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# A model of mathematicians' approach to the defining processes

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*This paper presents a modeling of the defining process used by mathematicians. This modeling has a strong epistemological background and has didactical outcomes. The aim here is to propose tools in view of studying the ways to implement a mathematical activity (close to the mathematicians' one) at the university level.*

**Keywords:** Mathematicians, epistemology, conceptions, definitions, processes.

## INTRODUCTION – ON THE IMPORTANCE OF STUDYING THE DEFINING PROCESSES OF THE MATHEMATICIANS

The study of defining activities is a discreet but constant didactical topic of research in mathematics education since the 90s. Mariotti & Fischbein (1997) have underscored the importance of such research: “(...) learning to define is a basic problem of mathematical education.” (Mariotti & Fischbein, 1997, p. 219). Characterizing the defining processes and the defining activities is a new way, in mathematics education, to explore the mathematical concepts, their learning and their teaching. To place the definitions in the core of the mathematical activity (i.e. an activity that builds new knowledge, brings new proofs and theories), actually reveals an epistemological interest and a didactical interest: besides, the construction of definitions is a component of the research process of the mathematicians. Some researchers in mathematics education have characterized heuristics and behaviors of mathematicians (e.g., Schoenfeld, 1985, Burton, 2004; Carlson & Bloom, 2005; Gardes, 2013), but little research propose a modeling of the defining processes used by the mathematicians. Moreover, the modeling of defining processes in mathematics and the characterization of problems that allow a defining activity should be fruitful from a didactical point of view. Indeed, it brings a new way to analyze the pro-

cesses of constructing new concepts, new proofs and new theories; then, the analysis and the guidance of the mathematical activity of students becomes feasible.

My research, i.e. the modeling of the defining processes in mathematics, fits into a wider field in sciences education that is the inquiry-based learning. The study of the implementation of mathematical and scientific processes in the classrooms (for mathematics, it means every process at stake in a mathematical inquiry of a mathematical problem) is a crucial point for the research in mathematics education. Besides, the question of the closeness between the results of the on-going mathematical research and the contents of university mathematics has to be studied both by mathematics educators and mathematicians.

In this paper, I propose first to synthesize the research in mathematics education that deal with the activity of mathematicians, with a short focus on the defining processes and the defining activities. I have developed in my research (Ouvrier-Bufferet, 2013) a complete modeling of the mathematical defining processes. This model is based upon an epistemological background with a didactical efficiency. In this paper, I have chosen to present the results of interviews with professional mathematicians regarding their defining processes: these interviews bring concrete features of the professional defining activity. They also confirm and enrich the epistemological choices I have made for the design of my model (see Ouvrier-Bufferet, 2006, 2011 and 2013 for the whole epistemological component). The last part of this paper opens new perspectives for university mathematics education.

## STUDYING THE PRACTICE OF MATHEMATICIANS: AN OVERVIEW

In mathematics education, a recent kind of research deals with the practice of mathematicians. The focus

is often made on proof and proving processes, which is legitimated by the fact that proofs are the holder of the mathematical knowledge (Rav, 1999) among other reasons. Hanna & Barbeau (2008) extend this point of view and underscore that the proofs can have other functions such as introducing new methods, tools and strategies to deal with new problems. It fits with Weber's conclusions: Weber (2011) shows that one of the reasons that mathematicians read the proofs of colleagues is to transpose ideas and techniques which may be useful in their own research. Wilkerson-Jerde & Wilensky (2011) analyze the reasoning of mathematicians and students dealing with an unfamiliar proof. They show several processes such as: the use of previous knowledge, the construction of examples, the deconstruction of a concept or an idea into sub-components in order to explore the different components of a concept, the tests and the explorations of definitions, the attempts to connect definitions. As for Shriki (2010), he considers the creativity in the construction of knowledge. He shows that teachers reckon that mathematics can be taught as a reconstruction: the same kind of defining activities (reconstruction of geometrical concepts) as in the research of Larsen & Zandieh (2008) and Zandieh & Rasmussen (2010) is at stake. The place of the study of the construction and the use of examples in mathematics is also important. The focus on the examples that can be produced by students when they try to understand new concepts and when they want to illustrate mathematical ideas and properties, a focus which has been made by Watson, Mason and other colleagues, leads now to the analysis of the construction and the use of examples in proof processes (Watson & Mason, 2002; Sandefur et al., 2013). Recently, Gardes (2013) models the concept of "gesture" and defines it in a new way based upon the contemporary epistemology in order to analyze the practices of mathematicians. This concept appears relevant (for further research) to analyze the processes of mathematicians and of students during a research, and to consider the question of the transposition of the work of mathematicians to the classroom. In fact, in mathematics education, the defining activities are usually evoked during the study of proofs and of problem solving processes, but are not much studied for themselves. Indeed, it is commonly accepted that a proof can imply the necessity of the reconstruction of a definition: in this case, it concerns the exploration of the meanings of a definition (it can lead to the need of a better definition) and/or the study of the consequences of an assumption

