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Olivier Pierre

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ANALYTIC CURRENT-VORTEX SHEETS IN INCOMPRESSIBLE MAGNETOHYDRODYNAMICS

OLIVIER PIERRE

March 15, 2016

ABSTRACT. In this paper, we address the problem of current-vortex sheets in ideal incompressible magnetohydrodynamics. More precisely, we prove a local-in-time existence and uniqueness result for analytic initial data using a Cauchy-Kowalevskaya theorem.

1. INTRODUCTION

1.1. Motivation. In this article, we are interested in a free boundary problem arising in magnetohydrodynamics (MHD), namely the current-vortex sheet problem. We consider a homogeneous plasma (the density is constant), assumed to be perfectly conducting, inviscid and incompressible. The equations of ideal incompressible MHD thus read:

$$\begin{cases} \partial_t u + \nabla \cdot (u \otimes u - H \otimes H) + \nabla q = 0, \\ \partial_t H - \nabla \times (u \times H) = 0, \\ \nabla \cdot u = \nabla \cdot H = 0, \end{cases} \quad (1.1)$$

where $u \in \mathbb{R}^3$ and $H \in \mathbb{R}^3$ stand for the velocity and the magnetic field of the plasma respectively. The unknown q defined by $q := p + \frac{|H|^2}{2}$ is called the “total pressure”, p being the physical pressure.

We are looking for a special class of (weak) solutions of (1.1): we want (u, H, q) to be smooth on either side of a hypersurface $\Gamma(t)$ (t is the time-variable), and to give rise to a *tangential* discontinuity across $\Gamma(t)$. We shall assume that the hypersurface $\Gamma(t)$ can be parametrized by $\Gamma(t) := \{x_3 = f(t, x')\}$, where x' is the tangential variable living in the 2-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, and x_3 denotes the normal variable. The unknown f is called the “front” of the discontinuity later on. For all $t \in [0, T]$, we shall consider the MHD system (1.1) in the time-dependent domain

$$\Omega(t) := \Omega^+(t) \sqcup \Omega^-(t), \quad \text{where} \quad \Omega^\pm(t) := \{x_3 \gtrless f(t, x')\},$$

with appropriate boundary conditions on $\Gamma(t)$:

$$\partial_t f = u^\pm \cdot N, \quad H^\pm \cdot N = 0, \quad [q] = 0, \quad \forall t \in (0, T). \quad (1.2)$$

The notation $[q]$ in (1.2) stands for the jump of q across $\Gamma(t)$:

$$[q] := q^+|_{\Gamma(t)} - q^-|_{\Gamma(t)},$$

and N is a normal vector to $\Gamma(t)$, chosen as follows:

$$N := (-\partial_1 f, -\partial_2 f, 1).$$

The boundary conditions (1.2) correspond to a tangential discontinuity: the velocity $\partial_t f$ of the front is given by the normal component of u , the normal magnetic field $H \cdot N$ as well as the total pressure q are continuous across $\Gamma(t)$, and $H \cdot N|_{\Gamma(t)} = 0$ on either side of $\Gamma(t)$. We refer to [BT02] for other types of tangential discontinuities in MHD. For simplicity, we assume that the normal variable x_3 belongs to $(-1, 1)$. Thus we shall assume that for all t and x' we have $-1 < f(t, x') < 1$, and impose an additional condition on the “exterior” boundaries $\Gamma_\pm := \{(x', \pm 1), x' \in \mathbb{T}^2\}$. The system of current-vortex sheets eventually reads as follows:

$$\begin{cases} \partial_t u^\pm + (u^\pm \cdot \nabla)u^\pm - (H^\pm \cdot \nabla)H^\pm + \nabla q^\pm = 0 & \text{in } \Omega^\pm(t), \quad t \in (0, T), \\ \partial_t H^\pm + (u^\pm \cdot \nabla)H^\pm - (H^\pm \cdot \nabla)u^\pm = 0 & \text{in } \Omega^\pm(t), \quad t \in (0, T), \\ \nabla \cdot u^\pm(t) = \nabla \cdot H^\pm(t) = 0 & \text{in } \Omega^\pm(t), \quad t \in (0, T), \\ \partial_t f = u^\pm \cdot N, \quad H^\pm \cdot N = 0, \quad [q] = 0 & \text{on } \Gamma(t), \quad t \in (0, T), \\ u_3^\pm = H_3^\pm = 0 & \text{on } \Gamma_\pm. \end{cases} \quad (1.3)$$

The superscript \pm denotes the unknown u , H and q restricted to $\Omega^\pm(t)$.

Remark: In order to solve problem (1.3), a common procedure will be to reduce this free boundary problem into the fixed domains $\Omega^+ := \mathbb{T}^2 \times (0, 1)$ and $\Omega^- := \mathbb{T}^2 \times (-1, 0)$, using a suitable change of variables (see Paragraph 1.3 below). Consequently, the reference domain we shall consider is $\Omega := \mathbb{T}^2 \times (-1, 1)$. Besides, the (fixed) boundaries will be noted as follows:

$$\Gamma := \mathbb{T}^2 \times \{x_3 = 0\}, \quad \text{and} \quad \Gamma_{\pm} := \mathbb{T}^2 \times \{x_3 = \pm 1\}. \quad (1.4)$$

System (1.3) is also supplemented with initial data $(u_0^{\pm}, H_0^{\pm}, f_0)$ satisfying the constraints $\nabla \cdot u_0^{\pm} = \nabla \cdot H_0^{\pm} = 0$ in $\Omega^{\pm}(0)$.

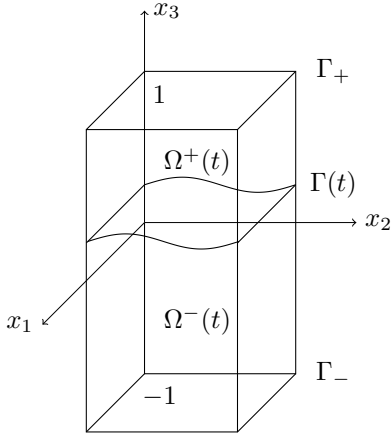


FIGURE 1. The time-dependent domains $\Omega^+(t)$ and $\Omega^-(t)$, and the sheet $\Gamma(t)$.

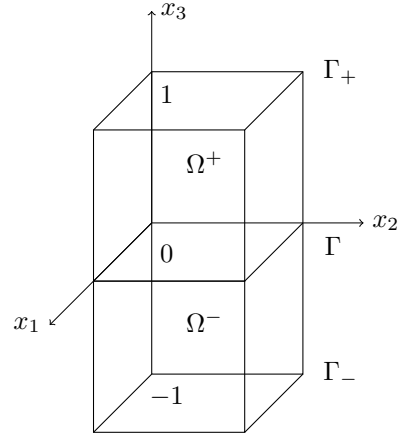


FIGURE 2. The fixed domains Ω^+ and Ω^- , and the fixed interface Γ .

1.2. Background. Current-vortex sheet in ideal incompressible MHD has been a well-known free boundary problem since the 1950's. It has been addressed for instance by Syrovatskiĭ [Syr54], Axford [Axf60] or Chandrasekhar [Cha61]. These references in particular deal with the (linear) stability of *planar* current-vortex sheets using the so-called normal modes analysis. The linear stability criterion derived in these references reads as follows:

$$|[u]|^2 \leq 2(|H^+|^2 + |H^-|^2), \quad (1.5a)$$

$$|[u] \times H^-|^2 + |[u] \times H^+|^2 \leq 2|H^+ \times H^-|^2. \quad (1.5b)$$

In (1.5), $[\cdot]$ denotes the jump across the planar sheet $\{x_3 = 0\}$. Besides, if we assume that $H^+ \times H^- \neq 0$ and

$$|[u] \times H^-|^2 + |[u] \times H^+|^2 < 2|H^+ \times H^-|^2, \quad (1.6)$$

then (1.5a) follows from (1.5b) with a strict inequality. However, if (1.5) is not satisfied, then it leads to the so-called *Kelvin-Helmholtz instabilities*. For instance, we refer to [Cha61], [Tra05], [CMST12, p.251-252] and references therein for a more detailed discussion about this stability condition. The latter turns out to be a *weak* linear stability condition, *i.e.* the *uniform Kreiss-Lopatinskiĭ condition* is not satisfied, leading to *weakly nonlinear surface waves* [AH03]. Without the magnetic field, system (1.3) is reduced to the *vortex sheet problem* for the Euler equations. The tangential discontinuity given by (1.2) in this case gives rise to the well-known Kelvin-Helmholtz instabilities in hydrodynamics. This problem is well-posed only in the *analytic* scale. We refer for example to [Cha61], [SSBF81], [Del91] and [Leb02] for more details. Nevertheless, when surface tension is involved, the vortex-sheet problem is well-posed in the *Sobolev* scale (see e.g. [AM07]).

Under a more restrictive stability condition, namely

$$\max(|[u] \times H^+|, |[u] \times H^-|) < |H^+ \times H^-|, \quad (1.7)$$

Coulombel, Morando, Secchi and Trebeschi proved in [CMST12] an *a priori* estimate without loss of derivatives for the nonlinear problem (1.3), based on a symmetry argument introduced by Trakhinin [Tra05]. Their approach thus gives some hope to state an existence and uniqueness result for system (1.3), without using a Nash-Moser iteration.

In that spirit, the aim of this article is to construct *analytic solutions* to (1.3) using a Cauchy-Kowalevskaya theorem. In other words, we shall use a *fixed point argument* in an appropriate scale of Banach spaces. In a future work, we plan to use the main result of [CMST12] to exhibit solutions of fixed *Sobolev regularity* to (1.3) *via* a compactness argument, by approximating Sobolev data by analytic data.

On the other hand, very recently Sun, Wang and Zhang [SWZ15] have used a completely different approach to solve the incompressible current-vortex sheet problem. The main idea of their proof is to reduce the whole problem on the free surface $\Gamma(t)$ only, as is common in water waves theory for instance [Lan13]. Then, using some elliptic arguments, they can reconstruct the complete solution inside both domains $\Omega^\pm(t)$. The advantage of their method is that they can prove a local-in-time existence and uniqueness theorem in the *whole* domain of stability given by (1.6), using an *elliptic problem* satisfied by the total pressure q^\pm in (1.3). Therefore the arguments in [SWZ15] require the velocity to be divergence-free. On the opposite, the method we advocate in this paper is based on *a priori* estimates only and could therefore apply to other hyperbolic systems. For instance, as a future work, we might avoid using the Nash-Moser theorem used by Trakhinin [Tra09] and by Chen and Wang [CW08] to prove local-in-time existence and uniqueness of *compressible* current-vortex sheets.

1.3. Reformulation on a fixed domain, lifting of the front f . We begin by recalling Lemma 1 of [CMST12], which shows how to lift the front f defined on $(0, T) \times \Gamma$ into a function ψ defined on $(0, T) \times \Omega$, in order to gain half a derivative in the Sobolev regularity scale. The strategy is inspired from Lannes [Lan05].

Lemma 1.1. *Let $r \in \mathbb{N}$ such that $r \geq 2$. Then there exists a continuous linear map*

$$f \in H^{r-\frac{1}{2}}(\mathbb{T}^2) \mapsto \psi \in H^r(\Omega)$$

such that, for all $x' \in \mathbb{T}^2$, we have:

$$\psi(x', 0) = f(x'), \quad \psi(x', \pm 1) = 0, \quad \partial_3 \psi(x', 0) = 0.$$

Lemma 1.1 allows to define the map $\Psi : (t, x) \mapsto (x', x_3 + \psi(t, x))$, where $\psi(t, \cdot)$ is given by the previous lemma applied to the function $f(t, \cdot)$. According to [CMST12], if we impose a smallness condition to the front $f(t, \cdot)$ in the space $H^{2.5}(\mathbb{T}^2)$, then $\Psi(t, \cdot)$ is an H^r -diffeomorphism of Ω . Indeed, the jacobian $1 + \partial_3 \psi$ of the change of variables satisfies for example $1 + \partial_3 \psi \geq \frac{1}{2}$. We will let A denote the inverse of the jacobian matrix of Ψ , and J the jacobian of $D\Psi$:

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{\partial_1 \psi}{J} & -\frac{\partial_2 \psi}{J} & \frac{1}{J} \end{pmatrix} \quad \text{and} \quad J := 1 + \partial_3 \psi. \quad (1.8)$$

We also define, for x in the *fixed domains* Ω^\pm , the new following unknowns:

$$v^\pm(t, x) := u^\pm(t, \Psi(t, x)), \quad B^\pm(t, x) := H^\pm(t, \Psi(t, x)), \quad Q^\pm(t, x) := q^\pm(t, \Psi(t, x)).$$

With these new unknowns, system (1.3) can be rewritten as follows:

$$\left\{ \begin{array}{ll} \partial_t v^\pm + (\tilde{v}^\pm \cdot \nabla) v^\pm - (\tilde{B}^\pm \cdot \nabla) B^\pm + A^T \nabla Q^\pm = 0 & \text{in } (0, T) \times \Omega^\pm, \\ \partial_t B^\pm + (\tilde{v}^\pm \cdot \nabla) B^\pm - (\tilde{B}^\pm \cdot \nabla) v^\pm = 0 & \text{in } (0, T) \times \Omega^\pm, \\ (A^T \nabla) \cdot v^\pm = (A^T \nabla) \cdot B^\pm = 0 & \text{in } (0, T) \times \Omega^\pm, \\ \partial_t f = v^\pm \cdot N, \quad B^\pm \cdot N = 0, \quad [Q] = 0 & \text{on } (0, T) \times \Gamma, \\ v_3^\pm = B_3^\pm = 0 & \text{on } \Gamma_\pm, \end{array} \right. \quad (1.9)$$

where we have set:

$$N := (-\partial_1 \psi, -\partial_2 \psi, 1), \quad \tilde{v}^\pm := \left(v_1^\pm, v_2^\pm, \frac{v^\pm \cdot N - \partial_t \psi}{J} \right), \quad \tilde{B}^\pm := \left(B_1^\pm, B_2^\pm, \frac{B^\pm \cdot N}{J} \right). \quad (1.10)$$

Here, the vector N is defined on the whole domain Ω , and not only on the interface Γ . From now on, the notation $[\cdot]$ stands for the jump across the **fixed** interface Γ :

$$[Q] := Q^+|_\Gamma - Q^-|_\Gamma.$$

The aim of this paper is to solve system (1.9) for *analytic* initial data (v_0^\pm, B_0^\pm, f_0) , using a Cauchy-Kowalevskaya theorem. Therefore, in the next paragraph we define spaces of analytic functions, whose definition relies on the usual Sobolev spaces H^r .

1.4. Scales of analytic Banach spaces. First of all, let us recall the basic properties of Sobolev spaces that we shall use later on in order to define some analytic spaces. We refer for instance to [BGS07], [Bre83], [Eva98], [Tri10] or [Zui02].

Periodic Sobolev spaces. Let us denote the 1-periodic torus by $\mathbb{T} := \mathbb{R}/\mathbb{Z}$. For $s \in \mathbb{R}_+$, we define the d -dimensional periodic Sobolev space by (see e.g. [Tri10]):

$$H^s(\mathbb{T}^d) := \left\{ u \in L^2(\mathbb{T}^d) \mid \sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^s |c_n(u)|^2 < +\infty \right\}, \quad (1.11)$$

equipped with the norm $\|\cdot\|_{H^s(\mathbb{T}^d)}$ defined by

$$\|u\|_{H^s(\mathbb{T}^d)}^2 := \sum_{n \in \mathbb{Z}^d} (1 + |n|^2)^s |c_n(u)|^2. \quad (1.12)$$

For $n \in \mathbb{Z}^d$, the quantity $c_n(u)$ corresponds to the n^{th} Fourier coefficient of u . Therefore, we have a convenient characterization of the spaces $H^s(\mathbb{T}^d)$ by means of the (discrete) Fourier transform. The Sobolev space $H^s(\mathbb{T}^d)$ is a Hilbert space.

Sobolev spaces on a bounded domain of \mathbb{R}^3 . Let us now consider Ω , a bounded and smooth domain of \mathbb{R}^3 . For all $r \in \mathbb{N}$, the Sobolev space $H^r(\Omega)$ we shall use in this article is defined by:

$$H^r(\Omega) := \left\{ u \in L^2(\Omega) \mid \forall \alpha \in \mathbb{N}^3, |\alpha| \leq r, \partial^\alpha u \in L^2(\Omega) \right\}, \quad (1.13)$$

equipped with the norm $\|\cdot\|_{H^r(\Omega)}$ defined by

$$\|u\|_{H^r(\Omega)}^2 := \sum_{|\alpha| \leq r} \|\partial^\alpha u\|_{L^2(\Omega)}^2. \quad (1.14)$$

The space $H^r(\Omega)$ is a Hilbert space.

A particular case: the Sobolev space $H^r(\mathbb{T}^2 \times (0, 1))$. The equations of magnetohydrodynamics we study are set on a domain of the form $\mathbb{T}^2 \times (0, 1)$, i.e. a horizontal strip $\{0 \leq x_3 \leq 1\}$, with periodic boundary conditions with respect to the tangential variable $x' := (x_1, x_2) \in \mathbb{T}^2$. Consequently, the functional framework used to establish a

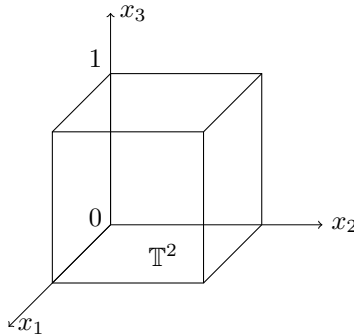


FIGURE 3. Periodic strip $\mathbb{T}^2 \times (0, 1)$.

a priori estimates in [CMST12] is the Sobolev space $H^r(\mathbb{T}^2 \times (0, 1))$, with $r \in \mathbb{N}$.

Using Fubini's theorem, we can write, for $u \in H^r(\mathbb{T}^2 \times (0, 1))$,

$$\|u\|_{H^r(\mathbb{T}^2 \times (0, 1))}^2 := \sum_{|\alpha| \leq r} \|\partial^\alpha u\|_{L^2(\mathbb{T}^2 \times (0, 1))}^2 = \sum_{k=0}^r \left\| \partial_3^k u \right\|_{L_{x_3}^2(H^{r-k}(\mathbb{T}^2))}^2. \quad (1.15)$$

As a consequence, the space $H^r(\mathbb{T}^2 \times (0, 1))$ also takes the form

$$H^r(\mathbb{T}^2 \times (0, 1)) = \bigcap_{k=0}^r H_{x_3}^k(H^{r-k}(\mathbb{T}^2)).$$

From a computational point a view, the norm given by (1.15) turns out to be useful. Indeed, for $k \in \{0, \dots, r\}$, we shall compute the quantities $\|\partial_3^k u\|_{L_{x_3}^2(H^{r-k}(\mathbb{T}^2))}$ thanks to Fourier series.

We eventually give a useful tame estimate for composite functions, in a very particular case (see e.g. [Lan13, p.283]).

Proposition 1.2. *Let $r \in \mathbb{N}$ and Ω be a smooth bounded domain of \mathbb{R}^d . We consider $u \in H^r(\Omega)$ such that:*

$$\exists c_0 > 0, \quad \forall x \in \Omega, \quad c_0^{-1} \leq 1 + u(x) \leq c_0.$$

Then the function $\frac{1}{1+u}$ belongs to $H^r(\Omega)$ and satisfies the estimate:

$$\left\| \frac{1}{1+u} \right\|_{H^r(\Omega)} \leq C_{r,c_0} (1 + \|u\|_{H^r(\Omega)}), \quad (1.16)$$

where $C_{r,c_0} > 0$ depends only on r , c_0 and Ω .

The fact we can estimate the inverse of a given function turns out to be useful when, for instance, we perform a change of variables. More precisely, some computations of derivatives can make the jacobian of this change of variables appear, as well as **its inverse**.

Now that we have recalled the main properties of Sobolev spaces, we are able to construct some spaces of analytic functions. Their definition relies on the spaces $H^r(\Omega)$.

Analytic spaces on $\mathbb{T}^2 \times (-1, 1)$. From now on, Ω denotes the reference domain $\mathbb{T}^2 \times (-1, 1)$. The norm of the Sobolev space $H^r(\Omega)$, for $r \in \mathbb{N}$, is simply noted as $\|\cdot\|_{H^r}$. We also consider a parameter $\rho_0 > 0$, fixed once for all. The following definitions and properties of the functional spaces below are inspired from [Sed94].

Definition 1.3. *Let $(r, k) \in \mathbb{N}^2$ and $\rho \in (0, \rho_0]$. We define the space $B_{\rho,r}^k$ by*

$$B_{\rho,r}^k := \left\{ u \in H_{x_3}^{k+r}(H^{+\infty}(\mathbb{T}^2)) \mid \sum_{n \geq 0} \frac{\rho^n}{n!} \max_{\substack{|\alpha|=n \\ \alpha_3 \leq k}} \|\partial^\alpha u\|_{H^r} < +\infty \right\},$$

where α will always denote a multi-index of \mathbb{N}^3 and $|\alpha|$ its length. We equip this space with the norm $\|\cdot\|_{\rho,r}^k$ given by

$$\|u\|_{\rho,r}^k := \sum_{n \geq 0} \frac{\rho^n}{n!} \max_{\substack{|\alpha|=n \\ \alpha_3 \leq k}} \|\partial^\alpha u\|_{H^r}.$$

Proposition 1.4. *For (r, k) fixed, the sequence $(B_{\rho,r}^k)_{0 < \rho \leq \rho_0}$ is a scale of Banach spaces, in other words:*

$$\forall 0 < \rho' \leq \rho \leq \rho_0, \quad B_{\rho',r}^k \supset B_{\rho,r}^k, \quad \text{with} \quad \|\cdot\|_{\rho',r}^k \leq \|\cdot\|_{\rho,r}^k.$$

We refer to [Sed94] for the proof. In order to apply a Cauchy-Kowalevskaya theorem, we will need algebra and differentiation properties in the spaces $B_{\rho,r}^k$. Both following propositions give such properties.

