithm. In turns 51 and 52 Jeff and Liz respectively highlight the iterative nature of the algorithm (“we have to do it again” and “then you start over again”). The use of “next” and “now” in the algorithm closely resembles students' verbal description of the process. This however, is only the first step in developing a comprehensive understanding of Euler's method.

We now further the theoretical and methodological advance for analysing individual and collective mathematical progress that was started in Tabach and colleagues (2014) and Hershkowitz and colleagues (2014). In particular, we use the previous episode to show how the various individual epistemic actions are intertwined with the collective production of arguments. This intertwining reflects the internal commonalities between the RBC+C and DCA methodologies.

### **RBC+C AND DCA INTERNAL COMMONALITIES**

We begin by relating each of the RBC constructs to the DCA approach and then we relate the three criteria of the DCA approach to consolidation.

*Relationship between Recognizing and Data.* Theoretically, we argue that Recognizing actions are largely associated with Data. One uses some piece of information as Data because that piece of information makes sense to him/her. Recognizing action means recognizing a piece of information as relevant as data. Empirically, in the above example, parts of students' talk which were coded as Data were also coded as Recognizing. However in some cases, when a construction takes place, it happens that part of the argument is coded as Data (e.g., turns 21–24). In the previous example, we see that Recognizing actions are primarily associated with Data. In some cases (e.g., turn 1), Recognizing actions can be associated with Warrants, which are at times difficult to disentangle from Data.

*Relationship between Building-with and Warrants.* Theoretically, Warrants establish a connection between data and claim; in order to establish such a connection, one needs to build-with what one has. In the example this commonality is largely the case. Sometimes Building-with is linked to Data, because oftentimes Warrants and data are interchangeable (e.g., turn 36). While the previous excerpt also shows some slight differences in the relationship between Building-with and Warrants (e.g., Building-with may be linked to Claims for which, by Criterion 1, the Data and/or Backing drops off (see turn 43), additional data sets are needed to empirically test the conjecture about the relationship between the two constructs.

*The relationship between Constructing and Arguments as a whole.* Constructing requires vertical mathematization. Constructing actions are more global than Recognizing or Building-with actions; they incorporate sequences of interweaving Recognizing and Building-with actions (plus the glue between them). Similarly, arguments interweave Data-Claims-Warrants and Backings as a whole. Hence, in a line by line coding it is not feasible to indicate the holistic nature of an argument and it is typically indicated after a line by line coding (see for example Tabach et al., 2014). Moreover, arguments are usually co-constructed by several participants over several turns. Such exchanges are similarly typical for constructing actions.

*Consolidating and the three criteria for identifying function-as-if-shared ideas.* In processes of consolidating as well as across the three criteria for identifying when

an idea functions as if shared, there is a repetition, reuse, revisiting, or repurposing of earlier ideas. To clarify, in Criterion 1 there is a repetition, but the repetition is partial in the sense that some parts of the argument (Data, Warrants) cease to be explicitly stated. In Criterion 2 there is repurposing of previous part of an argument (e.g., Claim) as either Data or Warrant. In this sense there is a repeating and reusing, but for a different purpose. In Criterion 3 there is a revisiting of either Data or Warrants to establish new Claims. In consolidation, previous constructs are recognized as relevant (i.e., revisited), and then built-with (i.e. used, possibly repeatedly) for example for solving a problem, reflecting on a situation or result, or even in the framework and for the purpose of an additional constructing action (for example, in lines 19–23, Csy is built-with as part of constructing Cpit).

Further commonalities between consolidating and the three criteria can be seen by considering characteristics of consolidation: awareness, self-evidence, flexibility, immediacy, and confidence (Dreyfus & Tsamir, 2004). Self-evidence links to Criterion 1 – the evidence is the Data, which drops off in subsequent arguments. The subsequent argument also then relates to immediacy and confidence in the validity of the idea. Flexibility links to Criterion 2 – components of an argument are being reused and repurposed (as sign of flexibility) in subsequent arguments. Similarly, Criterion 3 relates to flexibility in a different way. Flexibility lies in the fact that one is able to use an idea (e.g, Build-with) as Data or Warrant for a variety of different Claims. Hence close relationships exists between the criteria and Consolidation characteristics

## CONCLUSION

Students in undergraduate mathematics classrooms are increasingly experiencing inquiry based learning and research is pointing to the strong benefits on student success in terms of grades and subsequent coursework (Freeman et al., 2014; Kogan & Laursen, 2013)<sup>2</sup>. While broad measures of student success are needed, there is also a need for methodologies that provide a fine-grained analysis of the individual and collective processes that make inquiry learning possible, and may have the potential to explain at the micro-level how such learning works and why it is beneficial. This report makes a contribution in this direction. The DCA analysis helps illuminate what is happening on the social or discursive plane, while the

RBC+C analysis helps illuminate what is happening on the cognitive side.

In this report, we used the case of a group of students reinventing Euler's method and we used this case to explicate the environmental, underlying, and internal commonalities between the AiC and DCA approaches. This represents considerable progress toward the call for what Prediger and colleagues (2008) refer to as the local integration of different theoretical/methodological approaches as well as contributing to our understanding of how undergraduate students individually and collectively reinvent important mathematical ideas.

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## ENDNOTE

1. While acknowledging the teacher's crucial role, we did not relate to it here, as this is the next step in our research plan.