(Hanna, 2000). Besides, the way the students learn new concepts can be described with the enrichment of their concept images (Tall, 1991; Vinner, 1991). In fact, in these examples of the emergence of a defining activity, the emphasis is made on a part of the mathematical activity only, and not on the whole process that deals with definitions in the research activity of mathematicians. From an epistemological point of view, the work of Lakatos (1961, 1976) gives a unique example in the literature where the defining process and the proving process interact.

My research (see Ouvrier-Bufferet, 2013 for the outcomes) proposes a reference epistemological modeling of the mathematical defining activity, taking into account the dialectic between defining and proving. Theoretical frameworks from the didactic of mathematics (the model of conceptions (Balacheff, 2013) and mathematics (complexity theory, Garey & Johnson, 1979) are called upon. This modeling is also based upon several experiments (at secondary and university levels) and upon interviews with mathematicians. The first level of my research was to identify emblematic epistemological conceptions which can characterize the mathematical defining processes: I have taken on the Lakatosian, Aristotelician, and Popperian conceptions (see Ouvrier-Bufferet, 2006, 2011). Yet, these conceptions were not enough to describe the defining processes because some cognitive aspects, for instance, were not taken into account (the didactical experiments have shown this aspect). Besides, the characterization of problems which can lead to a defining activity had not been carried out. Moreover, it was difficult to understand how the epistemological conceptions coexist and interact. Then, I have enriched the epistemological component and I have conducted interviews with mathematicians to propose a complete overview of the defining processes. I have then design four main components to define my modeling of defining activities and processes: the characterization of three main epistemological conceptions regarding defining processes (the Lakatosian, Aristotelician, and Popperian conceptions; Ouvrier-Bufferet, 2013, p. 67); the definition of problems that can lead to a defining activity (Ouvrier-Bufferet, 2013, p. 72); the emphasis of four moments of work involving definitions (this part highlights the role and the place of the epistemological conceptions and the cognitive aspects; Ouvrier-Bufferet, 2013, p. 69); and a didactical methodology to build, to analyze and to guide defining

processes in classroom situations (Ouvrier-Bufferet, 2013, p. 76).

## DEFINING PROCESSES OF MATHEMATICIANS

### Research questions and methodology

The aim of the interviews with mathematicians was twofold: firstly, I wanted to enrich my model of the defining activity based upon three epistemological conceptions, and intrinsically to validate it. Secondly, I was searching for a way to complete the aforementioned conceptions and to grasp the whole defining activity of mathematicians. Therefore, my underlying research questions were: how do the conceptions interact? Can one identify different moments when the mathematicians work on definitions, and are the epistemological conceptions operational? What are the types of definitions which the mathematicians use?

Regarding my previous epistemological research, I have defined *six lines* to analyze the data. They deal with the different kinds of definitions which the mathematicians talked about; the features of their defining processes (a focus is made on the interplay between defining and proving); the reasons of the evolution of a definition during a research; the ways the mathematicians valid a definition; the view they have on the defining activity in university mathematics classes; the identification of the moments when the mathematicians work on definitions, and then of their moments of work dealing with defining processes. These features and the elements of the epistemological conceptions gave me a grid with several components to analyze the interviews. I have also focused on the actions described by the mathematicians in order *to connect them to the epistemological conceptions*. In this article, the aim is to provide concrete examples to give an overview of a professional defining activity (following the six lines above-mentioned) and to highlight the characterization of four moments of the work on definitions in particular.

### Data for this research

Eight professional mathematicians participated in this study. They come from different French universities and different fields of mathematics. Semi-structured interviews (1 hour or less long) were audiotaped and then transcribed. The questions were oriented towards four elements: their professional profiles, their practices of mathematics, their practices of the definitions and of the interplay between defi-

nitions and proofs, their representations of teaching at university level (do they think that implementing defining activities at university level can be useful and relevant?). The description of these interviews is available in Ouvrier-Bufferet (2013).