Proposition 1.5. *Let $r \in \mathbb{N}$ such that $r \geq 2$, $u, v \in B_{\rho,r}^k$ and $N \in \mathbb{N}$. Then $uv \in B_{\rho,r}^k$ and we have the estimates*

$$\|uv\|_{\rho,r}^{k,N} \leq C \|u\|_{\rho,r}^{k,N} \|v\|_{\rho,r}^{k,N}, \quad (1.17)$$

$$\|uv\|_{\rho,r}^{k,N} \leq C \sum_{j=0}^k \|u\|_{\rho,r}^{j,N} \|v\|_{\rho,r}^{k-j,N}, \quad (1.18)$$

where $C > 0$ depends only on r , and $\|\cdot\|_{\rho,r}^{k,N}$ denotes the partial sum of order N of $\|\cdot\|_{\rho,r}^k$. Furthermore, taking the supremum over N , we get the following algebra properties:

$$\|uv\|_{\rho,r}^k \leq C \|u\|_{\rho,r}^k \|v\|_{\rho,r}^k, \quad (1.19)$$

$$\|uv\|_{\rho,r}^k \leq C \sum_{j=0}^k \|u\|_{\rho,r}^j \|v\|_{\rho,r}^{k-j}, \quad (1.20)$$

where $C > 0$ depends only on r .

Estimate (1.19) gives the algebra property of the spaces $B_{\rho,r}^k$, whereas inequality (1.20) will turn out to be more convenient in what follows, because of the appearing Cauchy product. Once again, we refer to [Sed94] for the details. The next proposition, also adapted from [Sed94], shows how the differentiation behaves in the scale $(B_{\rho,r}^k)_{\rho \leq \rho_0}$.

Proposition 1.6. *Let $u \in B_{\rho,r}^k$. Then for all $\rho' < \rho$, u satisfies $\partial_1 u, \partial_2 u \in B_{\rho',r}^k$ with*

$$\|\partial_j u\|_{\rho',r}^k \leq \frac{1}{\rho - \rho'} \|u\|_{\rho,r}^k, \quad j = 1, 2. \quad (1.21)$$

Besides, if $k \geq 1$, then $\partial_3 u \in B_{\rho',r}^{k-1}$ with

$$\|\partial_3 u\|_{\rho',r}^{k-1} \leq \frac{1}{\rho - \rho'} \|u\|_{\rho,r}^k. \quad (1.22)$$

We can also express the loss of derivative thanks to the index r . If $r \geq 1$, then for $j = 1, 2, 3$, we have $\partial_j u \in B_{\rho,r-1}^k$ and the estimate

$$\|\partial_j u\|_{\rho,r-1}^k \leq \|u\|_{\rho,r}^k. \quad (1.23)$$

To take all the normal derivatives $\partial_3^{\alpha_3}$ (with $\alpha_3 \in \mathbb{N}$) into account, we now introduce a new scale of Banach spaces. Their construction is based on the previous Banach spaces $B_{\rho,r}^k$.

Definition 1.7. *Let $r \in \mathbb{N}$, $\sigma > 0$ (small parameter to fix later on) and $\rho \in (0, \rho_0]$. We define the space $B_{\rho,r,\sigma}$ by*

$$B_{\rho,r,\sigma} := \left\{ u \in H^{+\infty}(\Omega) \mid \sum_{k \geq 0} \sigma^k \|u\|_{\rho,r}^k < +\infty \right\}.$$

We equip $B_{\rho,r,\sigma}$ with the norm $\|\cdot\|_{\rho,r,\sigma}$ given by

$$\|u\|_{\rho,r,\sigma} := \sum_{k \geq 0} \sigma^k \|u\|_{\rho,r}^k.$$

We shall see that if $\sigma > 0$ is small enough (the choice will depend only on the parameters r and ρ_0), then we will be able to apply a Cauchy-Kowalevskaya theorem in the scale of Banach spaces $(B_{\rho,r,\sigma})_{\rho \leq \rho_0}$ for solving the equations of ideal inviscid incompressible MHD (1.9) in the fixed domain Ω . Sedenko [Sed94] applies a Cauchy-Kowalevskaya theorem in a similar scale of Banach spaces, for solving Euler equations for ideal incompressible nonhomogeneous and barotropic fluids, bounded by free surfaces. The required properties of the scale $(B_{\rho,r,\sigma})_{\rho \leq \rho_0}$ are given by the following theorem.

Theorem 1.8. *Let $r \in \mathbb{N}$ such that $r \geq 2$ and $\sigma \in (0, 1]$. Then the sequence $(B_{\rho,r,\sigma})_{0 < \rho \leq \rho_0}$ is a scale of Banach spaces:*

$$\forall 0 < \rho' \leq \rho \leq \rho_0, \quad B_{\rho',r,\sigma} \supset B_{\rho,r,\sigma}, \quad \text{with} \quad \|\cdot\|_{\rho',r,\sigma} \leq \|\cdot\|_{\rho,r,\sigma}.$$

Moreover, if $u \in B_{\rho,r,\sigma}$ then for all $\rho' < \rho$, $\partial_i u \in B_{\rho',r,\sigma}$ ($1 \leq i \leq 3$) with

$$\|\partial_i u\|_{\rho',r,\sigma} \leq \frac{C_\sigma}{\rho - \rho'} \|u\|_{\rho,r,\sigma}, \quad (1.24)$$

where $C_\sigma := \sigma^{-1} > 0$. To finish with, the spaces $B_{\rho,r,\sigma}$ are algebras, more precisely:

$$\forall u, v \in B_{\rho,r,\sigma}, \quad uv \in B_{\rho,r,\sigma} \quad \text{and} \quad \|uv\|_{\rho,r,\sigma} \leq C_r \|u\|_{\rho,r,\sigma} \|v\|_{\rho,r,\sigma}, \quad (1.25)$$

where $C_r > 0$ does not depend on ρ and σ .

Remarks:

- As a matter of fact, we can check that the spaces $B_{\rho,r,\sigma}$ consist in analytic functions on $\bar{\Omega}$.
- The spaces $B_{\rho,r,\sigma}$ turn out to be dense in $H^r(\Omega)$, which could be useful to handle initial data of Sobolev regularity.

The introduction of the analytic spaces $B_{\rho,r,\sigma}$ will allow us to construct analytic solutions to the equations of the current-vortex sheet problem in ideal incompressible MHD, using a Cauchy-Kowalevskaya theorem. Such a problem is represented by the coupled system of equations (1.9). Consequently, we similarly define analytic spaces on the sub-domains $\Omega^+ = \mathbb{T}^2 \times (0, 1)$ and $\Omega^- = \mathbb{T}^2 \times (-1, 0)$, respectively denoted as $B_{\rho,r,\sigma}(\Omega^+)$ and $B_{\rho,r,\sigma}(\Omega^-)$. The norm on $B_{\rho,r,\sigma}(\Omega^+)$ is noted $\|\cdot\|_{\rho,r,\sigma}^+$ and so on.

For $(u^+, u^-) \in H^r(\Omega^+) \times H^r(\Omega^-)$, we will note the norms as follows:

$$\begin{aligned} \|u^+\|_{r,+} &:= \|u^+\|_{H^r(\Omega^+)}, & \|u^-\|_{r,-} &:= \|u^-\|_{H^r(\Omega^-)}, \\ \|u^\pm\|_{r,\pm} &:= \|u^+\|_{r,+} + \|u^-\|_{r,-}. \end{aligned} \quad (1.26)$$

Furthermore, if $(u^+, u^-) \in B_{\rho,r,\sigma}(\Omega^+) \times B_{\rho,r,\sigma}(\Omega^-)$, we will write:

$$\|u^\pm\|_{\rho,r,\sigma}^\pm := \|u^+\|_{\rho,r,\sigma}^+ + \|u^-\|_{\rho,r,\sigma}^-. \quad (1.27)$$

The algebra and differentiation properties previously established still hold in the Banach spaces $B_{\rho,r}^k(U)$ and $B_{\rho,r,\sigma}(U)$ introduced above, where U stands for the domains Ω , Ω^+ or Ω^- .

Analytic spaces on $\Gamma \simeq \mathbb{T}^2$.

Definition 1.9. Let $s \in \mathbb{R}_+$ and $\rho \in (0, \rho_0]$. We define

$$B_{\rho,s}(\mathbb{T}^2) := \left\{ u \in H^{+\infty}(\mathbb{T}^2) \mid \|u\|_{\rho,s} := \sum_{n \geq 0} \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|\partial^{\alpha'} u\|_{H^s(\mathbb{T}^2)} < +\infty \right\},$$

where α' is here a multi-index of \mathbb{N}^2 . We recall that the norm on $H^s(\mathbb{T}^2)$ is defined by (1.12).

With such scales of Banach spaces, we shall be able in Section 5 to construct analytic solutions to (1.9).

1.5. Main theorem. The main theorem of this paper reads:

Theorem 1.10. Let $R > 0$ and let $U_0 := (v_0^\pm, B_0^\pm, f_0)$ be some initial data belonging to the space \mathbb{B}_{ρ_0} (defined by (5.5)), where $\rho_0 > 0$ is a small parameter given by Theorem 4.2. We assume that U_0 satisfies

$$\|v_0^\pm, B_0^\pm\|_{\rho_0} < R, \quad \text{and} \quad \|f_0\|_{\rho_0} < \eta_0,$$

where $\eta_0 > 0$ is a small parameter given by Corollary 2.3, and the initial constraints

$$\begin{cases} A^T \nabla \cdot v_0^\pm = A^T \nabla \cdot B_0^\pm = 0 & \text{in } \Omega^\pm, \\ (B^\pm \cdot N)|_{t=0} = 0 & \text{on } \Gamma, \\ [v \cdot N]|_{t=0} = 0 & \text{on } \Gamma. \end{cases}$$

Then there exist $a > 0$, depending only on ρ_0 and R , and a unique solution (v^\pm, B^\pm, Q^\pm, f) to the system (1.9), such that for all $\rho \in (0, \rho_0)$, we have:

$$\begin{aligned} v^\pm, B^\pm, Q^\pm &\in C_t^1([0, a(\rho_0 - \rho)]; B_{\rho,3,\sigma}(\Omega^\pm)), \\ f &\in C_t^1([0, a(\rho_0 - \rho)]; B_{\rho,\frac{7}{2}}(\mathbb{T}^2)), \\ \|v^\pm(t, \cdot), B^\pm(t, \cdot), Q^\pm(t, \cdot)\|_{\rho,3,\sigma}^\pm &< R \quad \text{and} \quad \|f(t, \cdot)\|_{\rho,\frac{7}{2}} < \eta_0, \quad \forall t \in [0, a(\rho_0 - \rho)], \\ \sum_{\pm} \int_{\Omega^\pm} Q^\pm(t, x) dx &= 0, \quad \forall t \in [0, a(\rho_0 - \rho)]. \end{aligned}$$

Strategy of the proof. We have reduced the current-vortex sheet problem into the fixed domains Ω^+ and Ω^- using the lifting ψ of the front f . Consequently, in Section 2 we give analytic estimates of the new unknown ψ with respect to the front f . In Section 3, we simplify the problem (1.9) to fit with the version of Cauchy-Kowalevskaya theorem [Nis77] we will apply. To do so, we show that some equations in (1.9) are restrictions on the initial data only. Afterwards, we deal with the total pressure Q^\pm in Section 4, which is the tricky part of this article. Using an appropriate elliptic problem satisfied by the pressure, we give analytic estimates on Q^\pm depending on the unknowns v^\pm and B^\pm . Finally, we apply in Section 5 Cauchy-Kowalevskaya theorem [Nis77] to construct analytic solutions to (1.9).

2. LIFTING OF THE FRONT AND ANALYTIC ESTIMATES

2.1. Analytic estimate of the lifting ψ . We wish to prove local existence of analytic solutions to the system (1.9). To do so, we will use the spaces introduced in Paragraph 1.4, namely $B_{\rho,r,\sigma}(\Omega)$, $B_{\rho,r,\sigma}(\Omega^\pm)$ and $B_{\rho,s}(\mathbb{T}^2)$. For the time being, ρ_0 stands for any real number of the interval $(0, 1]$, but it will be required to be small enough later on.

First, using such analytic spaces, we show that the gain of half a derivative for ψ with respect to f (see Lemma 1.1) persists within the analytic norms $\|\cdot\|_{\rho,r,\sigma}$.

Proposition 2.1. Let $\rho \in (0, \rho_0]$, $r \geq 2$ be an integer and $\sigma \in (0, 1)$. Then we can choose the linear map of Lemma 1.1 such that if it is restricted to the space $B_{\rho,r-\frac{1}{2}}(\mathbb{T}^2)$, then it maps continuously $B_{\rho,r-\frac{1}{2}}(\mathbb{T}^2)$ into $B_{\rho,r,\sigma}(\Omega)$. In other terms, for all $f \in B_{\rho,r-\frac{1}{2}}(\mathbb{T}^2)$, we have $\psi \in B_{\rho,r,\sigma}(\Omega)$ with the following estimate:

$$\|\psi\|_{\rho,r,\sigma} \leq \frac{C_r}{(1-\sigma)^3} \|f\|_{\rho,r-\frac{1}{2}}, \quad (2.1)$$

where $C_r > 0$ depends only on r . The choice of this linear map is independent of the parameters ρ , r and σ . We recall that the lifting ψ satisfies, for all $x' \in \mathbb{T}^2$,

$$\psi(x', 0) = f(x'), \quad \psi(x', \pm 1) = 0, \quad \text{and} \quad \partial_3 \psi(x', 0) = 0. \quad (2.2)$$

Proof. We adapt the proof of Lemma 1 in [CMST12]. We begin by recalling the construction of ψ from f .

Step 1: For $g \in B_{\rho, r-\frac{1}{2}}(\mathbb{T}^2)$, we define the function Φ_g by

$$\Phi_g(x', x_3) := \varphi(x_3|D|)g(x'), \quad \forall (x', x_3) \in \Omega, \quad (2.3)$$

where the operator $|D|$ corresponds to the Fourier multiplier in the tangential variable $x' \in \mathbb{T}^2$. For the time being, we want the function φ to satisfy two conditions:

$$\varphi(0) = 1 \quad \text{and} \quad \varphi \in H^{+\infty}(\mathbb{R}).$$

The aim of this step is to bound from above $\|\Phi_g\|_{\rho, r, \sigma}$ by $\|g\|_{\rho, r-\frac{1}{2}}$. First of all, we are going to estimate the norms $\|\Phi_g\|_{\rho, r}^k$, for all $k \in \mathbb{N}$. In the following computations, $C_0 > 0$ will denote any numerical constant, and $C_r > 0$ any constant that depends only on r . Let us start by estimating $\|\Phi_g\|_{H^r(\Omega)}$. To do so, we use Fubini's theorem:

$$\|\Phi_g\|_{H^r(\Omega)}^2 = \sum_{j=0}^r \|\partial_3^j \Phi_g\|_{L_{x_3}^2((-1,1); H^{r-j}(\mathbb{T}^2))}^2,$$

and we handle each term $\|\partial_3^j \Phi_g\|_{L_{x_3}^2(H^{r-j}(\mathbb{T}^2))}^2$ separately.

► For $j = 0$: by definition of the operator $|D|$, we can write:

$$\begin{aligned} \|\Phi_g(\cdot, x_3)\|_{H^r(\mathbb{T}^2)}^2 &= \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^r |\varphi(x_3|k|)|^2 |c_k(g)|^2 \\ &= |c_0(g)|^2 + \sum_{k \neq 0} (1 + |k|^2)^r |\varphi(x_3|k|)|^2 |c_k(g)|^2. \end{aligned}$$

Integrating over $x_3 \in (-1, 1)$, we get:

$$\begin{aligned} \|\Phi_g\|_{L_{x_3}^2(H^r(\mathbb{T}^2))}^2 &= 2|c_0(g)|^2 + \sum_{k \neq 0} (1 + |k|^2)^r |c_k(g)|^2 \int_{-1}^1 |\varphi(x_3|k|)|^2 dx_3 \\ &= 2|c_0(g)|^2 + \sum_{k \neq 0} (1 + |k|^2)^r |c_k(g)|^2 \int_{-|k|}^{|k|} |\varphi(s)|^2 \frac{ds}{|k|} \\ &\leq C_0 \left(|c_0(g)|^2 + \|\varphi\|_{L^2(\mathbb{R})}^2 \sum_{k \neq 0} (1 + |k|^2)^{r-\frac{1}{2}} |c_k(g)|^2 \right) \\ &\leq C_0 \left(1 + \|\varphi\|_{L^2(\mathbb{R})}^2 \right) \|g\|_{H^{r-\frac{1}{2}}(\mathbb{T}^2)}^2. \end{aligned}$$

► For $j \in \{1, \dots, r\}$, differentiating the function Φ_g , we have:

$$\partial_3^j \Phi_g(x) = \varphi^{(j)}(x_3|D|) (|D|^j g)(x').$$

Thus,

$$\begin{aligned} \|\partial_3^j \Phi_g(\cdot, x_3)\|_{H^{r-j}(\mathbb{T}^2)}^2 &= \sum_{k \in \mathbb{Z}^2} (1 + |k|^2)^{r-j} |\varphi^{(j)}(x_3|k|)|^2 |k|^{2j} |c_k(g)|^2 \\ &= \sum_{k \neq 0} (1 + |k|^2)^{r-j} |\varphi^{(j)}(x_3|k|)|^2 |k|^{2j} |c_k(g)|^2 \\ &\leq \sum_{k \neq 0} (1 + |k|^2)^r |\varphi^{(j)}(x_3|k|)|^2 |c_k(g)|^2. \end{aligned}$$

Integrating over x_3 , we obtain:

$$\begin{aligned} \|\partial_3^j \Phi_g\|_{L_{x_3}^2(H^{r-j}(\mathbb{T}^2))}^2 &\leq \sum_{k \neq 0} (1 + |k|^2)^r |c_k(g)|^2 \int_{-|k|}^{|k|} |\varphi^{(j)}(s)|^2 \frac{ds}{|k|} \\ &\leq C_0 \|\varphi^{(j)}\|_{L^2(\mathbb{R})}^2 \sum_{k \neq 0} (1 + |k|^2)^{r-\frac{1}{2}} |c_k(g)|^2 \\ &\leq C_0 \|\varphi^{(j)}\|_{L^2(\mathbb{R})}^2 \|g\|_{H^{r-\frac{1}{2}}(\mathbb{T}^2)}^2. \end{aligned}$$

Summing over j , we finally have:

$$\|\Phi_g\|_{H^r(\Omega)}^2 \leq C_0 \left(1 + \|\varphi\|_{H^r(\mathbb{R})}^2 \right) \|g\|_{H^{r-\frac{1}{2}}(\mathbb{T}^2)}^2,$$

hence

$$\|\Phi_g\|_{H^r(\Omega)} \leq C_0 (1 + \|\varphi\|_{H^r(\mathbb{R})}) \|g\|_{H^{r-\frac{1}{2}}(\mathbb{T}^2)}. \quad (2.4)$$

Step 2: Now, let us focus on the norms $\|\Phi_g\|_{\rho,r}^k$, for all $k \in \mathbb{N}$. We begin with the case $k = 0$, for which the tangential derivatives ∂^α , with $\alpha = (\alpha_1, \alpha_2, 0) = (\alpha', 0)$, commute with the operator $\varphi(x_3|D|)$. We easily have:

$$\|\Phi_g\|_{\rho,r}^0 = \sum_{n \geq 0} \frac{\rho^n}{n!} \max_{\substack{|\alpha|=n \\ \alpha_3=0}} \|\partial^\alpha \Phi_g\|_{H^r(\Omega)} = \sum_{n \geq 0} \frac{\rho^n}{n!} \max_{\substack{|\alpha|=n \\ \alpha_3=0}} \|\Phi_{\partial^{\alpha'} g}\|_{H^r(\Omega)}.$$

Therefore, we can apply estimate (2.4) to the function $\partial^{\alpha'} g$ (instead of g), and we get:

$$\begin{aligned} \|\Phi_g\|_{\rho,r}^0 &\leq C_0 (1 + \|\varphi\|_{H^r(\mathbb{R})}) \sum_{n \geq 0} \frac{\rho^n}{n!} \max_{|\alpha|=n} \|\partial^{\alpha'} g\|_{H^{r-\frac{1}{2}}(\mathbb{T}^2)} \\ &\leq C_0 (1 + \|\varphi\|_{H^r(\mathbb{R})}) \|g\|_{\rho,r-\frac{1}{2}}. \end{aligned} \quad (2.5)$$