### Several kinds of definitions

The interviewed researchers actually identify several kinds of definitions:

- the discrimination between the definitions one knows beforehand and the definitions one can deduce from other results;
- the distinction between the definitions which remain and will belong to the public domain and the local definitions which are used to shorten a talk.
- the working definitions: one starts from the intuition that one gets from objects and problems. With this kind of definitions, one can work with the mathematical objects, and the statement of these definitions can be put off.

These kinds of definitions are linked to different aspects of mathematics and then several moments of the mathematical activity already appear: the moment when the intuition of objects and problems catalyses a research; the moment when working definitions (which can be local ones) give legitimacy to a new object which becomes worthy of interest; and the moment when the theoretical, formal and logical definitions are at stake (here the axiomatic theory is concerned).

### The defining process and its interplay with proof

It is clear that the formalized definitions come “after” during a constructing process. The processes involved in the construction of a theory are not really dealt with by the mathematicians. By “defining process”, they circumscribe a heuristic domain, from an intuitive exploration of mathematical objects and problem to the validation of a result. The insight of the results is often present. The defining processes specific to the construction of a new theory with axiomatic, logical and linguistic constraints are not much developed. For the mathematicians, to define is motivated by different needs: to have a better understanding of a concept or a problem, to simplify,

to generalize, to explore different linked frames or connected fields than the first one, to communicate. The mathematicians consider that the proof is the master and that the proof can imply an evolution of the definitions. When they define or re-define objects during a proof, it is first to continue the research process, then to determine the domain of applicability of an idea or a proof, and finally to study more general cases or more particular cases. The definition of a really new concept (i.e. without an insight of it during a first exploration of a problem and/or a proof) can emerge during a proof. The proof process can have a local or a global impact: the significance of the new built concept will be proved later. Therefore it implies a long-term study of this new concept.

### **The reasons of the evolution of a definition**

The “communication” dimension is important for the mathematicians: they explain that a definition will evolve when they will communicate their results. It can be in different institutions (seminary, talk, publications, university textbook) and the context will lead to one definition in particular or another. The mathematicians underscore the difficult use of the examples and the counter-examples in research. Some mathematical fields, where the concrete and discrete dimensions are involved (such as discrete mathematics), seem to be more suitable for building and using examples and counter-examples. Some defining problems (i.e. problems that can lead to a defining process) are pointed out by the mathematicians and are shortly described through research questions such as: to search if the problem can become a more general one, to search analogies with linked mathematical fields or with other concepts including a similar structure, to define the dual concept (it also implies the construction of new problems).

### **The validation of a definition**

It occurs in different levels and places. The validation (i.e. when one considers that a definition is “correct”) can be an individual and personal process (it is a self-validation). The colleagues play the part of the validation too, as well as the mathematical community. The fact that the proof works is also an element for the validation of a definition. In the same way, when a conjecture is “almost proved”, or when there are no more counter-examples, or when the validity of a statement is controlled by the use of examples, the

mathematicians consider that a definition can be valid. And a definition can also be agreed upon when it has a good strength during mathematical natural transformations or the implementation in different mathematical frames or structures. This being said, a mathematician indicates that for some cases, one cannot bring a validation, in particular when one does not know if the research process is ending (this is consistent with the Lakatosian view). These elements go in the same directions as Weber's results (2008) regarding the validation of a proof by the mathematicians.

## **A MODEL OF HOW CONTEMPORARY MATHEMATICIANS DEFINE**

### **A basis: Three epistemological conceptions**

I have previously presented an epistemological framework taking into account several conceptions: the Aristotelian one, the Popperian one, and the Lakatosian one (Ouvrier-Bufferet, 2006, 2011). I have shown the ability of students to build definitions and to make a working definition evolve with the lack of counterexamples and with the reinvestment in a proof. I have also shown that the three aforesaid conceptions are useful to describe the students' defining processes, but not enough to grasp the whole processes, in particular the intuition and the in-action processes. I have then reintegrated the in-action dimension (this cognitive feature was missing in the model) with the in-action definitions and extended the modeling (Ouvrier-Bufferet, 2013).

### **Mathematicians dealing with defining processes: Four moments of work**

Four moments of work have been characterized during the interviews with the mathematicians. These moments do not describe a linear activity, but they are connected: they give a dynamic overall view of the defining activity in the mathematical research which also integrates the epistemological conceptions and underscores the different kinds of definitions. The names that I have chosen for these four moments of work are directly connected to the different kinds of definitions which exist in the speeches of the interviewed mathematicians.

### **The “in-action” moment of work**

This moment of work deals with the intuition of mathematical objects, ideas, and results. The mathematical activity is here mainly an exploration and an impregnation of one or several problems and

of objects in order to know them better (the use of examples, non-examples, counter-examples is here at stake). Analogies with other close mathematical fields can be used and new weak problems are stated. The Lakatosian conception is operational, with several operators: the statements of problems, the construction of examples and counter-examples, and the change of mathematical framework. In this “in-action” moment of work appear “in-action definitions” (Ouvrier-Bufferet, 2011) and concept images (Vinner, 1991). An “in-action definition” is a statement used as a tool (not an object) that enables students to be operational without an explicit definition.

### **A transitional moment of work between “in-action” and “zero” – A potential link with the “axiomatic” moment of work**

Two processes characterize this moment of work: to construct a first classification of mathematical objects and to re-use existing classifications; and to try to use analogies with existing concepts and theories. Here, the classifying, the categorizing activities and the denomination process of objects constitute the defining process. The Aristotelician and Lakatosian conceptions can be mobilized. If a link is made with the “axiomatic” moment of work, the Popperian conception can be used.

### **The “zero” moment of work**

This is the place of the “zero-definitions” (Lakatos, 1961; Ouvrier-Bufferet, 2011, 2013), but also of definitions that have a local impact. A “zero-definition” marks the beginning of the research process. A zero-definition can be modified in order to protect a primitive conjecture from a “monster” or because the concept is altered by the presentation of a proof. The Lakatosian operators (to use and to build examples and counter-examples, to use the method of monster-barring for instance) and other processes can be mobilized such as: to do false things, to reach an idea of the proof (the proof constraints the concepts and their definitions, quoting the mathematicians). Then, the zero-definitions and other local definitions have different functions: to denominate, to bring up several ways to grasp a concept, to work on a proof, to delimit the range of use of an idea or of a conjecture or of a proof, to communicate. The Lakatosian operator “to use another mathematical framework” can also be used and a link with the “axiomatic” moment of work made (in particular when a local or global theory pre-exists).

### **The “formalized” moment of work**

I would like to underscore the “communicational” aspect, which appears during both a heuristic research, and a need of formalization, then I use the term “formalized” for this moment of work of mathematicians. Mathematicians can have to communicate (local) results during seminary, prepublications, talks etc. or to write a more formalized paper. In these both cases, there is a gap – an abstraction jump – compare to the “zero” moment of work. During the “formalized” moment of work, the mathematical activity concerns the use of some Lakatosian controls such as: the end of counter-examples implies that definitions, conjectures and/or proofs are solid. Here, “proof-generated definitions” can emerge. The proof and the proof-generated definition work together. A proof-generated definition originates from a proof while stemming from the development of the potential of a zero-definition. The catalysis of a proof-generated definition is impossible without the proof idea. Other operators take part in this moment of work, such as Popperian ones dealing with the construction of local (or even global) axiomatic theories. The writing of formalized and successfully completed definitions can be partially described with the Aristotelician conception. This moment interact with the concept definition (Vinner, 1991). Besides, the statement of new problems (a Lakatosian operator) allows the continuation of the mathematical research. The understanding of the new built concepts, and the generalization and the use of definitions, problems and results can lead new questionings. The exploration of neighboring concepts leads to a new “in-action” moment of work.

### **The “axiomatic” moment of work**

The construction of a new theory (which can be momentarily local) and the construction of new concepts inscribed in this theory are at stake in this moment of work. I choose to call the definitions which are built during this moment “theoretical definitions” in order to underscore that there are inscribed in a theory. The Popperian conception is here clearly useful to characterize the research process of the “axiomatic” moment of work. In particular, the construction of the involved mathematical theory implies the search of the minimal number of rules (axioms) in order to generate results with a wide range. The axiomatic process can also unify concepts (see, for instance, the case of linear algebra, Dorier et al., 2000). The transposition of concepts to other mathematical fields brings opening questions for the research too.