In order to estimate the norms $\|\Phi_g\|_{\rho,r}^{k+1}$, with $k \geq 0$, we proceed by induction:

$$\begin{aligned} \|\Phi_g\|_{\rho,r}^{k+1} &= \sum_{n \geq 0} \frac{\rho^n}{n!} \max_{\substack{|\alpha|=n \\ \alpha_3 \leq k+1}} \|\partial^\alpha \Phi_g\|_{H^r(\Omega)} \\ &\leq \|\Phi_g\|_{\rho,r}^k + \sum_{n \geq k+1} \frac{\rho^n}{n!} \max_{\substack{|\alpha|=n \\ \alpha_3 = k+1}} \|\partial^\alpha \Phi_g\|_{H^r(\Omega)} \\ &= \|\Phi_g\|_{\rho,r}^k + \sum_{n \geq k+1} \frac{\rho^n}{n!} \max_{|\alpha'|=n-k-1} \|\partial_3^{k+1} \Phi_{\partial^{\alpha'} g}\|_{H^r(\Omega)}. \end{aligned} \quad (2.6)$$

As previously, we estimate the norms $\|\partial_3^j \partial_3^{k+1} \Phi_{\partial^{\alpha'} g}\|_{L_{x_3}^2(H^{r-j}(\mathbb{T}^2))}$, for all $j \in \{0, \dots, r\}$:

$$\begin{aligned} \|\partial_3^{j+k+1} \Phi_{\partial^{\alpha'} g}(\cdot, x_3)\|_{H^{r-j}(\mathbb{T}^2)}^2 &= \sum_{\ell \neq 0} (1 + |\ell|^2)^{r-j} |\varphi^{(j+k+1)}(x_3|\ell|)|^2 |\ell|^{2(j+k+1)} |c_\ell(\partial^{\alpha'} g)|^2 \\ &\leq \sum_{\ell \neq 0} (1 + |\ell|^2)^r |\ell|^{2(k+1)} (2\pi|\ell_1|)^{2\alpha_1} (2\pi|\ell_2|)^{2\alpha_2} |c_\ell(g)|^2 |\varphi^{(j+k+1)}(x_3|\ell|)|^2. \end{aligned}$$

Integrating over x_3 , we have:

$$\|\partial_3^{j+k+1} \Phi_{\partial^{\alpha'} g}\|_{L_{x_3}^2(H^{r-j}(\mathbb{T}^2))}^2 \leq C_0 \|\varphi^{(j+k+1)}\|_{L^2(\mathbb{R})}^2 \sum_{\ell \neq 0} (1 + |\ell|^2)^{r-\frac{1}{2}} |\ell|^{2(k+1)} (2\pi|\ell_1|)^{2\alpha_1} (2\pi|\ell_2|)^{2\alpha_2} |c_\ell(g)|^2. \quad (2.7)$$

Then, we rewrite the sum in (2.7) as $\sum_{|\ell_1| \leq |\ell_2|} + \sum_{|\ell_2| < |\ell_1|}$ (if $\ell = 0$, the general term of the series vanishes). Thus,

$$\begin{aligned} \sum_{|\ell_1| \leq |\ell_2|} (1 + |\ell|^2)^{r-\frac{1}{2}} |\ell|^{2(k+1)} (2\pi|\ell_1|)^{2\alpha_1} (2\pi|\ell_2|)^{2\alpha_2} |c_\ell(g)|^2 \\ \leq \sum_{|\ell_1| \leq |\ell_2|} (1 + |\ell|^2)^{r-\frac{1}{2}} (2\ell_2^2)^{k+1} (2\pi|\ell_1|)^{2\alpha_1} (2\pi|\ell_2|)^{2\alpha_2} |c_\ell(g)|^2 \\ \leq \sum_{|\ell_1| \leq |\ell_2|} (1 + |\ell|^2)^{r-\frac{1}{2}} (2\pi|\ell_1|)^{2\alpha_1} (2\pi|\ell_2|)^{2(\alpha_2+k+1)} |c_\ell(g)|^2. \end{aligned} \quad (2.8)$$

Let us set $\gamma' := (\alpha_1, \alpha_2 + k + 1)$. Then we have $|\gamma'| = n$, and inequality (2.8) gives:

$$\begin{aligned} \sum_{|\ell_1| \leq |\ell_2|} (1 + |\ell|^2)^{r-\frac{1}{2}} |\ell|^{2(k+1)} (2\pi|\ell_1|)^{2\alpha_1} (2\pi|\ell_2|)^{2\alpha_2} |c_\ell(g)|^2 \\ \leq \sum_{|\ell_1| \leq |\ell_2|} (1 + |\ell|^2)^{r-\frac{1}{2}} |c_\ell(\partial^{\gamma'} g)|^2 \\ \leq \|\partial^{\gamma'} g\|_{H^{r-\frac{1}{2}}(\mathbb{T}^2)}^2 \leq \max_{|\beta'|=n} \|\partial^{\beta'} g\|_{H^{r-\frac{1}{2}}(\mathbb{T}^2)}^2. \end{aligned} \quad (2.9)$$

The case of the second sum $\sum_{|\ell_2| < |\ell_1|}$ is symmetric, and gives the same upper bound as (2.9). Back to (2.7), we obtain:

$$\|\partial_3^{j+k+1} \Phi_{\partial^{\alpha'} g}\|_{L_{x_3}^2(H^{r-j}(\mathbb{T}^2))}^2 \leq C_0 \|\varphi^{(j+k+1)}\|_{L^2(\mathbb{R})}^2 \max_{|\beta'|=n} \|\partial^{\beta'} g\|_{H^{r-\frac{1}{2}}(\mathbb{T}^2)}^2.$$

Summing over j , it follows that

$$\|\partial_3^{k+1} \Phi_{\partial^{\alpha'} g}\|_{H^r(\Omega)} \leq C_0 \|\varphi^{(k+1)}\|_{H^r(\mathbb{R})} \max_{|\beta'|=n} \|\partial^{\beta'} g\|_{H^{r-\frac{1}{2}}(\mathbb{T}^2)}. \quad (2.10)$$

Using (2.10) in the estimate (2.6), we have:

$$\|\Phi_g\|_{\rho,r}^{k+1} \leq \|\Phi_g\|_{\rho,r}^k + C_0 \|\varphi^{(k+1)}\|_{H^r(\mathbb{R})} \|g\|_{\rho,r-\frac{1}{2}}.$$

By induction, and using (2.5), we end up with the following estimate, holding for all $k \geq 0$:

$$\|\Phi_g\|_{\rho,r}^k \leq C_0 \left(1 + \sum_{j=0}^k \|\varphi^{(j)}\|_{H^r(\mathbb{R})} \right) \|g\|_{\rho,r-\frac{1}{2}}. \quad (2.11)$$

Step 3: To conclude about the estimate of the norm $\|\Phi_g\|_{\rho,r,\sigma}$, it remains to multiply (2.11) by σ^k , and to sum over k . To do so, the Sobolev norms of the functions $\varphi^{(j)}$ must not increase “more” than geometrically. Thus, we choose $\varphi \in H^{+\infty}(\mathbb{R})$ such that:

$$\widehat{\varphi}(\xi) := \pi \mathbf{1}_{[-1,1]}(\xi), \quad \forall \xi \in \mathbb{R}.$$

The function φ is given by the following formula:

$$\varphi(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{ix\xi} \widehat{\varphi}(\xi) d\xi = \frac{\sin(x)}{x}.$$

In particular, we have $\varphi(0) = 1$ and $\varphi \in H^{+\infty}(\mathbb{R})$. Besides, we easily estimate the norms $\|\varphi^{(j)}\|_{H^r(\mathbb{R})}$ using the Fourier transform and the parity of $\widehat{\varphi}$:

$$\|\varphi^{(j)}\|_{H^r(\mathbb{R})}^2 \leq C_r \int_{\mathbb{R}} (1 + \xi^2)^r \xi^{2j} \widehat{\varphi}(\xi)^2 d\xi \leq C_r \int_0^1 (1 + \xi^2)^r \xi^{2j} d\xi \leq C_r. \quad (2.12)$$

Estimate (2.11) finally reads:

$$\|\Phi_g\|_{\rho,r}^k \leq C_r (k+1) \|g\|_{\rho,r-\frac{1}{2}}.$$

Now, let $K \in \mathbb{N}$, and write $\|\Phi_g\|_{\rho,r,\sigma}^K$ the partial sum of order K of $\|\Phi_g\|_{\rho,r,\sigma}$. Then, for $\sigma \in (0, 1)$,

$$\|\Phi_g\|_{\rho,r,\sigma}^K = \sum_{k=0}^K \sigma^k \|\Phi_g\|_{\rho,r}^k \leq C_r \sum_{k=0}^K (k+1) \sigma^k \|g\|_{\rho,r-\frac{1}{2}} \leq \frac{C_r}{(1-\sigma)^2} \|g\|_{\rho,r-\frac{1}{2}}.$$

Taking the supremum over K , we conclude that $\Phi_g \in B_{\rho,r,\sigma}$ and

$$\|\Phi_g\|_{\rho,r,\sigma} \leq \frac{C_r}{(1-\sigma)^2} \|g\|_{\rho,r-\frac{1}{2}}. \quad (2.13)$$

Conclusion: Given $f \in B_{\rho,r-\frac{1}{2}}(\mathbb{T}^2)$, we define the function $\Psi_f = \psi$ by:

$$\Psi_f(x', x_3) := (1 - x_3^2) \Phi_f(x) = (1 - x_3^2) \varphi(x_3 |D|) f(x'), \quad \forall (x', x_3) \in \Omega. \quad (2.14)$$

The definition of ψ above can also be considered in the proof done by [CMST12], and does not change any result about the lifting of the front f recalled by Lemma 1.1.

We remind that we wish to estimate $\|\Psi_f\|_{\rho,r,\sigma}$ by $\|f\|_{\rho,r-\frac{1}{2}}$. To do so, it suffices to use the algebra property of the spaces $B_{\rho,r,\sigma}(\Omega)$ given by Theorem 1.8, and estimate (2.13) obtained in step 3. Setting $\omega(x_3) := 1 - x_3^2$, we have:

$$\|\Psi_f\|_{\rho,r,\sigma} \leq C_r \|\omega\|_{\rho,r,\sigma} \|\Phi_f\|_{\rho,r,\sigma} \leq \frac{C_r}{(1-\sigma)^2} \|\omega\|_{\rho,r,\sigma} \|f\|_{\rho,r-\frac{1}{2}}.$$

To finish with, we can explicitly estimate $\|\omega\|_{\rho,r,\sigma}$, since all the tangential derivatives of ω vanish. For all $k \geq 0$, we get:

$$\|\omega\|_{\rho,r}^k = \sum_{n=0}^k \frac{\rho^n}{n!} \|\partial_3^n \omega\|_{H^r(\Omega)} \leq \sum_{n=0}^2 \frac{\rho^n}{n!} \|\partial_3^n \omega\|_{H^r(\Omega)} \leq C_0,$$

because all the derivatives of ω of order larger than 3 vanish, and ρ can be directly bounded above by 1. As a consequence, we have the estimate

$$\|\omega\|_{\rho,r,\sigma} \leq \frac{C_0}{1-\sigma}.$$

Thus, we can conclude that there exists a constant $C_r > 0$, depending only on r , such that:

$$\|\Psi_f\|_{\rho,r,\sigma} \leq \frac{C_r}{(1-\sigma)^3} \|f\|_{\rho,r-\frac{1}{2}}. \quad (2.15)$$

□

2.2. Analytic estimate of the inverse of the jacobian J . The change of unknown $f \mapsto \psi$ required the continuous estimate (2.1) in order to finally apply a Cauchy-Kowalevskaya theorem to solve (1.9). With the new unknowns (v^\pm, B^\pm, ψ) , the equations of (1.9) make the inverse of the jacobian J appear. To construct analytic solutions to this problem, we will have to be able to properly estimate the analytic norm $\|\cdot\|_{\rho,r,\sigma}$ of $\frac{1}{J}$ (recall that $J = 1 + \partial_3 \psi$). To do so, we will use the algebra property of the Banach spaces $B_{\rho,r,\sigma}(\Omega)$.

Proposition 2.2. *Let $\rho \in (0, \rho_0]$, $r \geq 2$ be an integer and $\sigma \in (0, 1)$. There exists $\varepsilon_1 = \varepsilon_1(r) > 0$ depending only on r , such that for all $g \in B_{\rho,r,\sigma}(\Omega)$ satisfying $\|g\|_{\rho,r,\sigma} \leq \varepsilon_1$, we have:*

$$\frac{1}{1+g} \in B_{\rho,r,\sigma}(\Omega) \quad \text{and} \quad \left\| \frac{1}{1+g} \right\|_{\rho,r,\sigma} \leq \frac{C_r}{1-\sigma}, \quad (2.16)$$

where $C_r > 0$ depends only on r .

Proof. We start by recalling the algebra property of the spaces $B_{\rho,r,\sigma}(\Omega)$. There exists a constant $C_r > 0$ depending only on r , such that for all $u, v \in B_{\rho,r,\sigma}(\Omega)$, we have:

$$uv \in B_{\rho,r,\sigma}(\Omega) \quad \text{and} \quad \|uv\|_{\rho,r,\sigma} \leq C_r \|u\|_{\rho,r,\sigma} \|v\|_{\rho,r,\sigma}.$$

We define $\varepsilon_1 := \frac{1}{2C_r}$, and consider $g \in B_{\rho,r,\sigma}(\Omega)$ such that $\|g\|_{\rho,r,\sigma} \leq \varepsilon_1$. Then the series $\sum_{m \geq 0} (-1)^m g^m$ is absolutely convergent in the Banach space $B_{\rho,r,\sigma}(\Omega)$. Indeed,

$$\sum_{m \geq 0} \|(-1)^m g^m\|_{\rho,r,\sigma} \leq \|1\|_{\rho,r,\sigma} + \sum_{m \geq 1} C_r^{m-1} \|g\|_{\rho,r,\sigma}^m \leq \|1\|_{\rho,r,\sigma} + C_r^{-1}.$$

Then, we estimate the constant function 1 in $B_{\rho,r,\sigma}(\Omega)$ by a straightforward computation:

$$\|1\|_{\rho,r,\sigma} = \frac{C_0}{1-\sigma},$$

where $C_0 > 0$ is a numerical constant. Eventually, we deduce that

$$\sum_{m \geq 0} \|(-1)^m g^m\|_{\rho,r,\sigma} \leq \frac{C_0}{1-\sigma} + C_r^{-1} \leq \frac{C'_r}{1-\sigma} < +\infty,$$

where $C'_r := C_0 + C_r^{-1} > 0$ depends only on r . The completeness of the spaces $B_{\rho,r,\sigma}(\Omega)$ allows to write the identity

$$\frac{1}{1+g} = \sum_{m \geq 0} (-1)^m g^m.$$

Consequently, $\frac{1}{1+g}$ belongs to the space $B_{\rho,r,\sigma}(\Omega)$, and satisfies the estimate

$$\left\| \frac{1}{1+g} \right\|_{\rho,r,\sigma} \leq \frac{C'_r}{1-\sigma}.$$

□

Using Proposition 2.1, we deduce a straightforward corollary about the analytic estimate of the inverse of the jacobian J . For the sake of clarity, we will assume from now on that $\sigma \in (0, \sigma_0]$, where σ_0 is a numerical constant fixed once for all in the interval $(0, 1)$, for instance $\sigma_0 := \frac{1}{2}$. It will allow us to bound from above the quantity $\frac{1}{1-\sigma}$ uniformly with respect to σ , which will simplify the future estimates. This restriction will be harmless for our work, since σ will have to be chosen small enough in the end.

Corollary 2.3. *Let $\rho \in (0, \rho_0]$, $r \geq 2$ be an integer and $\sigma \in (0, \sigma_0]$. Let also $f \in B_{\rho,r+\frac{1}{2}}(\mathbb{T}^2)$. There exists a constant $\eta_0 = \eta_0(r) > 0$ such that:*

$$\|f\|_{\rho,r+\frac{1}{2}} \leq \eta_0 \quad \implies \quad \frac{1}{J} \in B_{\rho,r,\sigma}(\Omega) \quad \text{and} \quad \left\| \frac{1}{J} \right\|_{\rho,r,\sigma} \leq M_r, \quad (2.17)$$

where $M_r > 0$ depends only on r .

Proof. To begin with, let us rewrite the results (2.15) and (2.16) taking the inequality $\sigma \leq \sigma_0$ into account. There exists a constant $\tilde{C}_r > 0$, depending only on r , such that:

$$\|\psi\|_{\rho, r+1, \sigma} \leq \tilde{C}_r \|f\|_{\rho, r+\frac{1}{2}}. \quad (2.18)$$

Consequently,

$$\|\partial_3 \psi\|_{\rho, r, \sigma} \leq \tilde{C}_r \|f\|_{\rho, r+\frac{1}{2}}. \quad (2.19)$$

Likewise, there exists a constant $M_r > 0$, depending only on r , such that:

$$\|g\|_{\rho, r, \sigma} \leq \varepsilon_1 \implies \frac{1}{1+g} \in B_{\rho, r, \sigma}(\Omega) \quad \text{and} \quad \left\| \frac{1}{1+g} \right\|_{\rho, r, \sigma} \leq M_r. \quad (2.20)$$

We define $\eta_0 := \frac{\varepsilon_1}{\tilde{C}_r}$, and assume that $\|f\|_{\rho, r+\frac{1}{2}} \leq \eta_0$. Using (2.19), we have $\|\partial_3 \psi\|_{\rho, r, \sigma} \leq \varepsilon_1$. So, applying (2.20) to the function $g := \partial_3 \psi$, we get:

$$\left\| \frac{1}{1+\partial_3 \psi} \right\|_{\rho, r, \sigma} \leq M_r.$$

□

In order to apply a Cauchy-Kowalevskaya theorem, we wish to rewrite the problem (1.9) under the form of a “differential equation” in time. That is why the next section is devoted to simplify the equations in (1.9). As a matter of fact, some equations in (1.9) turn out to be restrictions on the initial data only, and will be propagated in time.

3. SIMPLIFICATION OF THE PROBLEM

3.1. Elliptic problem satisfied by the total pressure. The total pressure Q^\pm appearing in problem (1.9) can be implicitly expressed in terms of the unknowns (v^\pm, B^\pm, f) , using the “divergence”-free constraint fulfilled by the velocity. With the same computations as in [CMST12, p.266], we show that the couple (Q^+, Q^-) satisfies the following elliptic problem:

$$\begin{cases} -A^T \nabla \cdot (A^T \nabla Q^\pm) = \mathcal{F}^\pm & \text{in } (0, T) \times \Omega^\pm, \\ [Q] = 0 & \text{on } (0, T) \times \Gamma, \\ [A^T \nabla Q \cdot N] = \mathcal{G} & \text{on } (0, T) \times \Gamma, \\ \partial_3 Q^\pm = 0 & \text{on } (0, T) \times \Gamma_\pm. \end{cases} \quad (3.1)$$

The source terms \mathcal{F}^\pm are defined by:

$$\mathcal{F}^\pm := -\partial_t A_{ki} \partial_k v_i^\pm + A_{ki} \partial_k \tilde{v}^\pm \cdot \nabla v_i^\pm - \tilde{v}^\pm \cdot \nabla A_{ki} \partial_k v_i^\pm - A_{ki} \partial_k \tilde{B}^\pm \cdot \nabla B_i^\pm + \tilde{B}^\pm \cdot \nabla A_{ki} \partial_k B_i^\pm, \quad (3.2)$$

where we have used Einstein’s summation convention for repeated indices. The boundary source term \mathcal{G} in (3.1) is defined as follows:

$$\mathcal{G} := -[2v' \cdot \nabla' \partial_t f + (v' \cdot \nabla') \nabla' f \cdot v' - (B' \cdot \nabla') \nabla' f \cdot B'], \quad (3.3)$$

where the notation “ ’ ” stands for the tangential coordinates:

$$\nabla' := (\partial_1, \partial_2), \quad v' := (v_1, v_2), \quad B' := (B_1, B_2).$$

We want to solve the problem (1.9),(3.1),(3.2),(3.3). In order to use Cauchy-Kowalevskaya theorem, we wish to have exactly five evolution equations, corresponding to the five unknowns (v^\pm, B^\pm, f) , with prescribed initial data. The latter must contain enough information to keep the equivalence with the original problem (1.9). We therefore focus on the resolution of the problem thereafter (3.4),(3.1),(3.2),(3.3) together with the constraints (3.5) on the initial data. Adapting the analysis of Trakhinin [Tra05], we are going to check that the “reduced” problem (3.4) below is equivalent to the first one given by (1.9),(3.1),(3.2),(3.3). The “reduced” problem reads as follows:

$$\begin{cases} \partial_t v^\pm + (\tilde{v}^\pm \cdot \nabla) v^\pm - (\tilde{B}^\pm \cdot \nabla) B^\pm + A^T \nabla Q^\pm = 0 & \text{in } (0, T) \times \Omega^\pm, \\ \partial_t B^\pm + (\tilde{v}^\pm \cdot \nabla) B^\pm - (\tilde{B}^\pm \cdot \nabla) v^\pm = 0 & \text{in } (0, T) \times \Omega^\pm, \\ \partial_t f = v^+ \cdot N & \text{on } (0, T) \times \Gamma, \\ v_3^\pm = B_3^\pm = 0 & \text{on } \Gamma_\pm, \end{cases} \quad (3.4)$$

where Q^\pm satisfies the elliptic problem (3.1) with the source terms \mathcal{F}^\pm and \mathcal{G} given by (3.2),(3.3). We recall that the quantities A , N , \tilde{v}^\pm and \tilde{B}^\pm are defined by (1.8) and (1.10). We prescribe initial data (v_0^\pm, B_0^\pm, f_0) satisfying

the constraints:

$$\begin{cases} A^T \nabla \cdot v_0^\pm = A^T \nabla \cdot B_0^\pm = 0 & \text{in } \Omega^\pm, \\ (B^\pm \cdot N)|_{t=0} = 0 & \text{on } \Gamma, \\ [v \cdot N]|_{t=0} = 0 & \text{on } \Gamma. \end{cases} \quad (3.5)$$

Let us observe that the matrix $A|_{t=0}$ and the vector $N|_{t=0}$ are computed after lifting the initial front f_0 . The last condition in (3.5) will allow to get the second evolution equation on f , namely $\partial_t f = v^- \cdot N$ on $(0, T) \times \Gamma$, as we shall see later on. Taking the definition of N given by (1.10) into account, we have in particular

$$N|_{t=0} = (-\partial_1 f_0, -\partial_2 f_0, 1) \quad \text{on } \Gamma.$$

3.2. Propagation of the condition related to the normal component of the upper magnetic field B^+ . Within this paragraph, we show that the initial constraint $B^+ \cdot N|_{t=0} = 0$ on Γ is propagated in time by the solutions of (3.4). The method is inspired from [Tra05].