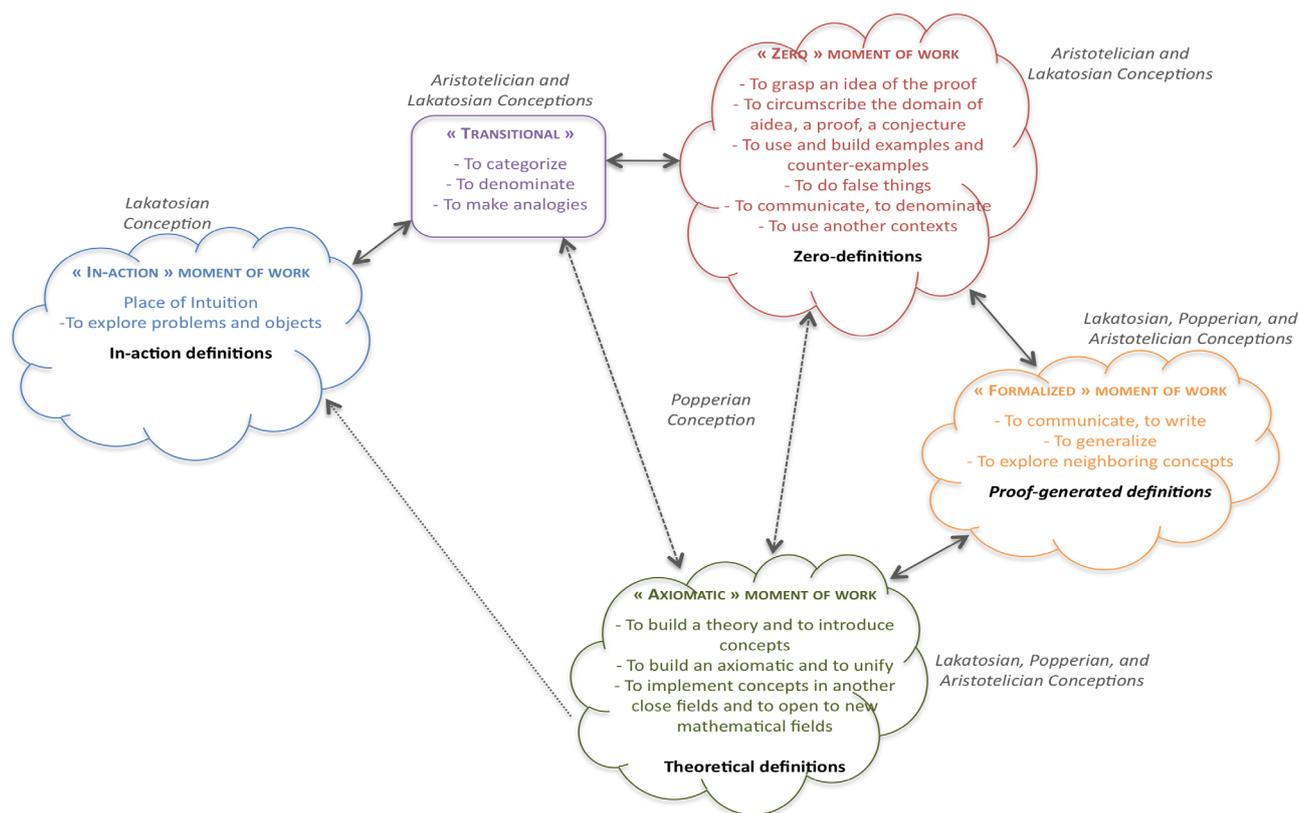


Figure 1: Overview of the defining process in mathematical research

**The epistemological conceptions are operational in the moments of work**

In the different moments of work on the definitions, as well as in the transition from one moment to another, the mathematicians mobilize one or more conceptions (mainly, they express in words different elements which fit with operators and control structures of the epistemological conceptions). Then, Figure 1 synthesizes the features of each moment (in the coloured clouds), and shows the places where the epistemological conceptions are operational (in a cloud or in a transition between two clouds). The arrows mark the transition between two moments and sometimes the efficiency of a conception in such a transition. According to the interviewed researchers, one researcher cannot deal with all these moments, except for brilliant mathematicians. We can continue this research with new interviews, taking into account the different fields of mathematics.

**OPENINGS**

Ouvrier-Bufferet (2011) has shown how a mathematical experience with a problem involving a defining activity can be conducted at university level. Such experiments also bring opportunities to reinvest the constructed concepts and the ways of reasoning in

other mathematical fields. Then the question of the in-service teacher education (of university mathematics teachers) becomes crucial, as well as the definition of the contents of the university mathematics i.e. the concepts and the processes involving a real mathematical activity (and not only proof). There are clearly needs to engage collaborative research between university mathematics teachers and researchers in mathematics education (following Nardi, 2008). The interviewed mathematicians think suitable to implement defining activities at the university level, but they cannot conceptualize the way it can be implemented with students. They are very interested in the didactical research that can lead to new situations for the university. One can also extend this idea to sciences, especially with the study of the inquiry-based learning.

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