Proposition 3.1. *Let (v^\pm, B^\pm, f) be a smooth solution of (3.4), (3.1), (3.2), (3.3), that is:*

$$(v^\pm, B^\pm, f) \in \mathcal{C}([0, T]; H^3(\Omega^\pm))^2 \times \mathcal{C}([0, T]; H^{3,5}(\mathbb{T}^2)).$$

If the initial data (v_0^\pm, B_0^\pm, f_0) satisfy $B_0^+ \cdot N_0 = 0$ on Γ , then for all $t \in [0, T]$, we have:

$$B^+ \cdot N(t, \cdot) = 0 \quad \text{on } \Gamma.$$

Proof. We set, for $(t, x) \in [0, T] \times \Omega^+$,

$$B_N^+(t, x) := (B^+ \cdot N)(t, x) \quad \text{and} \quad v_N^+(t, x) := (v^+ \cdot N)(t, x).$$

We are looking for an evolution equation satisfied by B_N^+ . So, we compute:

$$\begin{aligned} \partial_t B_N^+ &= \partial_t B^+ \cdot N + B^+ \cdot \partial_t N \\ &= \left(-(\tilde{v}^+ \cdot \nabla) B^+ + (\tilde{B}^+ \cdot \nabla) v^+ \right) \cdot N + B^+ \cdot (-\partial_1 \partial_t \psi, -\partial_2 \partial_t \psi, 0) \\ &= -v_j^+ \partial_j B^+ \cdot N - \left(\frac{v_N^+ - \partial_t \psi}{J} \right) \partial_3 B^+ \cdot N + B_j^+ \partial_j v^+ \cdot N + \left(\frac{B_N^+}{J} \right) \partial_3 v^+ \cdot N - B_j^+ \partial_j \partial_t \psi, \end{aligned}$$

where we have used Einstein's summation convention for the indices between 1 and 2 only. Now, we commute the derivatives ∂_j (for $j = 1, 2$) in order to make the desired quantities $\partial_j B_N^+$ appear:

$$\begin{aligned} \partial_t B_N^+ &= -v_j^+ (\partial_j B_N^+ - B^+ \cdot \partial_j N) - \left(\frac{v_N^+ - \partial_t \psi}{J} \right) \partial_3 B^+ \cdot N \\ &\quad + B_j^+ (\partial_j v_N^+ - v^+ \cdot \partial_j N) + \left(\frac{B_N^+}{J} \right) \partial_3 v^+ \cdot N - B_j^+ \partial_j \partial_t \psi. \end{aligned}$$

We pass to the limit $x_3 \rightarrow 0_+$. Then, because $\psi = f$, $\partial_t f = v^+ \cdot N$ and $J = 1$ on Γ (see (3.4) and (2.2)), we get the following transport equation in $(0, T) \times \Gamma$:

$$\partial_t B_N^+ = -v_j^+ \partial_j B_N^+ + v_j^+ B^+ \cdot \partial_j N + B_j^+ \partial_j v_N^+ - B_j^+ v^+ \cdot \partial_j N + (\partial_3 v^+ \cdot N) B_N^+ - B_j^+ \partial_j v_N^+.$$

We simplify, and finally obtain:

$$\begin{cases} \partial_t B_N^+ + v_1^+ \partial_1 B_N^+ + v_2^+ \partial_2 B_N^+ = (\partial_3 v^+ \cdot N) B_N^+ & \text{in } (0, T) \times \Gamma, \\ B_N^+(0, \cdot) = 0 & \text{in } \Gamma. \end{cases} \quad (3.6)$$

The method of characteristics concludes the proof. \square

3.3. Propagation of the conditions related to the jump of the normal velocity, and the normal magnetic field B_N^- .

Proposition 3.2. *Let (v^\pm, B^\pm, f) be a smooth solution of (3.4), (3.1), (3.2), (3.3), that is:*

$$(v^\pm, B^\pm, f) \in \mathcal{C}([0, T]; H^3(\Omega^\pm))^2 \times \mathcal{C}([0, T]; H^{3,5}(\mathbb{T}^2)).$$

If the initial data (v_0^\pm, B_0^\pm, f_0) satisfy $B_0^+ \cdot N_0 = 0$ and $B_0^- \cdot N_0 = [v_0 \cdot N_0] = 0$ on Γ , then for all $t \in [0, T]$ we have:

$$B^- \cdot N(t, \cdot) = [v \cdot N](t, \cdot) = 0 \quad \text{on } \Gamma.$$

Proof. As previously, we define for all $(t, x) \in [0, T] \times \Omega^\pm$,

$$B_N^\pm(t, x) := (B^\pm \cdot N)(t, x) \quad \text{and} \quad v_N^\pm(t, x) := (v^\pm \cdot N)(t, x).$$

We are looking for two evolution equations for $B_N^-|_\Gamma$ and $[v_N]$. With the same computations as B_N^+ , we can write:

$$\begin{aligned} \partial_t B_N^- &= -v_j^- (\partial_j B_N^- - B^- \cdot \partial_j N) - \left(\frac{v_N^- - \partial_t \psi}{J} \right) \partial_3 B^- \cdot N \\ &\quad + B_j^- (\partial_j v_N^- - v^- \cdot \partial_j N) + \left(\frac{B_N^-}{J} \right) \partial_3 v^- \cdot N - B_j^- \partial_j \partial_t \psi. \end{aligned}$$

Passing to the limit $x_3 \rightarrow 0_-$, we get this time (recall that we did not include the equation $\partial_t f = v^- \cdot N$ in (3.4)):

$$\partial_t B_N^- + v_j^- \partial_j B_N^- + B_j^- \partial_j [v_N] = (\partial_3 v^- \cdot N) B_N^- + (\partial_3 B^- \cdot N) [v_N] \quad \text{in } (0, T) \times \Gamma, \quad (3.7)$$

where we have omitted the sum over $j = 1, 2$. Next, we compute an evolution equation for the jump of the normal velocity $[v_N]$. To do so, we write in Ω^\pm :

$$\begin{aligned} \partial_t v_N^\pm &= \partial_t v^\pm \cdot N + v^\pm \cdot \partial_t N \\ &= \left(-(\tilde{v}^\pm \cdot \nabla) v^\pm + (\tilde{B}^\pm \cdot \nabla) B^\pm - A^T \nabla Q^\pm \right) \cdot N + v^\pm \cdot \partial_t N \\ &= -v_j^\pm \partial_j v^\pm \cdot N - \left(\frac{v_N^\pm - \partial_t \psi}{J} \right) \partial_3 v^\pm \cdot N \\ &\quad + B_j^\pm \partial_j B^\pm \cdot N + \left(\frac{B_N^\pm}{J} \right) \partial_3 B^\pm \cdot N \\ &\quad + v^\pm \cdot (-\partial_1 \partial_t \psi, -\partial_2 \partial_t \psi, 0) - A^T \nabla Q^\pm \cdot N. \end{aligned}$$

We commute the derivatives ∂_j ($j = 1, 2$) to make the quantities $\partial_j B_N^\pm$ and $\partial_j v_N^\pm$ appear:

$$\begin{aligned} \partial_t v_N^\pm &= -v_j^\pm (\partial_j v_N^\pm - v^\pm \cdot \partial_j N) - \left(\frac{v_N^\pm - \partial_t \psi}{J} \right) \partial_3 v^\pm \cdot N \\ &\quad + B_j^\pm (\partial_j B_N^\pm - B^\pm \cdot \partial_j N) + \left(\frac{B_N^\pm}{J} \right) \partial_3 B^\pm \cdot N \\ &\quad + v^\pm \cdot (-\partial_1 \partial_t \psi, -\partial_2 \partial_t \psi, 0) - A^T \nabla Q^\pm \cdot N. \end{aligned}$$

Now, we pass to the limit $x_3 \rightarrow 0_\pm$. On the one hand, using Proposition 3.1, we get:

$$\partial_t v_N^+ = -2v_j^+ \partial_j v_N^+ + v_j^+ v^+ \cdot \partial_j N - B_j^+ B^+ \cdot \partial_j N - A^T \nabla Q^+ \cdot N. \quad (3.8)$$

On the other hand, we have:

$$\begin{aligned} \partial_t v_N^- &= -v_j^- \partial_j v_N^- + v_j^- v^- \cdot \partial_j N + [v_N] (\partial_3 v^- \cdot N) \\ &\quad + B_j^- \partial_j B_N^- - B_j^- B^- \cdot \partial_j N + B_N^- (\partial_3 B^- \cdot N) \\ &\quad - v_j^- \partial_j v_N^+ - A^T \nabla Q^- \cdot N. \end{aligned} \quad (3.9)$$

We subtract (3.9) to (3.8), and use the definition of $\mathcal{G} = [A^T \nabla Q \cdot N]$ given by (3.3) to finally deduce that

$$\partial_t [v_N] + v_j^- \partial_j [v_N] + B_j^- \partial_j B_N^- = -(\partial_3 v^- \cdot N) [v_N] - (\partial_3 B^- \cdot N) B_N^- \quad \text{in } (0, T) \times \Gamma. \quad (3.10)$$

Let us define in $[0, T] \times \mathbb{T}^2$:

$$V := \begin{pmatrix} B_N^- \\ [v_N] \end{pmatrix}, \quad \mathcal{B}_j := \begin{pmatrix} v_j^- & B_j^- \\ B_j^- & v_j^- \end{pmatrix}, \quad \text{and} \quad \mathcal{C} := \begin{pmatrix} \partial_3 v^- \cdot N & \partial_3 B^- \cdot N \\ -\partial_3 B^- \cdot N & -\partial_3 v^- \cdot N \end{pmatrix}.$$

Then V is solution to the following symmetric hyperbolic system:

$$\begin{cases} \partial_t V + \sum_{j=1}^2 \mathcal{B}_j(t, x') \partial_j V = \mathcal{C}(t, x') V & \text{in } (0, T) \times \mathbb{T}^2, \\ V(0, \cdot) = 0 & \text{in } \mathbb{T}^2. \end{cases} \quad (3.11)$$

The solutions (v^\pm, B^\pm, f) being smooth enough, the matrix \mathcal{C} belongs to the space $L^\infty([0, T] \times \mathbb{T}^2)$ and the matrices \mathcal{B}_j to the space $L^\infty(0, T; W^{1, \infty}(\mathbb{T}^2))$. Using standard energy arguments (see e.g. [BGS07]), we thus deduce that $V \equiv 0$.

□

3.4. The divergence-free constraints. The divergence-free constraints in the new coordinates, namely $(A^T \nabla) \cdot v^\pm = 0$ and $(A^T \nabla) \cdot B^\pm = 0$, also turn out to be only restrictions on the initial data. Indeed, if all the constraints (3.5) hold at $t = 0$, then they will be propagated by the solutions of (3.4).

Proposition 3.3. *Let (v^\pm, B^\pm, f) be a smooth solution of (3.4), (3.1), (3.2), (3.3), in other words such that:*

$$(v^\pm, B^\pm, f) \in \mathcal{C}([0, T]; H^3(\Omega^\pm))^2 \times \mathcal{C}([0, T]; H^{3,5}(\mathbb{T}^2)).$$

If the initial data (v_0^\pm, B_0^\pm, f_0) satisfy the constraints (3.5), then for all $t \in [0, T]$ we have:

$$(A^T \nabla) \cdot v^\pm(t, \cdot) = (A^T \nabla) \cdot B^\pm(t, \cdot) = 0 \quad \text{in } \Omega^\pm.$$

Proof. We apply the operator $(A^T \nabla) \cdot$ to the first equation of (3.4):

$$\begin{aligned} \partial_t(A^T \nabla \cdot v^\pm) + \tilde{v}^\pm \cdot \nabla(A^T \nabla \cdot v^\pm) - \tilde{B}^\pm \cdot \nabla(A^T \nabla \cdot B^\pm) \\ + [A^T \nabla \cdot, \partial_t]v^\pm + [A^T \nabla \cdot, \tilde{v}^\pm \cdot \nabla]v^\pm - [A^T \nabla \cdot, \tilde{B}^\pm \cdot \nabla]B^\pm - \mathcal{F}^\pm = 0, \end{aligned} \quad (3.12)$$

where \mathcal{F}^\pm is given by (3.2). Let us check that

$$\mathcal{F}^\pm = [A^T \nabla \cdot, \partial_t]v^\pm + [A^T \nabla \cdot, \tilde{v}^\pm \cdot \nabla]v^\pm - [A^T \nabla \cdot, \tilde{B}^\pm \cdot \nabla]B^\pm, \quad (3.13)$$

which will simplify (3.12). The computation of the first commutator gives:

$$\begin{aligned} [A^T \nabla \cdot, \partial_t]v^\pm &= A^T \nabla \cdot (\partial_t v^\pm) - \partial_t(A^T \nabla \cdot v^\pm) \\ &= A_{ki} \partial_k \partial_t v_i^\pm - \partial_t(A_{ki} \partial_k v_i^\pm) = -\partial_t A_{ki} \partial_k v_i^\pm. \end{aligned}$$

Then, we have:

$$\begin{aligned} [A^T \nabla \cdot, \tilde{v}^\pm \cdot \nabla]v^\pm &= A^T \nabla \cdot ((\tilde{v}^\pm \cdot \nabla)v^\pm) - \tilde{v}^\pm \cdot \nabla(A^T \nabla \cdot v^\pm) \\ &= A_{ki} \partial_k (\tilde{v}_j \partial_j v_i^\pm) - \tilde{v}_j^\pm \partial_j (A_{ki} \partial_k v_i^\pm) \\ &= A_{ki} \partial_k \tilde{v}_j^\pm \partial_j v_i^\pm + A_{ki} \tilde{v}_j^\pm \partial_{jk} v_i^\pm - \tilde{v}_j^\pm A_{ki} \partial_{jk} v_i^\pm - \tilde{v}_j^\pm \partial_j A_{ki} \partial_k v_i^\pm \\ &= A_{ki} \partial_k \tilde{v}_j^\pm \partial_j v_i^\pm - \tilde{v}_j^\pm \partial_j A_{ki} \partial_k v_i^\pm. \end{aligned}$$

Likewise, we obtain:

$$[A^T \nabla \cdot, \tilde{B}^\pm \cdot \nabla]B^\pm = A_{ki} \partial_k \tilde{B}_j^\pm \partial_j B_i^\pm - \tilde{B}_j^\pm \partial_j A_{ki} \partial_k B_i^\pm.$$

Taking the definition (3.2) of \mathcal{F}^\pm into account, the equality (3.13) is satisfied. We deduce a first evolution equation:

$$\partial_t(A^T \nabla \cdot v^\pm) + \tilde{v}^\pm \cdot \nabla(A^T \nabla \cdot v^\pm) - \tilde{B}^\pm \cdot \nabla(A^T \nabla \cdot B^\pm) = 0. \quad (3.14)$$

Now, we apply the operator $(A^T \nabla) \cdot$ to the second equation of (3.4):

$$\begin{aligned} \partial_t(A^T \nabla \cdot B^\pm) + \tilde{v}^\pm \cdot \nabla(A^T \nabla \cdot B^\pm) - \tilde{B}^\pm \cdot \nabla(A^T \nabla \cdot v^\pm) \\ + [A^T \nabla \cdot, \partial_t]B^\pm + [A^T \nabla \cdot, \tilde{v}^\pm \cdot \nabla]B^\pm - [A^T \nabla \cdot, \tilde{B}^\pm \cdot \nabla]v^\pm = 0. \end{aligned} \quad (3.15)$$

First, we have:

$$[A^T \nabla \cdot, \partial_t]B^\pm = -\partial_t A_{ki} \partial_k B_i^\pm.$$

Then, the computation of the second commutator appearing in the second line of (3.15) gives:

$$\begin{aligned} [A^T \nabla \cdot, \tilde{v}^\pm \cdot \nabla]B^\pm &= A_{ki} \partial_k (\tilde{v}_j^\pm \partial_j B_i^\pm) - \tilde{v}_j^\pm \partial_j (A_{ki} \partial_k B_i^\pm) \\ &= A_{ki} \partial_k \tilde{v}_j^\pm \partial_j B_i^\pm + A_{ki} \tilde{v}_j^\pm \partial_{jk} B_i^\pm - \tilde{v}_j^\pm \partial_j A_{ki} \partial_k B_i^\pm - \tilde{v}_j^\pm A_{ki} \partial_{jk} B_i^\pm. \end{aligned}$$

Likewise, using the symmetry of the roles played by v^\pm and B^\pm , we obtain:

$$[A^T \nabla \cdot, \tilde{B}^\pm \cdot \nabla]v^\pm = A_{ki} \partial_k \tilde{B}_j^\pm \partial_j v_i^\pm + A_{ki} \tilde{B}_j^\pm \partial_{jk} v_i^\pm - \tilde{B}_j^\pm \partial_j A_{ki} \partial_k v_i^\pm - \tilde{B}_j^\pm A_{ki} \partial_{jk} v_i^\pm.$$

Finally, using the definitions of \tilde{v}^\pm and \tilde{B}^\pm given by (1.10), we check that the sum of the commutators in the second line of (3.15) vanishes:

$$[A^T \nabla \cdot, \partial_t]B^\pm + [A^T \nabla \cdot, \tilde{v}^\pm \cdot \nabla]B^\pm - [A^T \nabla \cdot, \tilde{B}^\pm \cdot \nabla]v^\pm = 0.$$

Therefore we deduce a second evolution equation:

$$\partial_t(A^T \nabla \cdot B^\pm) + \tilde{v}^\pm \cdot \nabla(A^T \nabla \cdot B^\pm) - \tilde{B}^\pm \cdot \nabla(A^T \nabla \cdot v^\pm) = 0. \quad (3.16)$$

For all $(t, x) \in (0, T) \times \Omega^\pm$, let us define

$$U^\pm(t, x) := \begin{pmatrix} A^T \nabla \cdot v^\pm(t, x) \\ A^T \nabla \cdot B^\pm(t, x) \end{pmatrix} \quad \text{and} \quad \mathcal{A}_j^\pm(t, x) := \begin{pmatrix} \tilde{v}_j^\pm(t, x) & -\tilde{B}_j^\pm(t, x) \\ -\tilde{B}_j^\pm(t, x) & \tilde{v}_j^\pm(t, x) \end{pmatrix}.$$

Then (3.14),(3.16) can be rewritten as a hyperbolic symmetric system with variable coefficients:

$$\begin{cases} \partial_t U^\pm + \sum_{j=1}^3 \mathcal{A}_j^\pm(t, x) \partial_j U^\pm = 0 & \text{in } (0, T) \times \Omega^\pm, \\ U^\pm(0, \cdot) = 0 & \text{in } \Omega^\pm. \end{cases} \quad (3.17)$$

We now proceed using the energy method. First, we begin with the case of U^+ . Taking the dot product (in $L^2(\Omega^+)$) of the first equation of (3.17) with U^+ , and integrating by parts, we get the following energy estimate (where we have used the symmetry of the matrices \mathcal{A}_j^+):

$$\frac{d}{dt} \int_{\Omega^+} |U^+(t, x)|^2 dx = \sum_{j=1}^3 \int_{\Omega^+} U^+ \cdot \partial_j \mathcal{A}_j^+ U^+ dx - \int_{\Gamma_+} (U^+ \cdot \mathcal{A}_3^+ U^+) |_{\Gamma_+} dx' + \int_{\Gamma} (U^+ \cdot \mathcal{A}_3^+ U^+) |_{\Gamma} dx'. \quad (3.18)$$

On Γ_+ , we know that $\psi = 0$ (see Proposition 2.1) and $v_3^+ = B_3^+ = 0$. As a consequence, the matrix $\mathcal{A}_3^+ |_{\Gamma_+}$ is **identically zero**. Next, the boundary conditions of the system (3.4) allow to write:

$$\tilde{v}_3^+ |_{\Gamma} = \frac{v^+ \cdot N - \partial_t f}{J} = 0.$$

Besides, using Proposition 3.1, we have:

$$\tilde{B}_3^+ |_{\Gamma} = \frac{B^+ \cdot N}{J} = 0.$$

Thus, the matrix $\mathcal{A}_3^+ |_{\Gamma}$ is also **identically zero**. The equality (3.18) readily gives:

$$\frac{d}{dt} \int_{\Omega^+} |U^+(t, x)|^2 dx = \sum_{j=1}^3 \int_{\Omega^+} U^+ \cdot \partial_j \mathcal{A}_j^+ U^+ dx \leq C \int_{\Omega^+} |U^+(t, x)|^2 dx, \quad (3.19)$$

so $U^+ \equiv 0$. Now, let us deal with the case of U^- . The method is similar, and we obtain first:

$$\frac{d}{dt} \int_{\Omega^-} |U^-(t, x)|^2 dx = \sum_{j=1}^3 \int_{\Omega^-} U^- \cdot \partial_j \mathcal{A}_j^- U^- dx + \int_{\Gamma_-} (U^- \cdot \mathcal{A}_3^- U^-) |_{\Gamma_-} dx' - \int_{\Gamma} (U^- \cdot \mathcal{A}_3^- U^-) |_{\Gamma} dx'. \quad (3.20)$$

On Γ_- , we can use the same arguments as previously, because of the two identities $v_3^- = B_3^- = 0$ and $\psi = 0$. Thus, the matrix $\mathcal{A}_3^- |_{\Gamma_-}$ vanishes. On Γ , we have to proceed more carefully: using Proposition 3.2, from now on we can use the equation $\partial_t f = v^- \cdot N$ (because $v^- \cdot N = v^+ \cdot N = \partial_t f$), to get:

$$\tilde{v}_3^- |_{\Gamma} = \frac{v^- \cdot N - \partial_t f}{J} = 0.$$

Likewise, also using Proposition 3.2, we have

$$\tilde{B}_3^- |_{\Gamma} = \frac{B_N^-}{J} = 0.$$

The identity (3.20) can be simplified like U^+ :

$$\frac{d}{dt} \int_{\Omega^-} |U^-(t, x)|^2 dx = \sum_{j=1}^3 \int_{\Omega^-} U^- \cdot \partial_j \mathcal{A}_j^- U^- dx. \quad (3.21)$$

Therefore $U^- \equiv 0$, and the proposition is proved. \square

We have thus derived an easier problem to study, namely (3.4),(3.1),(3.2),(3.3), with the constraints (3.5) on the initial data. In the next section we will be interested in the particular case of the pressure, which satisfies the coupled elliptic problem (3.1).

4. ANALYTIC ESTIMATE OF THE TOTAL PRESSURE

Within Section 3, we have shown that the total pressure Q^\pm can be implicitly expressed as a function of (v^\pm, B^\pm) by solving an elliptic problem. Consequently, we will need to get analytic estimates for Q^\pm depending on the analytic norms of v^\pm and B^\pm .

Let us recall the elliptic problem (3.1) satisfied by the total pressure Q^\pm , rewritten as follows:

$$\begin{cases} -A_{ji}\partial_j(A_{ki}\partial_k Q^\pm) = \mathcal{F}^\pm & \text{in } (0, T) \times \Omega^\pm, \\ [Q] = 0 & \text{on } (0, T) \times \Gamma, \\ (1 + |\nabla' f|^2)[\partial_3 Q] = \mathcal{G} & \text{on } (0, T) \times \Gamma, \\ \partial_3 Q^\pm = 0 & \text{on } (0, T) \times \Gamma_\pm. \end{cases} \quad (4.1)$$

We have used Einstein's summation convention over the indices i, j and k belonging to the set $\{1, 2, 3\}$. We recall the expression of the matrix A given by (1.8):

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{\partial_1 \psi}{J} & \frac{\partial_2 \psi}{J} & \frac{1}{J} \end{pmatrix}, \quad \text{with } J = 1 + \partial_3 \psi.$$

For the time being, the source terms \mathcal{F}^\pm and \mathcal{G} in (4.1) are any analytic functions belonging to one of the spaces $B_{\rho, r, \sigma}$ introduced earlier. In the end, we will come back to the particular case where \mathcal{F}^\pm and \mathcal{G} depend on (v^\pm, B^\pm) and are defined by (3.2) and (3.3).

To begin with, we admit that¹:

- if the coefficients A_{ji} given by (1.8) belong to $H^3(\Omega)$,
- if $\|\psi\|_{H^3} \leq \varepsilon_0$, for some numerical constant $\varepsilon_0 \in (0, 1)$,
- if the source terms \mathcal{F}^\pm and \mathcal{G} respectively belong to $H^1(\Omega^\pm)$ and $H^{1.5}(\Gamma)$,

then the problem (4.1) is well-posed in $H^3(\Omega^+) \times H^3(\Omega^-)$. In other terms, there exists a unique solution $(Q^+, Q^-) \in H^3(\Omega^+) \times H^3(\Omega^-)$ to the problem (4.1) such that $\sum_{\pm} \int_{\Omega^\pm} Q^\pm = 0$. Moreover, the solution satisfies a classical interior regularity property, namely:

$$\|Q^\pm\|_{3, \pm} \leq C_0 (\|\mathcal{F}^\pm\|_{1, \pm} + \|\mathcal{G}\|_{H^{1.5}(\Gamma)}), \quad (4.2)$$

where $C_0 > 0$ is a constant depending only on the norm $\|\psi\|_{H^3}$.

Furthermore, we will admit that the problem (4.1) is still well-posed in $H^\infty(\Omega^+) \times H^\infty(\Omega^-)$. In other words, if the coefficients A_{ji} and the source terms \mathcal{F}^\pm and \mathcal{G} are from now on in H^∞ , then there exists a unique zero mean solution (Q^+, Q^-) to the problem (4.1) that belongs to H^∞ (see [Eva98] for the general method). In the following, we shall require more than the H^∞ regularity. Indeed, we will take an analytic front f in the space $B_{\rho, \frac{\sigma}{2}}(\mathbb{T}^2)$. Therefore the lifting ψ will be in $B_{\rho, 4, \sigma}(\Omega)$ according to Proposition 2.1. Thus, the coefficients A_{ji} will belong to $B_{\rho, 3, \sigma}(\Omega)$. The source terms \mathcal{F}^\pm and \mathcal{G} will be taken respectively in the analytic spaces $B_{\rho, 1, \sigma}(\Omega^\pm)$ and $B_{\rho, \frac{3}{2}}(\Gamma)$. Working in such spaces remains consistent with estimate (4.2). Intuitively, if the radius of analyticity ρ and the parameter σ go to zero, these spaces “degenerate” into the Sobolev spaces $H^3(\Omega)$ (for the coefficients A_{ji}), $H^1(\Omega^\pm)$ and $H^{1.5}(\Gamma)$ (for the source terms \mathcal{F}^\pm and \mathcal{G}).

The purpose of this section is to extend estimate (4.2) to the analytic spaces mentioned above. In particular, we shall see that if we take both analytic coefficients and source terms, then the solution (Q^+, Q^-) to (4.1) will also be analytic.

4.1. Estimate of the tangential derivatives.

Remark: in the following, the notation $f \lesssim g$ means that there exists a numerical constant $C_0 > 0$ (that does not depend on the functions f and g , neither on other parameters like ρ or σ), such that $f \leq C_0 g$.

Within this paragraph, we focus on estimating Q^\pm in the space $B_{\rho, 3}^0(\Omega^\pm)$. Thus, we wish to estimate the following quantity:

$$\|Q^\pm\|_{\rho, 3}^{0, \pm} = \sum_{n \geq 0} \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|\partial^{\alpha'} Q^\pm\|_{3, \pm}.$$

¹The existence and uniqueness result for the solution (Q^+, Q^-) together with the estimate (4.2) can be obtained in a classical way, applying Lax-Milgram theorem in the space $H^1(\Omega)$ (with a zero mean condition). Next, to get an estimate in $H^2(\Omega^\pm)$, we estimate the difference quotients of order 2 (see [Eva98]). To gain one more derivative, we proceed by induction, and estimate in a suitable way all the appearing commutators (see [CMST12, p.268]). Let us notice that thanks to the flatness of the front f in $H^{2.5}(\mathbb{T}^2)$ (leading to a flatness of the norm $\|\psi\|_{H^3(\Omega)}$ using Proposition 2.1), the jacobian $J = 1 + \partial_3 \psi$ will satisfy $J \in [\frac{1}{2}, \frac{3}{2}]$.

Using (4.2), we shall be able to estimate the tangential derivatives $\partial^{\alpha'} Q^\pm$ in $H^3(\Omega^\pm)$. Multi-indices α in \mathbb{N}^3 will be noted $\alpha := (\alpha', \alpha_3)$, where $\alpha' \in \mathbb{N}^2$. In the following, the notation α' will always stand for a multi-index of \mathbb{N}^2 , to refer to the tangential derivatives ∂_1 and ∂_2 . For all $N \in \mathbb{N}$, we will note $\|Q\|_{\rho,3}^{0,\pm,N}$ the partial sum of order N of the norm $\|Q^\pm\|_{\rho,3}^{0,\pm}$. To estimate $\partial^{\alpha'} Q^\pm$ in $H^3(\Omega^\pm)$, we shall commute the problem (4.1) with $\partial^{\alpha'}$. Thus, estimate (4.2) will hold for $\partial^{\alpha'} Q^\pm$, with new source terms \mathcal{F}'^\pm and \mathcal{G}' . These latter contain commutators, composed of derivatives of Q^\pm and coefficients of the matrix A . To estimate these commutators in a suitable way, we will have to take the flatness condition required on the front f in the space $H^{2.5}(\mathbb{T}^2)$ into account. So, the norm $\|\psi\|_{H^3(\Omega)}$ will also be small according to Lemma 1.1. Consequently, if we write $A = I_3 - \tilde{A}$, with

$$\tilde{A} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial_1 \psi}{J} & \frac{\partial_2 \psi}{J} & \frac{\partial_3 \psi}{J} \end{pmatrix}, \quad (4.3)$$

then $\|\tilde{A}\|_{H^2(\Omega)}$ will also be a small quantity. It explains why during the commutators estimates thereafter, we will have to distinguish the cases where low derivatives of A appear, to take advantage of the flatness of the front f .

Now, let $\alpha' \in \mathbb{N}^2$ such that $|\alpha'| = n \geq 1$. The case $n = 0$ is a straightforward application of estimate (4.2), that gives:

$$\|Q^\pm\|_{3,\pm} \lesssim \|\mathcal{F}^\pm\|_{1,\pm} + \|\mathcal{G}\|_{H^{1.5}(\Gamma)} \lesssim \|\mathcal{F}^\pm\|_{\rho,1}^{0,\pm} + \|\mathcal{G}\|_{\rho,\frac{3}{2}}. \quad (4.4)$$

From now on, we shall note $Q'^\pm := \partial^{\alpha'} Q^\pm$. Let us commute problem (4.1) with $\partial^{\alpha'}$:

$$\begin{cases} -A_{ji} \partial_j (A_{ki} \partial_k Q'^\pm) = \mathcal{F}'^\pm & \text{in } (0, T) \times \Omega^\pm, \\ [Q'^\pm] = 0 & \text{on } (0, T) \times \Gamma, \\ (1 + |\nabla' f|^2) [\partial_3 Q'] = \mathcal{G}' & \text{on } (0, T) \times \Gamma, \\ \partial_3 Q' = 0 & \text{on } (0, T) \times \Gamma_\pm. \end{cases} \quad (4.5)$$

The source term \mathcal{F}'^\pm is defined by

$$\mathcal{F}'^\pm := \partial^{\alpha'} \mathcal{F}^\pm + [\partial^{\alpha'} ; A_{ji} \partial_j (A_{ki} \partial_k \cdot)] Q^\pm.$$

Writing $A = I_3 - \tilde{A}$, we can split \mathcal{F}'^\pm into four quantities as follows:

$$\begin{aligned} \mathcal{F}'^\pm &= \partial^{\alpha'} \mathcal{F}^\pm + [\partial^{\alpha'} ; \tilde{A}_{ji} \partial_j (\tilde{A}_{ki} \partial_k \cdot)] Q^\pm - [\partial^{\alpha'} ; \partial_i \tilde{A}_{ki} \partial_k \cdot] Q^\pm - 2[\partial^{\alpha'} ; \tilde{A}_{ji} \partial_{ji} \cdot] Q^\pm. \\ &=: F_1 + F_2 - F_3 + 2F_4. \end{aligned} \quad (4.6)$$

The boundary source term \mathcal{G}' is given by

$$\begin{aligned} \mathcal{G}' &:= \partial^{\alpha'} \mathcal{G} - [[\partial^{\alpha'} ; (1 + |\nabla' f|^2) \partial_3] Q] \\ &=: G_1 - G_2. \end{aligned} \quad (4.7)$$

Let us be careful: the exterior bracket in (4.7) stands for the jump across Γ , whereas the interior bracket corresponds to a commutator. Applying estimate (4.2) to the problem (4.5), we have:

$$\|Q'^\pm\|_{3,\pm} \lesssim \|\mathcal{F}'^\pm\|_{1,\pm} + \|\mathcal{G}'\|_{H^{1.5}(\Gamma)}. \quad (4.8)$$

Thus, the partial sum of order N of $\|Q^\pm\|_{\rho,3}^{0,\pm}$ satisfies:

$$\|Q^\pm\|_{\rho,3}^{0,\pm,N} \lesssim \sum_{n=0}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|\mathcal{F}'^\pm\|_{1,\pm} + \sum_{n=0}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|\mathcal{G}'\|_{H^{1.5}(\Gamma)}. \quad (4.9)$$

Estimate of \mathcal{F}'^\pm :

The case of F_1 is easy. Indeed, we have assumed that \mathcal{F}^\pm is an analytic function belonging to the space $B_{\rho,1,\sigma}(\Omega^\pm)$. In particular, the quantity $\|\mathcal{F}^\pm\|_{\rho,1}^{0,\pm}$ is finite, and we can write:

$$\sum_{n=0}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|F_1\|_{1,\pm} = \sum_{n=0}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|\partial^{\alpha'} \mathcal{F}^\pm\|_{1,\pm} \leq \|\mathcal{F}^\pm\|_{\rho,1}^{0,\pm}. \quad (4.10)$$

Let us deal with the first commutator given by $F_2 := [\partial^{\alpha'} ; \tilde{A}_{ji} \partial_j (\tilde{A}_{ki} \partial_k \cdot)] Q^\pm$. Expanding the latter expression, we have:

$$F_2 = \sum_{\substack{\beta'+\gamma'+\delta'=\alpha' \\ |\beta'+\gamma'| \geq 1}} \frac{\alpha'!}{\beta'! \gamma'! \delta'!} \left(\partial^{\beta'} \tilde{A}_{ji} \partial^{\gamma'} \partial_j \tilde{A}_{ki} \partial^{\delta'} \partial_k Q^\pm + \partial^{\beta'} \tilde{A}_{ji} \partial^{\gamma'} \tilde{A}_{ki} \partial^{\delta'} \partial_{jk} Q^\pm \right). \quad (4.11)$$

Let us set $T_{01} := \partial^{\beta'} \tilde{A}_{ji} \partial^{\gamma'} \partial_j \tilde{A}_{ki} \partial^{\delta'} \partial_k Q^\pm$ and $T_{02} := \partial^{\beta'} \tilde{A}_{ji} \partial^{\gamma'} \tilde{A}_{ki} \partial^{\delta'} \partial_{jk} Q^\pm$. In order to estimate the terms T_{01} and T_{02} in $H^1(\Omega^\pm)$, we first estimate the 0^{th} -order derivative, and then the 1^{st} -order derivatives in $L^2(\Omega^\pm)$. In the following, ∂^m denotes any m^{th} -order derivative, for all $m \geq 1$. All the following estimates are computed using Hölder's inequality and Sobolev embeddings recalled below (U stands for any domain among Ω , Ω^+ or Ω^-):

$$\begin{aligned} H^1(U) &\hookrightarrow L^p(U), \quad \forall 1 \leq p \leq 6, \\ H^2(U) &\hookrightarrow L^\infty(U). \end{aligned}$$

We start with the 0^{th} -order derivative:

$$\begin{aligned} \|T_{01}\|_{0,\pm} &= \|\partial^{\beta'} \tilde{A}_{ji} \partial^{\gamma'} \partial_j \tilde{A}_{ki} \partial^{\delta'} \partial_k Q^\pm\|_{0,\pm} \\ &\lesssim \|\partial^{\beta'} \tilde{A}\|_{H^2} \|\partial^{\gamma'} \tilde{A}\|_{H^1} \|\partial^{\delta'} Q^\pm\|_{3,\pm}, \end{aligned} \quad (\text{Hölder's inequality in } L^\infty \times L^2 \times L^\infty).$$

In the norms, we will write H^r to denote the space $H^r(\Omega)$ (and only for the domain Ω). Likewise, we get:

$$\|T_{02}\|_{0,\pm} \lesssim \|\partial^{\beta'} \tilde{A}\|_{H^2} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \|\partial^{\delta'} Q^\pm\|_{2,\pm}, \quad (\text{Hölder's inequality in } L^\infty \times L^\infty \times L^2).$$

Then, we compute the 1^{st} -order derivatives. We write $\partial^1 T_{01} = T_{11} + T_{12} + T_{13}$, with:

$$T_{11} = \partial^{\beta'} \partial^1 \tilde{A}_{ji} \partial^{\gamma'} \partial^1 \tilde{A}_{ki} \partial^{\delta'} \partial^1 Q^\pm, \quad T_{12} = \partial^{\beta'} \tilde{A}_{ji} \partial^{\gamma'} \partial^2 \tilde{A}_{ki} \partial^{\delta'} \partial^1 Q^\pm, \quad T_{13} = \partial^{\beta'} \tilde{A}_{ji} \partial^{\gamma'} \partial^1 \tilde{A}_{ki} \partial^{\delta'} \partial^2 Q^\pm.$$

As done before, applying Hölder's inequality in $L^4 \times L^4 \times L^\infty$ for T_{11} , in $L^\infty \times L^2 \times L^\infty$ for T_{12} and in $L^\infty \times L^4 \times L^4$ for T_{13} , we get:

$$\|T_{11}\|_{0,\pm}, \|T_{12}\|_{0,\pm}, \|T_{13}\|_{0,\pm} \lesssim \|\partial^{\beta'} \tilde{A}\|_{H^2} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \|\partial^{\delta'} Q^\pm\|_{3,\pm}. \quad (4.12)$$

The case of T_{02} is analogous. We write $\partial^1 T_{02} = T_{14} + T_{15} + T_{16}$, with:

$$T_{14} = \partial^{\beta'} \partial^1 \tilde{A}_{ji} \partial^{\gamma'} \tilde{A}_{ki} \partial^{\delta'} \partial^2 Q^\pm, \quad T_{15} = \partial^{\beta'} \tilde{A}_{ji} \partial^{\gamma'} \partial^1 \tilde{A}_{ki} \partial^{\delta'} \partial^2 Q^\pm, \quad T_{16} = \partial^{\beta'} \tilde{A}_{ji} \partial^{\gamma'} \tilde{A}_{ki} \partial^{\delta'} \partial^3 Q^\pm.$$

Using Hölder's inequality in $L^4 \times L^\infty \times L^4$ for T_{14} , in $L^\infty \times L^4 \times L^4$ for T_{15} and in $L^\infty \times L^\infty \times L^2$ for T_{16} , we also show that the terms T_{14} , T_{15} and T_{16} satisfy the estimate (4.12). To sum up, we can write:

$$\|T_{01} + T_{02}\|_{1,\pm} \lesssim \|\partial^{\beta'} \tilde{A}\|_{H^2} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \|\partial^{\delta'} Q^\pm\|_{3,\pm}. \quad (4.13)$$

From now on, we can estimate the commutator F_2 given by (4.11). Applying (4.13), we have:

$$\|F_2\|_{1,\pm} \lesssim \sum_{\substack{\beta'+\gamma'+\delta'=\alpha' \\ |\beta'+\gamma'| \geq 1}} \frac{\alpha'!}{\beta'! \gamma'! \delta'!} \|\partial^{\beta'} \tilde{A}\|_{H^2} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \|\partial^{\delta'} Q^\pm\|_{3,\pm}.$$

Summing over the length of multi-indices (recall that $|\alpha'| = n$), we obtain:

$$\|F_2\|_{1,\pm} \lesssim \sum_{\substack{i_1+i_2+i_3=n \\ i_1+i_2 \geq 1}} \left(\sum_{\substack{\beta'+\gamma'+\delta'=\alpha' \\ |\beta'|=i_1 \\ |\gamma'|=i_2 \\ |\delta'|=i_3}} \frac{\alpha'!}{\beta'! \gamma'! \delta'!} \right) \max_{\substack{\beta'+\gamma'+\delta'=\alpha' \\ |\beta'|=i_1 \\ |\gamma'|=i_2 \\ |\delta'|=i_3}} \left(\|\partial^{\beta'} \tilde{A}\|_{H^2} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \|\partial^{\delta'} Q^\pm\|_{3,\pm} \right). \quad (4.14)$$

Next, we use the following identity to simplify the second sum in (4.14):

$$\sum_{\substack{\beta'+\gamma'+\delta'=\alpha' \\ |\beta'|=i_1, |\gamma'|=i_2, |\delta'|=i_3}} \frac{\alpha'!}{\beta'! \gamma'! \delta'!} = \frac{n!}{i_1! i_2! i_3!}.$$

Then, bounding from above the maximum of the products by the product of the maximums, we get:

$$\|F_2\|_{1,\pm} \lesssim \sum_{\substack{i_1+i_2+i_3=n \\ i_1+i_2 \geq 1}} \frac{n!}{i_1! i_2! i_3!} \max_{|\beta'|=i_1} \|\partial^{\beta'} \tilde{A}\|_{H^2} \max_{|\gamma'|=i_2} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \max_{|\delta'|=i_3} \|\partial^{\delta'} Q^\pm\|_{3,\pm}.$$

Now we have to use that i_3 can not be larger than $n-1$. So, we rewrite the previous sum using a summation over the values of $i_1 + i_2$:

$$\begin{aligned} \frac{1}{n!} \|F_2\|_{1,\pm} &\lesssim \sum_{i=1}^n \sum_{i_1+i_2=i} \frac{1}{i_1!} \max_{|\beta'|=i_1} \|\partial^{\beta'} \tilde{A}\|_{H^2} \frac{1}{i_2!} \max_{|\gamma'|=i_2} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \frac{1}{(n-i)!} \max_{|\delta'|=n-i} \|\partial^{\delta'} Q^\pm\|_{3,\pm} \\ &\lesssim \sum_{i=0}^{n-1} \sum_{i_1+i_2=i+1} \frac{1}{i_1!} \max_{|\beta'|=i_1} \|\partial^{\beta'} \tilde{A}\|_{H^2} \frac{1}{i_2!} \max_{|\gamma'|=i_2} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \frac{1}{(n-1-i)!} \max_{|\delta'|=n-1-i} \|\partial^{\delta'} Q^\pm\|_{3,\pm}, \end{aligned} \quad (4.15)$$

where the sum in (4.15) has been re-indexed over i . For the sake of clarity, we rewrite the sum $\sum_{i_1+i_2=i+1}$ using only one index $i_1 \in \{0, \dots, i+1\}$:

$$\begin{aligned} \frac{1}{n!} \|F_2\|_{1,\pm} &\lesssim \sum_{i=0}^{n-1} \sum_{i_1=0}^{i+1} \frac{1}{i_1!} \max_{|\beta'|=i_1} \|\partial^{\beta'} \tilde{A}\|_{H^2} \frac{1}{(i+1-i_1)!} \max_{|\gamma'|=i+1-i_1} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \\ &\quad \times \frac{1}{(n-1-i)!} \max_{|\delta'|=n-1-i} \|\partial^{\delta'} Q^\pm\|_{3,\pm}. \end{aligned} \quad (4.16)$$

We recall that we have to distinguish the cases where we can exhibit the norms $\|\tilde{A}\|_{H^2}$ because of their smallness. So, in the estimate (4.16), we deal with the case $i=0$ for which we have the norm $\|\tilde{A}\|_{H^2}$, and the case $i \geq 1$ which will give some analytic norms. The latter will be handled thereafter using the radius of analyticity $\rho > 0$ as a small parameter. We rewrite the double sum in (4.16) as $S_{i=0} + S_{i \geq 1}$. The term $S_{i=0}$ corresponds to the case $i=0$, and the term $S_{i \geq 1}$ to the case where we sum over $i \geq 1$.

► **Treatment of $S_{i=0}$:**

We directly have the following estimate

$$\begin{aligned} S_{i=0} &= 2\|\tilde{A}\|_{H^2} \max_{|\gamma'|=1} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \frac{1}{(n-1)!} \max_{|\delta'|=n-1} \|\partial^{\delta'} Q^\pm\|_{3,\pm} \\ &\lesssim \|\tilde{A}\|_{H^2} \|\tilde{A}\|_{H^3} \frac{1}{(n-1)!} \max_{|\delta'|=n-1} \|\partial^{\delta'} Q^\pm\|_{3,\pm}. \end{aligned} \quad (4.17)$$

► **Treatment of $S_{i \geq 1}$:**

Remark: if $n=1$, the term $S_{i \geq 1}$ does not appear in the right side of (4.16). Consequently, the case $n=1$ is completely treated using the estimate (4.17). So, we can assume in the following that $n \geq 2$.

Re-indexing over i , we can write:

$$S_{i \geq 1} = \sum_{i=0}^{n-2} \sum_{i_1=0}^{i+2} \frac{1}{i_1!} \max_{|\beta'|=i_1} \|\partial^{\beta'} \tilde{A}\|_{H^2} \frac{1}{(i+2-i_1)!} \max_{|\gamma'|=i+2-i_1} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \frac{1}{(n-2-i)!} \max_{|\delta'|=n-2-i} \|\partial^{\delta'} Q^\pm\|_{3,\pm}.$$

Once more, we isolate the case $i_1 \in \{0, i+2\}$, which gives the norm $\|\tilde{A}\|_{H^2}$. Thus we get:

$$S_{i \geq 1} = 2\|\tilde{A}\|_{H^2} \sum_{i=0}^{n-2} \frac{1}{(i+2)!} \max_{|\gamma'|=i+2} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \frac{1}{(n-2-i)!} \max_{|\delta'|=n-2-i} \|\partial^{\delta'} Q^\pm\|_{3,\pm} \quad (4.18)$$

$$+ \sum_{i=0}^{n-2} \sum_{i_1=1}^{i+1} \frac{1}{i_1!} \max_{|\beta'|=i_1} \|\partial^{\beta'} \tilde{A}\|_{H^2} \frac{1}{(i+2-i_1)!} \max_{|\gamma'|=i+2-i_1} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \frac{1}{(n-2-i)!} \max_{|\delta'|=n-2-i} \|\partial^{\delta'} Q^\pm\|_{3,\pm}. \quad (4.19)$$

We shall note $S_{i_1=0, i+2}$ (resp. $S_{1 \leq i_1 \leq i+1}$) the right term of (4.18) (resp. (4.19)).

◆ **Treatment of $S_{i_1=0, i+2}$:**

Using both following trivial estimates

$$\max_{|\gamma'|=i+2} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \leq \max_{|\gamma'|=i+1} \|\partial^{\gamma'} \tilde{A}\|_{H^3} \quad \text{and} \quad (i+2)! \geq (i+1)!, \quad (4.20)$$

and re-indexing over i , we obtain:

$$S_{i_1=0, i+2} \lesssim \|\tilde{A}\|_{H^2} \sum_{i=1}^{n-1} \frac{1}{i!} \max_{|\gamma'|=i} \|\partial^{\gamma'} \tilde{A}\|_{H^3} \frac{1}{(n-1-i)!} \max_{|\delta'|=n-1-i} \|\partial^{\delta'} Q^\pm\|_{3,\pm}. \quad (4.21)$$

Eventually, the estimate (4.21) is rewritten as follows:

$$S_{i_1=0, i+2} \lesssim \|\tilde{A}\|_{H^2} \sum_{i_1+i_2=n-1} \frac{1}{i_1!} \max_{|\gamma'|=i_1} \|\partial^{\gamma'} \tilde{A}\|_{H^3} \frac{1}{i_2!} \max_{|\delta'|=i_2} \|\partial^{\delta'} Q^\pm\|_{3,\pm}. \quad (4.22)$$

This concludes the case of $S_{i_1=0, i+2}$.

◆ **Treatment of $S_{1 \leq i_1 \leq i+1}$:**

Re-indexing over i_1 , we have:

$$S_{1 \leq i_1 \leq i+1} = \sum_{i=0}^{n-2} \sum_{i_1=0}^i \frac{1}{(i_1+1)!} \max_{|\beta'|=i_1+1} \|\partial^{\beta'} \tilde{A}\|_{H^2} \frac{1}{(i+1-i_1)!} \max_{|\gamma'|=i+1-i_1} \|\partial^{\gamma'} \tilde{A}\|_{H^2} \\ \times \frac{1}{(n-2-i)!} \max_{|\delta'|=n-2-i} \|\partial^{\delta'} Q^\pm\|_{3,\pm}.$$

Using the same kind of inequalities given by (4.20), we deduce that

$$S_{1 \leq i_1 \leq i+1} \lesssim \sum_{i=0}^{n-2} \sum_{i_1=0}^i \frac{1}{i_1!} \max_{|\beta'|=i_1} \|\partial^{\beta'} \tilde{A}\|_{H^3} \frac{1}{(i-i_1)!} \max_{|\gamma'|=i-i_1} \|\partial^{\gamma'} \tilde{A}\|_{H^3} \frac{1}{(n-2-i)!} \max_{|\delta'|=n-2-i} \|\partial^{\delta'} Q^\pm\|_{3,\pm}.$$

Finally, rewriting the sum over i_1 with two indices i_1 and i_2 such that $i_1 + i_2 = i$, we end up with:

$$S_{1 \leq i_1 \leq i+1} \lesssim \sum_{i=0}^{n-2} \sum_{i_1+i_2=i} \frac{1}{i_1!} \max_{|\beta'|=i_1} \|\partial^{\beta'} \tilde{A}\|_{H^3} \frac{1}{i_2!} \max_{|\gamma'|=i_2} \|\partial^{\gamma'} \tilde{A}\|_{H^3} \frac{1}{(n-2-i)!} \max_{|\delta'|=n-2-i} \|\partial^{\delta'} Q^\pm\|_{3,\pm} \\ \lesssim \sum_{i_1+i_2+i_3=n-2} \frac{1}{i_1!} \max_{|\beta'|=i_1} \|\partial^{\beta'} \tilde{A}\|_{H^3} \frac{1}{i_2!} \max_{|\gamma'|=i_2} \|\partial^{\gamma'} \tilde{A}\|_{H^3} \frac{1}{i_3!} \max_{|\delta'|=i_3} \|\partial^{\delta'} Q^\pm\|_{3,\pm}, \quad (4.23)$$

which concludes the estimate of $S_{1 \leq i_1 \leq i+1}$.

Let us come back to the estimate (4.16) of the commutator F_2 . We use (4.17) together with (4.22) and (4.23) to get, for $|\alpha'| = n \geq 2$:

$$\frac{\rho^n}{n!} \max_{|\alpha'|=n} \|F_2\|_{1,\pm} \lesssim \rho \|\tilde{A}\|_{H^2} \|\tilde{A}\|_{H^3} \frac{\rho^{n-1}}{(n-1)!} \max_{|\delta'|=n-1} \|\partial^{\delta'} Q^\pm\|_{3,\pm} \quad (4.24)$$

$$+ \rho \|\tilde{A}\|_{H^2} \sum_{i_1+i_2=n-1} \frac{\rho^{i_1}}{i_1!} \max_{|\gamma'|=i_1} \|\partial^{\gamma'} \tilde{A}\|_{H^3} \frac{\rho^{i_2}}{i_2!} \max_{|\delta'|=i_2} \|\partial^{\delta'} Q^\pm\|_{3,\pm} \quad (4.25)$$

$$+ \rho^2 \sum_{i_1+i_2+i_3=n-2} \frac{\rho^{i_1}}{i_1!} \max_{|\beta'|=i_1} \|\partial^{\beta'} \tilde{A}\|_{H^3} \frac{\rho^{i_2}}{i_2!} \max_{|\gamma'|=i_2} \|\partial^{\gamma'} \tilde{A}\|_{H^3} \frac{\rho^{i_3}}{i_3!} \max_{|\delta'|=i_3} \|\partial^{\delta'} Q^\pm\|_{3,\pm}. \quad (4.26)$$

If $n = 1$, we recall that we have the simpler estimate:

$$\frac{\rho^n}{n!} \max_{|\alpha'|=n} \|F_2\|_{1,\pm} \lesssim \rho \|\tilde{A}\|_{H^2} \|\tilde{A}\|_{H^3} \frac{\rho^{n-1}}{(n-1)!} \max_{|\delta'|=n-1} \|\partial^{\delta'} Q^\pm\|_{3,\pm},$$

which turns out to be the same as (4.24). In the sums appearing in (4.25) and (4.26), we recognize ‘‘partial’’ Cauchy products. Thus, summing over $n \in \{1, \dots, N\}$ and using the straightforward inequality $\|\tilde{A}\|_{H^3} \leq \|\tilde{A}\|_{\rho,3}^0$, we obtain:

$$\sum_{n=1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|F_2\|_{1,\pm} \lesssim \rho \|\tilde{A}\|_{H^2} \|\tilde{A}\|_{\rho,3}^0 \|Q^\pm\|_{\rho,3}^{0,\pm,N-1} + \rho \|\tilde{A}\|_{H^2} \|\tilde{A}\|_{\rho,3}^{0,N-1} \|Q^\pm\|_{\rho,3}^{0,\pm,N-1} \\ + \rho^2 \|\tilde{A}\|_{\rho,3}^{0,N-2} \|\tilde{A}\|_{\rho,3}^{0,N-2} \|Q^\pm\|_{\rho,3}^{0,\pm,N-2}.$$

Since the coefficients of \tilde{A} belong to the space $B_{\rho,3,\sigma}(\Omega)$ (see page 17), we can bound from above the partial sums $\|\tilde{A}\|_{\rho,3}^{0,M}$ by the finite quantity $\|\tilde{A}\|_{\rho,3}^0$. Eventually, the commutator F_2 satisfies the estimate:

$$\sum_{n=1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|F_2\|_{1,\pm} \lesssim \rho \|\tilde{A}\|_{H^2} \|\tilde{A}\|_{\rho,3}^0 \|Q^\pm\|_{\rho,3}^{0,\pm,N} + \left(\rho \|\tilde{A}\|_{\rho,3}^0\right)^2 \|Q^\pm\|_{\rho,3}^{0,\pm,N}. \quad (4.27)$$

This concludes the case of the first commutator F_2 .

Let us move on to the second commutator F_3 given by (see (4.6)):

$$F_3 := [\partial^{\alpha'} ; \partial_i \tilde{A}_{ki} \partial_k \cdot] Q^\pm = \sum_{\substack{\beta'+\gamma'=\alpha' \\ |\beta'| \geq 1}} \frac{\alpha'!}{\beta'! \gamma'!} \partial^{\beta'} \partial_i \tilde{A}_{ki} \partial^{\gamma'} \partial_k Q^\pm. \quad (4.28)$$

We estimate F_3 with the same tools used for F_2 . First of all, considering the derivatives of order 0 and 1, we show that

$$\|\partial^{\beta'} \partial_i \tilde{A}_{ki} \partial^{\gamma'} \partial_k Q^\pm\|_{1,\pm} \lesssim \|\partial^{\beta'} \tilde{A}\|_{H^2} \|\partial^{\gamma'} Q^\pm\|_{3,\pm}. \quad (4.29)$$

It leads to the following estimate ($n \geq 1$):

$$\frac{1}{n!} \max_{|\alpha'|=n} \|F_3\|_{1,\pm} \lesssim \sum_{i_1=1}^n \frac{1}{i_1!} \max_{|\beta'|=i_1} \|\partial^{\beta'} \tilde{A}\|_{H^2} \frac{1}{(n-i_1)!} \max_{|\gamma'|=n-i_1} \|\partial^{\gamma'} Q^\pm\|_{3,\pm}.$$

Using similar inequalities as in (4.20), we can write

$$\frac{1}{n!} \max_{|\alpha'|=n} \|F_3\|_{1,\pm} \lesssim \sum_{i_1+i_2=n-1} \frac{1}{i_1!} \max_{|\beta'|=i_1} \|\partial^{\beta'} \tilde{A}\|_{H^3} \frac{1}{i_2!} \max_{|\gamma'|=i_2} \|\partial^{\gamma'} Q^\pm\|_{3,\pm}.$$

Summing over $n \in \{1, \dots, N\}$, we end up with:

$$\sum_{n=1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|F_3\|_{1,\pm} \lesssim \rho \|\tilde{A}\|_{\rho,3}^0 \|Q^\pm\|_{\rho,3}^{0,\pm,N}. \quad (4.30)$$

Let us finish with the last commutator F_4 of \mathcal{F}' :

$$F_4 := [\partial^{\alpha'} ; \tilde{A}_{ji} \partial_{ji} \cdot] Q^\pm = \sum_{\substack{\beta'+\gamma'=\alpha' \\ |\beta'| \geq 1}} \frac{\alpha!}{\beta! \gamma!} \partial^{\beta'} \tilde{A}_{ki} \partial^{\gamma'} \partial_{ki} Q^\pm. \quad (4.31)$$

In the same spirit as (4.29), we show that

$$\|\partial^{\beta'} \tilde{A}_{ki} \partial^{\gamma'} \partial_{ki} Q^\pm\|_{1,\pm} \lesssim \|\partial^{\beta'} \tilde{A}\|_{H^2} \|\partial^{\gamma'} Q^\pm\|_{3,\pm}.$$

This estimate is the same as (4.29). Consequently, we also conclude that

$$\sum_{n=1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|F_4\|_{1,\pm} \lesssim \rho \|\tilde{A}\|_{\rho,3}^0 \|Q^\pm\|_{\rho,3}^{0,\pm,N}. \quad (4.32)$$

This achieves the estimate of the source term \mathcal{F}^\pm . Let us sum up below the final estimate satisfied by \mathcal{F}^\pm , combining (4.10), (4.27), (4.30) and (4.32):

$$\begin{aligned} \sum_{n=1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|\mathcal{F}^\pm\|_{1,\pm} &\lesssim \|\mathcal{F}^\pm\|_{\rho,1}^0 + \rho \|\tilde{A}\|_{\rho,3}^0 \left(1 + \|\tilde{A}\|_{H^2}\right) \|Q^\pm\|_{\rho,3}^{0,\pm,N} \\ &\quad + \left(\rho \|\tilde{A}\|_{\rho,3}^0\right)^2 \|Q^\pm\|_{\rho,3}^{0,\pm,N}. \end{aligned} \quad (4.33)$$

Now, we shall proceed in the same way to estimate the boundary source term \mathcal{G}' .

Estimate of \mathcal{G}' :

We recall that \mathcal{G}' is defined by (4.7). The case of G_1 is identical to the interior source term F_1 . We obtain the same type of estimate as (4.10), namely:

$$\sum_{n=0}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|G_1\|_{H^{1,5}(\Gamma)} = \sum_{n=0}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|\partial^{\alpha'} \mathcal{G}\|_{H^{1,5}(\Gamma)} \leq \|\mathcal{G}\|_{\rho, \frac{3}{2}}. \quad (4.34)$$

Now, we treat the commutator G_2 . To do so, we use the continuity of the trace map from $H^2(\Omega^\pm)$ to $H^{1,5}(\Gamma)$, in order to eliminate the jump across Γ . Then, we also use Proposition 2.1 to make ψ appear in the estimates:

$$\|G_2\|_{H^{1,5}(\Gamma)} = \left\| \left[\partial^{\alpha'} ; (1 + |\nabla' f|^2) \partial_3 \right] Q \right\|_{H^{1,5}(\Gamma)} \lesssim \left\| \left[\partial^{\alpha'} ; (1 + \partial_h \psi \partial_h \psi) \partial_3 \right] Q^\pm \right\|_{2,\pm},$$

where we have adopted Einstein's summation convention over the index $h \in \{1, 2\}$. Thus we set

$$G_2^\pm := \left[\partial^{\alpha'} ; (1 + \partial_h \psi \partial_h \psi) \partial_3 \right] Q^\pm.$$

We expand to obtain:

$$G_2^\pm = \sum_{\substack{\beta'+\gamma'+\delta'=\alpha' \\ |\beta'+\gamma'| \geq 1}} \frac{\alpha!}{\beta! \gamma! \delta!} \partial^{\beta'} \partial_h \psi \partial^{\gamma'} \partial_h \psi \partial^{\delta'} \partial_3 Q^\pm.$$

Using the algebra property of the space $H^2(\Omega^\pm)$, we have

$$\|G_2\|_{H^{1,5}(\Gamma)} \lesssim \sum_{\substack{\beta'+\gamma'+\delta'=\alpha' \\ |\beta'+\gamma'| \geq 1}} \frac{\alpha!}{\beta! \gamma! \delta!} \|\partial^{\beta'} \psi\|_{H^3} \|\partial^{\gamma'} \psi\|_{H^3} \|\partial^{\delta'} Q^\pm\|_{3,\pm}.$$

The computations are exactly the same as the commutator F_2 defined by (4.11). Here, ψ plays the role of \tilde{A} , whose tangential derivatives are estimated in H^3 instead of H^2 for \tilde{A} . It remains consistent, because the definition of \tilde{A} only gives 1st-order derivatives of ψ . Thus the functions \tilde{A} and ψ are estimated in the same Sobolev scale. We get the following estimate for G_2 (analogous to (4.27)):

$$\sum_{n=2}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|G_2\|_{H^{1.5}(\Gamma)} \lesssim \rho \|\psi\|_{H^3} \|\psi\|_{\rho,4}^0 \|Q^\pm\|_{\rho,3}^{0,\pm,N} + \left(\rho \|\psi\|_{\rho,4}^0\right)^2 \|Q^\pm\|_{\rho,3}^{0,\pm,N}. \quad (4.35)$$

We notice that the norm $\|\psi\|_{H^3}$ as well as $\|\tilde{A}\|_{H^2}$ will be small quantities, because of the flatness of the front f in $H^{2.5}(\mathbb{T}^2)$. Below, we sum up the final estimate of \mathcal{G}' , combining (4.34) together with (4.35):

$$\sum_{n=1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n} \|\mathcal{G}'\|_{H^{1.5}(\Gamma)} \lesssim \|\mathcal{G}\|_{\rho,\frac{3}{2}} + \rho \|\psi\|_{H^3} \|\psi\|_{\rho,4}^0 \|Q^\pm\|_{\rho,3}^{0,\pm,N} + \left(\rho \|\psi\|_{\rho,4}^0\right)^2 \|Q^\pm\|_{\rho,3}^{0,\pm,N}. \quad (4.36)$$

Estimate of Q'^\pm :

Let us come back to the estimate (4.9). Using the inequalities (4.33) and (4.36) holding for $n \geq 1$, and (4.4) holding for $n = 0$, we deduce that:

$$\begin{aligned} \|Q^\pm\|_{\rho,3}^{0,\pm,N} &\lesssim \|\mathcal{F}^\pm\|_{\rho,1}^{0,\pm} + \|\mathcal{G}\|_{\rho,\frac{3}{2}} \\ &\quad + \rho \left((1 + \|\tilde{A}\|_{H^2}) \|\tilde{A}\|_{\rho,3}^0 + \|\psi\|_{H^3} \|\psi\|_{\rho,4}^0 \right) \|Q^\pm\|_{\rho,3}^{0,\pm,N} \\ &\quad + \rho^2 \left(\|\tilde{A}\|_{\rho,3}^0 \|\tilde{A}\|_{\rho,3}^0 + \|\psi\|_{\rho,4}^0 \|\psi\|_{\rho,4}^0 \right) \|Q^\pm\|_{\rho,3}^{0,\pm,N}. \end{aligned} \quad (4.37)$$

Conclusion:

Now we complete the estimate (4.37), absorbing the analytic terms on the right hand side. We consider $f \in B_{\rho,\frac{7}{2}}(\mathbb{T}^2)$ satisfying both following conditions:

$$\|f\|_{H^{2.5}(\mathbb{T}^2)} < \varepsilon_0, \quad (4.38a)$$

$$\|f\|_{\rho,\frac{7}{2}} < \eta_0, \quad (4.38b)$$

where the numerical constant $\eta_0 > 0$ is given by Corollary 2.3. The parameter $\varepsilon_0 > 0$ (a numerical constant) is chosen small enough in order to have $\frac{1}{2} \leq J \leq \frac{3}{2}$ and the interior regularity estimate (4.2) (see [CMST12, p.250, 271]). From now on, $C_0 > 0$ will denote any numerical constant that may depend on ε_0 , but not on the parameters ρ and σ . Using the assumption (4.38a), we have the inequality $\|J^{-1}\|_{H^2} \leq C_0$. Therefore the algebra property of $H^2(\Omega)$ together with Lemma 1.1 give:

$$\|\tilde{A}\|_{H^2} \leq C_0.$$

Next, we estimate the analytic norms in the following way:

$$\begin{aligned} \|\tilde{A}\|_{\rho,3}^0 &= \|J^{-1}\nabla\psi\|_{\rho,3}^0 \leq \|J^{-1}\nabla\psi\|_{\rho,3,\sigma} \\ &\leq C_0 \|J^{-1}\|_{\rho,3,\sigma} \|\psi\|_{\rho,4,\sigma} && \text{(algebra property of } B_{\rho,3,\sigma}(\Omega)\text{)} \\ &\leq C_0 \|J^{-1}\|_{\rho,3,\sigma} \|f\|_{\rho,\frac{7}{2}} && \text{(Proposition 2.1)} \\ &\leq C_0 \eta_0. && \text{(Corollary 2.3 and assumption (4.38b))} \end{aligned}$$

So, we have first:

$$\rho \left((1 + \|\tilde{A}\|_{H^2}) \|\tilde{A}\|_{\rho,3}^0 + \|\psi\|_{H^3} \|\psi\|_{\rho,4}^0 \right) \leq C_0 \eta_0 \rho.$$

Then, we also get:

$$\rho^2 \left(\|\tilde{A}\|_{\rho,3}^0 \|\tilde{A}\|_{\rho,3}^0 + \|\psi\|_{\rho,4}^0 \|\psi\|_{\rho,4}^0 \right) \leq C_0 \eta_0^2 \rho^2.$$

For instance, let us define

$$\rho_0 := \min \left\{ 1, \frac{1}{2C_0\eta_0} \right\} \in (0, 1], \quad (4.39)$$

so that for all $\rho \in (0, \rho_0]$, we eventually get:

$$\|Q^\pm\|_{\rho,3}^{0,\pm,N} \lesssim \|\mathcal{F}^\pm\|_{\rho,1}^{0,\pm} + \|\mathcal{G}\|_{\rho,\frac{3}{2}}. \quad (4.40)$$

Taking the supremum over N , we obtain $Q^\pm \in B_{\rho,3}^0(\Omega^\pm)$ and the same estimate as above for the norm $\|Q^\pm\|_{\rho,3}^{0,\pm}$.

We summarise the main result of this paragraph in the following theorem.

Theorem 4.1. *There exist $\varepsilon_0 \in (0, 1]$, $\eta_0 > 0$ and $\rho_0 \in (0, 1]$ such that, for all $\rho \in (0, \rho_0]$ and $\sigma \in (0, \sigma_0]$, if we consider a front $f \in B_{\rho, \frac{7}{2}}(\mathbb{T}^2)$ satisfying*

$$\|f\|_{H^{2.5}(\mathbb{T}^2)} < \varepsilon_0 \quad \text{and} \quad \|f\|_{\rho, \frac{7}{2}} < \eta_0, \quad (4.41)$$

and analytic source terms $\mathcal{F}^\pm \in B_{\rho, 1, \sigma}(\Omega^\pm)$ and $\mathcal{G} \in B_{\rho, \frac{3}{2}}(\Gamma)$, then the unique solution (Q^+, Q^-) to the elliptic problem (4.1) (with zero mean on Ω) belongs to the space $B_{\rho, 3}^0(\Omega^\pm)$; furthermore, it satisfies the following estimate:

$$\|Q^\pm\|_{\rho, 3}^{0, \pm} \leq C \left(\|\mathcal{F}^\pm\|_{\rho, 1}^{0, \pm} + \|\mathcal{G}\|_{\rho, \frac{3}{2}} \right), \quad (4.42)$$

where $C > 0$ is a constant depending only on ε_0 and η_0 .

Let us observe that we have already admitted that the elliptic problem (4.1) has a solution Q^\pm in $H^\infty(\Omega^\pm)$ (see page 17). Theorem 4.1 gives an additional information about the tangential derivatives of Q^\pm when the source terms and the coefficients of the elliptic operator are in appropriate analytic spaces. The purpose of the next paragraph is to handle all the remaining normal derivatives. We shall see that if the source terms \mathcal{F}^\pm and \mathcal{G} together with the coefficients A_{ji} of problem (4.1) are in analytic spaces, then the solution (Q^+, Q^-) will also be analytic, and satisfies a similar estimate as (4.42).

4.2. Estimate of the normal derivatives. Within this paragraph, we still consider analytic source terms $\mathcal{F}^\pm \in B_{\rho, 1, \sigma}(\Omega^\pm)$ and $\mathcal{G} \in B_{\rho, \frac{3}{2}}(\Gamma)$, together with a front $f \in B_{\rho, \frac{7}{2}}(\mathbb{T}^2)$ satisfying (4.41). From now on, the purpose is to show that the (zero mean) solution Q^\pm of problem (4.1) belongs to the space $B_{\rho, 3, \sigma}(\Omega^\pm)$, for $\sigma > 0$ small enough. To do so, we first have to handle all the normal derivatives, in order to show that for all $k \geq 1$, we have $Q^\pm \in B_{\rho, 3}^k(\Omega^\pm)$. Eventually, to prove that $Q^\pm \in B_{\rho, 3, \sigma}(\Omega^\pm)$, it will remain to estimate the partial sums $\|Q^\pm\|_{\rho, 3, \sigma}^{\pm, N}$.

We proceed by induction over $k \geq 0$. Let us assume that for all $j \in \{0, \dots, k\}$, we have $Q^\pm \in B_{\rho, 3}^j(\Omega^\pm)$. First, we know that $Q^\pm \in H^\infty(\Omega^\pm)$. Therefore, in particular, $Q^\pm \in H_{x_3}^{k+4}(H^\infty(\mathbb{T}^2))$. It remains to prove that the norm $\|Q^\pm\|_{\rho, 3}^{k+1, \pm}$ is a finite quantity. For all $N \geq 0$, we shall write $\|Q^\pm\|_{\rho, 3}^{k+1, \pm, N}$ its partial sum of order N . Using the induction assumption, we split the partial sum as follows (we assume that $N \geq k+1$):

$$\begin{aligned} \|Q^\pm\|_{\rho, 3}^{k+1, \pm, N} &:= \sum_{k=0}^N \frac{\rho^n}{n!} \max_{\substack{|\alpha|=n \\ \alpha_3 \leq k+1}} \|\partial^\alpha Q^\pm\|_{3, \pm} \\ &\leq \|Q^\pm\|_{\rho, 3}^{k, \pm} + \sum_{n=k+1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n-k-1} \|\partial^{\alpha'} \partial_3^{k+1} Q^\pm\|_{3, \pm}. \end{aligned} \quad (4.43)$$

In order to estimate the right sum in (4.43), we rewrite the norm $\|\partial^{\alpha'} \partial_3^{k+1} Q^\pm\|_{3, \pm}$ as follows:

$$\|\partial^{\alpha'} \partial_3^{k+1} Q^\pm\|_{3, \pm}^2 = \|\partial^{\alpha'} \partial_3^{k+1} Q^\pm\|_{2, \pm}^2 + \sum_{\substack{|\beta|=3 \\ |\beta'| \geq 1}} \|\partial^\beta \partial^{\alpha'} \partial_3^{k+1} Q^\pm\|_{0, \pm}^2 + \|\partial^{\alpha'} \partial_3^{k+4} Q^\pm\|_{0, \pm}^2. \quad (4.44)$$

Both first right terms of (4.44) will allow to get the quantity $\|Q^\pm\|_{\rho, 3}^{k, \pm}$. Indeed, we begin by writing the estimate

$$\|\partial^{\alpha'} \partial_3^{k+1} Q^\pm\|_{2, \pm} \leq \|\partial^{\alpha'} \partial_3^k Q^\pm\|_{3, \pm} \leq \max_{|\alpha'|=n-k-1} \|\partial^{\alpha'} \partial_3^k Q^\pm\|_{3, \pm}.$$

Thus summing over n we have

$$\sum_{n=k+1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n-k-1} \|\partial^{\alpha'} \partial_3^{k+1} Q^\pm\|_{2, \pm} \leq \sum_{n=k+1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n-k-1} \|\partial^{\alpha'} \partial_3^k Q^\pm\|_{3, \pm} \leq \rho \|Q^\pm\|_{\rho, 3}^{k, \pm}.$$

On the other hand, for $|\beta| = 3$ and $|\beta'| \geq 1$, the terms $\|\partial^\beta \partial^{\alpha'} \partial_3^{k+1} Q^\pm\|_{0, \pm}$ of (4.44) give the same kind of estimate. Indeed, since β is a multi-index of length 3, with at least one tangential derivative $\bar{\partial}$, we can write $\partial^\beta = \bar{\partial} \partial^2$. Therefore we get:

$$\|\partial^\beta \partial^{\alpha'} \partial_3^{k+1} Q^\pm\|_{0, \pm} = \|\bar{\partial} \partial^2 (\partial^{\alpha'} \partial_3^{k+1} Q^\pm)\|_{0, \pm} \leq \|\bar{\partial} \partial^{\alpha'} \partial_3^k Q^\pm\|_{3, \pm} \leq \max_{|\alpha'|=n-k} \|\partial^{\alpha'} \partial_3^k Q^\pm\|_{3, \pm}.$$

After summing over n , we obtain:

$$\sum_{\substack{|\beta|=3 \\ |\beta'| \geq 1}} \sum_{n=k+1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n-k-1} \|\partial^\beta \partial^{\alpha'} \partial_3^{k+1} Q^\pm\|_{0, \pm} \lesssim \sum_{n=k+1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n-k} \|\partial^{\alpha'} \partial_3^k Q^\pm\|_{3, \pm} \lesssim \|Q^\pm\|_{\rho, 3}^{k, \pm}.$$

Back to (4.43), we deduce that

$$\|Q^\pm\|_{\rho,3}^{k+1,\pm,N} \lesssim (1+\rho) \|Q^\pm\|_{\rho,3}^{k,\pm} + \sum_{n=k+1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n-k-1} \|\partial^{\alpha'} \partial_3^{k+4} Q^\pm\|_{0,\pm}. \quad (4.45)$$

The only tricky term to estimate is the one containing the most normal derivatives, namely $\partial^{\alpha'} \partial_3^{k+4} Q^\pm$. We have to estimate the latter in $L^2(\Omega^\pm)$. So, we begin by using the following bound:

$$\|\partial^{\alpha'} \partial_3^{k+4} Q^\pm\|_{0,\pm} \leq \|\partial^{\alpha'} \partial_3^k (\partial_3^2 Q^\pm)\|_{2,\pm}.$$

This highlights exactly 2 normal derivatives of Q^\pm , that we will be able to eliminate using the equation satisfied by Q^\pm in problem (4.1). Then, it will suffice to apply the operator $\partial^{\alpha'} \partial_3^k$ to this equation: it will exhibit at most k normal derivatives. Consequently, using the induction assumption, we shall be able to use all the quantities $\|Q^\pm\|_{\rho,3}^{j,\pm}$, for $j \leq k$. The difference of exactly one normal derivative between the steps k and $k+1$ will allow to gain one power of σ later on, after summing over k . Eventually, this gain will turn out to be crucial to absorb some analytic norms, in order to complete the estimate of the pressure. This method is similar as the previous paragraph, when the computations of $\|Q^\pm\|_{\rho,3}^{0,\pm}$ gave us some powers of ρ .

Expanding the first equation of (4.1), we get:

$$\frac{1+|\nabla'\psi|^2}{J^2} \partial_3^2 Q^\pm = -\Delta' Q^\pm + 2 \frac{\partial_h \psi}{J} \partial_h \partial_3 Q^\pm - \mathcal{F}^\pm - A_{ji} \partial_j \tilde{A}_{ki} \partial_k Q^\pm. \quad (4.46)$$

Let us define

$$\zeta := \frac{J^2}{1+|\nabla'\psi|^2}, \quad (4.47)$$

so that we can rewrite (4.46) as follows:

$$\partial_3^2 Q^\pm = -\zeta \Delta' Q^\pm + 2 \frac{\zeta}{J} \partial_h \psi \partial_h \partial_3 Q^\pm - \zeta \mathcal{F}^\pm - \zeta A \partial^1 \tilde{A} \partial^1 Q^\pm. \quad (4.48)$$

Let us set $\alpha := (\alpha', k)$ (remark that $|\alpha| = n-1$), and apply ∂^α to equation (4.48):

$$\begin{aligned} \partial^\alpha \partial_3^2 Q^\pm &= -\partial^\alpha (\zeta \Delta' Q^\pm) + 2 \partial^\alpha \left(\frac{\zeta}{J} \partial_h \psi \partial_h \partial_3 Q^\pm \right) - \partial^\alpha (\zeta \mathcal{F}^\pm) - \partial^\alpha (\zeta A \partial^1 \tilde{A} \partial^1 Q^\pm) \\ &=: -T_1 + 2T_2 - T_3 - T_4. \end{aligned} \quad (4.49)$$

To finish with, it remains to estimate each term T_i in $H^2(\Omega^\pm)$.

► Treatment of T_1 :

We have:

$$T_1 := \partial^\alpha (\zeta \Delta' Q^\pm) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta! \gamma!} \partial^\beta \zeta \partial^\gamma \Delta' Q^\pm.$$

As for the case of $\|Q^\pm\|_{\rho,3}^{0,\pm}$, we get:

$$\|T_1\|_{2,\pm} \lesssim \sum_{i_1+i_2=n-1} \frac{(n-1)!}{i_1! i_2!} \max_{\substack{|\beta|=i_1 \\ |\gamma|=i_2}} \left(\|\partial^\beta \zeta\|_{H^2} \|\partial^\gamma \Delta' Q^\pm\|_{2,\pm} \right). \quad (4.50)$$

Now, we have to use the fact that $\beta_3 + \gamma_3 = k$. Consequently, we can bound from above the right term of (4.50) as follows:

$$\|T_1\|_{2,\pm} \lesssim \sum_{j_1+j_2=k} \sum_{i_1+i_2=n-1} \frac{(n-1)!}{i_1! i_2!} \max_{\substack{|\beta|=i_1 \\ \beta_3 \leq j_1}} \|\partial^\beta \zeta\|_{H^2} \max_{\substack{|\gamma|=i_2 \\ \gamma_3 \leq j_2}} \|\partial^\gamma \Delta' Q^\pm\|_{2,\pm}. \quad (4.51)$$

The estimate of Q^\pm is reduced to $H^3(\Omega^\pm)$ using the following inequality:

$$\max_{\substack{|\gamma|=i_2 \\ \gamma_3 \leq j_2}} \|\partial^\gamma \Delta' Q^\pm\|_{2,\pm} \lesssim \max_{\substack{|\gamma|=i_2+1 \\ \gamma_3 \leq j_2}} \|\partial^\gamma Q^\pm\|_{3,\pm}.$$

Then, re-indexing over i_2 , we obtain:

$$\|T_1\|_{2,\pm} \lesssim \sum_{j_1+j_2=k} \sum_{\substack{i_1+i_2=n \\ i_2 \geq 1}} \frac{(n-1)!}{i_1! (i_2-1)!} \max_{\substack{|\beta|=i_1 \\ \beta_3 \leq j_1}} \|\partial^\beta \zeta\|_{H^2} \max_{\substack{|\gamma|=i_2 \\ \gamma_3 \leq j_2}} \|\partial^\gamma Q^\pm\|_{3,\pm}.$$

Finally, multiplying by $\frac{\rho^n}{n!}$ and summing over $n \in \{k+1, \dots, N\}$, we have:

$$\begin{aligned} \sum_{n=k+1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n-k-1} \|T_1\|_{2,\pm} &\lesssim \sum_{j_1+j_2=k} \sum_{n=k+1}^N \sum_{\substack{i_1+i_2=n \\ i_2 \geq 1}} \frac{(n-1)! i_2}{n!} \frac{\rho^{i_1}}{i_1!} \max_{\substack{|\beta|=i_1 \\ \beta_3 \leq j_1}} \|\partial^\beta \zeta\|_{H^2} \frac{\rho^{j_2}}{i_2!} \max_{\substack{|\gamma|=i_2 \\ \gamma_3 \leq j_2}} \|\partial^\gamma Q^\pm\|_{3,\pm} \\ &\lesssim \sum_{j_1+j_2=k} \sum_{n=k+1}^N \sum_{i_1+i_2=n} \frac{\rho^{i_1}}{i_1!} \max_{\substack{|\beta|=i_1 \\ \beta_3 \leq j_1}} \|\partial^\beta \zeta\|_{H^2} \frac{\rho^{i_2}}{i_2!} \max_{\substack{|\gamma|=i_2 \\ \gamma_3 \leq j_2}} \|\partial^\gamma Q^\pm\|_{3,\pm}, \end{aligned}$$

where we have used the inequality $\frac{(n-1)! i_2}{n!} \leq 1$. To conclude, we recognize a ‘‘partial’’ Cauchy product with respect to n , and write:

$$\sum_{n=k+1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n-k-1} \|T_1\|_{2,\pm} \lesssim \sum_{j_1+j_2=k} \|\zeta\|_{\rho,2}^{j_1,N} \|Q^\pm\|_{\rho,3}^{j_2,\pm,N} \lesssim \sum_{j_1+j_2=k} \|\zeta\|_{\rho,2}^{j_1} \|Q^\pm\|_{\rho,3}^{j_2,\pm}, \quad (4.52)$$

which ends the case of T_1 .

► **Treatment of T_2 :**

In the same way as T_2 , we get:

$$T_2 := \partial^\alpha \left(\frac{\zeta}{J} \partial_h \psi \partial_h \partial_3 Q^\pm \right) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta! \gamma!} \partial^\beta \left(\frac{\zeta}{J} \partial_h \psi \right) \partial^\gamma \partial_h \partial_3 Q^\pm.$$

Then, the estimate of $\|T_2\|_{2,\pm}$ is identical to T_1 , estimating Q^\pm in $H^3(\Omega^\pm)$ via the inequality

$$\max_{\substack{|\gamma|=i_2 \\ \gamma_3 \leq j_2}} \|\partial^\gamma \partial_h \partial_3 Q^\pm\|_{2,\pm} \leq \max_{\substack{|\gamma|=i_2+1 \\ \gamma_3 \leq j_2}} \|\partial^\gamma Q^\pm\|_{3,\pm}.$$

We conclude that

$$\sum_{n=k+1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n-k-1} \|T_2\|_{2,\pm} \lesssim \sum_{j_1+j_2=k} \left\| \frac{\zeta}{J} \partial_h \psi \right\|_{\rho,2}^{j_1} \|Q^\pm\|_{\rho,3}^{j_2,\pm}. \quad (4.53)$$

► **Treatment of T_3 :**

By definition, we have

$$T_3 := \partial^\alpha (\zeta \mathcal{F}^\pm) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta! \gamma!} \partial^\beta \zeta \partial^\gamma \mathcal{F}^\pm.$$

As previously, we get:

$$\|T_3\|_{2,\pm} \lesssim \sum_{j_1+j_2=k} \sum_{i_1+i_2=n-1} \frac{(n-1)!}{i_1! i_2!} \max_{\substack{|\beta|=i_1 \\ \beta_3 \leq j_1}} \|\partial^\beta \zeta\|_{H^2} \max_{\substack{|\gamma|=i_2 \\ \gamma_3 \leq j_2}} \|\partial^\gamma \mathcal{F}^\pm\|_{2,\pm}.$$

By assumption, we have $\mathcal{F}^\pm \in B_{\rho,1,\sigma}(\Omega^\pm)$. As a consequence, we are reduced to the norm $\|\cdot\|_{1,\pm}$ using the following inequality:

$$\max_{\substack{|\gamma|=i_2 \\ \gamma_3 \leq j_2}} \|\partial^\gamma \mathcal{F}^\pm\|_{2,\pm} \lesssim \max_{\substack{|\gamma|=i_2 \\ \gamma_3 \leq j_2+1}} \|\partial^\gamma \mathcal{F}^\pm\|_{1,\pm} + \max_{\substack{|\gamma|=i_2+1 \\ \gamma_3 \leq j_2+1}} \|\partial^\gamma \mathcal{F}^\pm\|_{1,\pm}.$$

Next, the arguments are completely analogous to the cases of T_1 and T_2 , and we end up with:

$$\sum_{n=k+1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n-k-1} \|T_3\|_{2,\pm} \lesssim (1+\rho) \sum_{j_1+j_2=k} \|\zeta\|_{\rho,2}^{j_1} \|\mathcal{F}^\pm\|_{\rho,1}^{j_2+1,\pm}. \quad (4.54)$$

► **Treatment of T_4 :**

The case of T_4 is the same as T_1 , but with an additional term to differentiate:

$$T_4 := \partial^\alpha (\zeta A \partial^1 \tilde{A} \partial^1 Q^\pm) = \sum_{\beta+\gamma+\delta=\alpha} \frac{\alpha!}{\beta! \gamma! \delta!} \partial^\beta (\zeta A) \partial^\gamma \partial^1 \tilde{A} \partial^\delta \partial^1 Q^\pm.$$

We generalize estimate (4.51) with three indices i_1 , i_2 and i_3 :

$$\|T_4\|_{2,\pm} \lesssim \sum_{j_1+j_2+j_3=k} \sum_{i_1+i_2+i_3=n-1} \frac{(n-1)!}{i_1! i_2! i_3!} \max_{\substack{|\beta|=i_1 \\ \beta_3 \leq j_1}} \|\partial^\beta (\zeta A)\|_{H^2} \max_{\substack{|\gamma|=i_2 \\ \gamma_3 \leq j_2}} \|\partial^\gamma \partial^1 \tilde{A}\|_{H^2} \max_{\substack{|\delta|=i_3 \\ \delta_3 \leq j_3}} \|\partial^\delta \partial^1 Q^\pm\|_{2,\pm}.$$

Then, using both estimates

$$\|\partial^\gamma \partial^1 \tilde{A}\|_{H^2} \leq \|\partial^\gamma \tilde{A}\|_{H^3} \quad \text{and} \quad \|\partial^\delta \partial^1 Q^\pm\|_{2,\pm} \leq \|\partial^\delta Q^\pm\|_{3,\pm},$$

we can deduce that

$$\|T_4\|_{2,\pm} \lesssim \sum_{j_1+j_2+j_3=k} \sum_{i_1+i_2+i_3=n-1} \frac{(n-1)!}{i_1! i_2! i_3!} \max_{\substack{|\beta|=i_1 \\ \beta_3 \leq j_1}} \|\partial^\beta (\zeta A)\|_{H^2} \max_{\substack{|\gamma|=i_2 \\ \gamma_3 \leq j_2}} \|\partial^\gamma \tilde{A}\|_{H^3} \max_{\substack{|\delta|=i_3 \\ \beta_3 \leq j_3}} \|\partial^\delta Q^\pm\|_{3,\pm}.$$

Eventually, we obtain:

$$\sum_{n=k+1}^N \frac{\rho^n}{n!} \max_{|\alpha'|=n-k-1} \|T_4\|_{2,\pm} \lesssim \rho \sum_{j_1+j_2+j_3=k} \|\zeta A\|_{\rho,2}^{j_1} \|\tilde{A}\|_{\rho,3}^{j_2} \|Q^\pm\|_{\rho,3}^{j_3,\pm}. \quad (4.55)$$

Back to estimate (4.45), and using (4.52)-(4.55), we have proved that

$$\begin{aligned} \|Q^\pm\|_{\rho,3}^{k+1,\pm,N} &\lesssim (1+\rho) \|Q^\pm\|_{\rho,3}^{k,\pm} + \sum_{j_1+j_2=k} \left(\|\zeta\|_{\rho,2}^{j_1} + \left\| \frac{\zeta}{J} \nabla' \psi \right\|_{\rho,2}^{j_1} \right) \|Q^\pm\|_{\rho,3}^{j_2,\pm} \\ &\quad + (1+\rho) \sum_{j_1+j_2=k} \|\zeta\|_{\rho,2}^{j_1} \|\mathcal{F}^\pm\|_{\rho,1}^{j_2+1,\pm} \\ &\quad + \rho \sum_{j_1+j_2+j_3=k} \|\zeta A\|_{\rho,2}^{j_1} \|\tilde{A}\|_{\rho,3}^{j_2} \|Q^\pm\|_{\rho,3}^{j_3,\pm}. \end{aligned} \quad (4.56)$$

In particular, we can see that the norm $\|Q^\pm\|_{\rho,3}^{k+1,\pm,N}$ is uniformly bounded with respect to N . Thus, taking the supremum over N , we have $\|Q^\pm\|_{\rho,3}^{k+1,\pm} < +\infty$. By induction, we obtain $Q^\pm \in \cap_{k \geq 0} B_{\rho,3}^k(\Omega^\pm)$, with estimate (4.56) which also holds for $\|Q^\pm\|_{\rho,3}^{k+1,\pm}$. Multiplying by σ^{k+1} and summing over $k \in \{0, \dots, K-1\}$ (where $K \geq 1$ is any integer), we end up with the following estimate for the partial sum $\|Q^\pm\|_{\rho,3,\sigma}^{\pm,K}$ of the norm $\|Q^\pm\|_{\rho,3,\sigma}^\pm$:

$$\|Q^\pm\|_{\rho,3,\sigma}^{\pm,K} \lesssim \|Q^\pm\|_{\rho,3}^{0,\pm} + (1+\rho) \|\zeta\|_{\rho,2,\sigma} \|\mathcal{F}^\pm\|_{\rho,1,\sigma}^\pm \quad (4.57)$$

$$+ \sigma \left(\|\zeta\|_{\rho,2,\sigma} + \left\| \frac{\zeta}{J} \nabla' \psi \right\|_{\rho,2,\sigma} + \rho \|\zeta\|_{\rho,2,\sigma} \|A\|_{\rho,2,\sigma} \|\tilde{A}\|_{\rho,3,\sigma} \right) \|Q^\pm\|_{\rho,3,\sigma}^{\pm,K}. \quad (4.58)$$

Using (4.42) satisfied by $\|Q^\pm\|_{\rho,3}^{0,\pm}$, and bounding from above ρ by $\rho_0 \leq 1$, we can estimate the right term of (4.57) by

$$C_0 \left(\|Q^\pm\|_{\rho,3}^{0,\pm} + \|\zeta\|_{\rho,2,\sigma} \|\mathcal{F}^\pm\|_{\rho,1,\sigma}^\pm \right),$$

where $C_0 > 0$ is a numerical constant. Estimate (4.57)-(4.58) is rewritten as follows:

$$\|Q^\pm\|_{\rho,3,\sigma}^{\pm,K} \leq C_1 \left(\|Q^\pm\|_{\rho,3}^{0,\pm} + \|\zeta\|_{\rho,2,\sigma} \|\mathcal{F}^\pm\|_{\rho,1,\sigma}^\pm \right) \quad (4.59)$$

$$+ \sigma C_1 \left(\|\zeta\|_{\rho,2,\sigma} + \left\| \frac{\zeta}{J} \nabla' \psi \right\|_{\rho,2,\sigma} + \rho \|\zeta\|_{\rho,2,\sigma} \|A\|_{\rho,2,\sigma} \|\tilde{A}\|_{\rho,3,\sigma} \right) \|Q^\pm\|_{\rho,3,\sigma}^{\pm,K}, \quad (4.60)$$

where $C_1 > 0$ is a numerical constant. From now on, it remains to absorb the analytic terms in (4.60), in order to complete the estimate of $\|Q^\pm\|_{\rho,3,\sigma}^{\pm,K}$.

Absorption of the analytic terms.

► Treatment of $\|\zeta\|_{\rho,2,\sigma}$:

We proceed in the same way as $\|Q^\pm\|_{\rho,3}^{0,\pm}$. We use the gain of one power of σ in order to absorb the analytic terms in (4.60), choosing σ small enough. Let $f \in B_{\rho,\frac{\sigma}{2}}(\mathbb{T}^2)$ still satisfy assumption (4.41) of Theorem 4.1. As a consequence, we obtain both following estimates (we have just rewritten (2.17) and (2.19)):

$$\|\nabla \psi\|_{\rho,3,\sigma} \leq \|\psi\|_{\rho,4,\sigma} \leq C_0 \|f\|_{\rho,\frac{\sigma}{2}} \leq C_0, \quad (4.61)$$

$$\left\| \frac{1}{J} \right\|_{\rho,3,\sigma} \leq C_0. \quad (4.62)$$

Here, and from now on, $C_0 > 0$ will stand for any numerical constant. We notice that both estimates (4.61) and (4.62) hold for all $\rho \in (0, \rho_0]$ and $\sigma \in (0, \sigma_0]$ (recall that σ_0 has been fixed, equal to $\frac{1}{2}$). Using the algebra property

of $B_{\rho,2,\sigma}(\Omega)$, we have:

$$\|\zeta\|_{\rho,2,\sigma} \leq C_0 \left\| \frac{1}{1 + |\nabla' \psi|^2} \right\|_{\rho,2,\sigma} \|J\|_{\rho,2,\sigma}^2.$$

On the one hand, using (4.61), we can write:

$$\|J\|_{\rho,2,\sigma} = \|1 + \partial_3 \psi\|_{\rho,2,\sigma} \leq C_0 (1 + \|\psi\|_{\rho,3,\sigma}) \leq C_0.$$

On the other hand, up to reduce a bit more $\eta_0 > 0$ if necessary (see assumption (4.41)), we get:

$$\left\| \frac{1}{1 + |\nabla' \psi|^2} \right\|_{\rho,2,\sigma} \leq C_0.$$

Indeed it suffices to use Proposition 2.2 and to adapt the proof of Corollary 2.3. Eventually, we can state that ζ is bounded in $B_{\rho,2,\sigma}(\Omega)$:

$$\|\zeta\|_{\rho,2,\sigma} \leq C_0. \quad (4.63)$$

► **Treatment of** $\left\| \frac{\zeta}{J} \nabla' \psi \right\|_{\rho,2,\sigma}$ **and** $\rho \|A\|_{\rho,2,\sigma} \|\zeta\|_{\rho,2,\sigma} \|\tilde{A}\|_{\rho,3,\sigma}$:

In the following, we do not detail the computations of these terms, since we use exactly the same method as before. It gives the same type of estimate as (4.63), namely:

$$\left\| \frac{\zeta}{J} \nabla' \psi \right\|_{\rho,2,\sigma} \leq C_0 \quad \text{and} \quad \rho \|\zeta\|_{\rho,2,\sigma} \|A\|_{\rho,2,\sigma} \|\tilde{A}\|_{\rho,3,\sigma} \leq C_0, \quad (4.64)$$

where we have used $\rho \leq \rho_0 \leq 1$.

► **Limitation of the parameter σ :**

From now on, we redefine σ_0 as follows:

$$\sigma_0 := \frac{1}{6C_0C_1},$$

where $C_0 > 0$ is given by (4.63)-(4.64) and C_1 by (4.60). Without loss of generality, we can increase C_0 to still have $\sigma_0 \leq \frac{1}{2}$. Therefore all the previous estimates obtained with $\sigma \in (0, \frac{1}{2}]$ remain valid. For all $\rho \in (0, \rho_0]$ and $\sigma \in (0, \sigma_0]$, the right term of (4.60) is estimated as follows:

$$\begin{aligned} \sigma C_1 \left(\|\zeta\|_{\rho,2,\sigma} + \left\| \frac{\zeta}{J} \nabla' \psi \right\|_{\rho,2,\sigma} + \rho \|\zeta\|_{\rho,2,\sigma} \|A\|_{\rho,2,\sigma} \|\tilde{A}\|_{\rho,3,\sigma} \right) \|Q^\pm\|_{\rho,3,\sigma}^{\pm,K} &\leq 3C_0C_1\sigma \|Q^\pm\|_{\rho,3,\sigma}^{\pm,K} \\ &\leq \frac{1}{2} \|Q^\pm\|_{\rho,3,\sigma}^{\pm,K}. \end{aligned}$$

Consequently, we get the following estimate for the norm $\|Q^\pm\|_{\rho,3,\sigma}^{\pm,K}$:

$$\|Q^\pm\|_{\rho,3,\sigma}^{\pm,K} \leq C_1 \left(\|Q^\pm\|_{\rho,3}^{0,\pm} + \|\zeta\|_{\rho,2,\sigma} \|\mathcal{F}^\pm\|_{\rho,1,\sigma}^\pm \right). \quad (4.65)$$

To conclude, we use (4.63) together with (4.42), and simplify (4.65) to end up with

$$\|Q^\pm\|_{\rho,3,\sigma}^{\pm,K} \leq C_2 \left(\|\mathcal{F}^\pm\|_{\rho,1,\sigma}^\pm + \|\mathcal{G}\|_{\rho,\frac{3}{2}} \right),$$

where $C_2 > 0$ is a numerical constant. Taking the supremum over K , we deduce the same estimate satisfied by the norm $\|Q^\pm\|_{\rho,3,\sigma}^\pm$. We sum up in the following theorem the main result of this paragraph about the pressure estimate.

Theorem 4.2. *There exist $\varepsilon_0 \in (0, 1]$, $\eta_0 > 0$, $\rho_0 \in (0, 1]$ and $\sigma_0 \in (0, \frac{1}{2}]$ such that, for all $\rho \in (0, \rho_0]$ and $\sigma \in (0, \sigma_0]$, if we consider a front $f \in B_{\rho,\frac{7}{2}}(\mathbb{T}^2)$ satisfying*

$$\|f\|_{H^{2,5}(\mathbb{T}^2)} < \varepsilon_0 \quad \text{and} \quad \|f\|_{\rho,\frac{7}{2}} < \eta_0,$$

and analytic source terms $\mathcal{F}^\pm \in B_{\rho,1,\sigma}(\Omega^\pm)$ and $\mathcal{G} \in B_{\rho,\frac{3}{2}}(\Gamma)$, then the unique solution (Q^+, Q^-) to the problem (4.1) with zero mean on Ω belongs to the space $B_{\rho,3,\sigma}(\Omega^+) \times B_{\rho,3,\sigma}(\Omega^-)$. Moreover, it satisfies the following estimate:

$$\|Q^\pm\|_{\rho,3,\sigma}^\pm \leq C \left(\|\mathcal{F}^\pm\|_{\rho,1,\sigma}^\pm + \|\mathcal{G}\|_{\rho,\frac{3}{2}} \right), \quad (4.66)$$

where $C > 0$ is a constant depending only on ε_0 and η_0 .

5. EXISTENCE OF ANALYTIC SOLUTIONS TO THE CURRENT-VORTEX SHEET PROBLEM

In order to apply Cauchy-Kowalevskaya theorem, we first have to estimate the convective terms appearing in the equations of problem (3.4). Then, we have to estimate the pressure term, namely $A^T \nabla Q^\pm$. In view of the estimates satisfied by the pressure (see Theorem 4.2), we will restrict the analysis in the analytic spaces $B_{\rho,r,\sigma}$ with $r = 3$, and the parameter σ will be fixed in the interval $(0, \sigma_0]$.

5.1. Estimate of the convective terms. To begin with, we rewrite problem (3.4) as a ‘‘differential equation’’ in time, in order to be consistent with the notations of Cauchy-Kowalevskaya theorem [Nis77]. The equations of (3.4) read as follows:

$$\begin{cases} \frac{d}{dt} U = F(U), & t \in (0, T), \\ U(0) = U_0, \end{cases} \quad (5.1)$$

where $U := (v^\pm, B^\pm, f)$ and

$$F(U) := \begin{pmatrix} -(\tilde{v}^\pm \cdot \nabla) v^\pm + (\tilde{B}^\pm \cdot \nabla) B^\pm - A^T \nabla Q^\pm \\ -(\tilde{v}^\pm \cdot \nabla) B^\pm + (\tilde{B}^\pm \cdot \nabla) v^\pm \\ (v^+ \cdot N)|_\Gamma \end{pmatrix}. \quad (5.2)$$

Remark: In the definition of the term \tilde{v}^\pm given by (1.10), the quantity $\partial_t \psi$ is appearing. This is only a notation standing for the quantity $\Psi_{(v^+ \cdot N)|_\Gamma}$, where Ψ is the lifting map defined by Proposition 2.1. Indeed, we have

$$\partial_t \psi = \partial_t \Psi_f = \Psi_{\partial_t f} = \Psi_{(v^+ \cdot N)|_\Gamma},$$

since the front f satisfies the third equation of (3.4). Thus the variable t plays the role of a parameter in the definition of $F(U)$.

To take the boundary conditions on Γ_\pm in problem (3.4) into account, we include them into the spaces $B_{\rho,3,\sigma}(\Omega^\pm)$. Thus we define:

$$\mathbb{B}_{\rho,3,\sigma}(\Omega^\pm) := \left\{ u \in (B_{\rho,3,\sigma}(\Omega^\pm))^3 \mid u_3 = 0 \text{ on } \Gamma_\pm \right\}, \quad (5.3)$$

whose norm will be noted $\| \cdot \|_{\rho,3,\sigma}$. Clearly, the space $\mathbb{B}_{\rho,3,\sigma}(\Omega^\pm)$ is closed in $B_{\rho,3,\sigma}(\Omega^\pm)$, because of the continuity of the trace map from $H^3(\Omega^\pm)$ to $H^{\frac{5}{2}}(\Gamma \cup \Gamma_\pm)$. Therefore $\mathbb{B}_{\rho,3,\sigma}(\Omega^\pm)$ is a Banach space.

To take the flatness of the front f in $H^{2.5}(\mathbb{T}^2)$ and $B_{\rho,\frac{7}{2}}(\mathbb{T}^2)$ into account, which gives estimate (4.66), we shall apply in the end Cauchy-Kowalevskaya theorem with a front f in the ball

$$\left\{ u \in B_{\rho,\frac{7}{2}}(\mathbb{T}^2) \mid \|u\|_{H^{2.5}(\mathbb{T}^2)} < \varepsilon_0 \text{ and } \|u\|_{\rho,\frac{7}{2}} < \eta_0 \right\}, \quad (5.4)$$

with ε_0 and η_0 given by Theorem 4.2.

Therefore we will apply Cauchy-Kowalevskaya theorem in the following scale:

$$\mathbb{B}_\rho := \mathbb{B}_{\rho,3,\sigma}(\Omega^+) \times \mathbb{B}_{\rho,3,\sigma}(\Omega^-) \times B_{\rho,\frac{7}{2}}(\mathbb{T}^2), \quad (5.5)$$

whose norm will be noted $\| \cdot \|_\rho$ (the definition is obvious). In (5.5), $\mathbb{B}_{\rho,3,\sigma}(\Omega^\pm)$ stands for the product space $\mathbb{B}_{\rho,3,\sigma}(\Omega^+) \times \mathbb{B}_{\rho,3,\sigma}(\Omega^-)$. Within this scale, all the algebra and differentiation properties still hold. However, although the spaces $\mathbb{B}_{\rho,3,\sigma}(\Omega^\pm)$ are not invariant under the normal derivative ∂_3 , the following result inspired from Theorem 1.8 remains valid. If $u \in \mathbb{B}_{\rho,3,\sigma}(\Omega^\pm)$, then for all $\rho' < \rho$ we have $\partial_3 u \in (B_{\rho',3,\sigma}(\Omega^\pm))^3$, with the estimate

$$\| \partial_3 u \|_{\rho',3,\sigma} \leq \frac{C_\sigma}{\rho - \rho'} \| u \|_{\rho,3,\sigma}, \quad (5.6)$$

where $C_\sigma := \sigma^{-1} > 0$.

Now let $\rho \in (0, \rho_0]$, and let $U := (v^\pm, B^\pm, f)$ and $V := (w^\pm, D^\pm, g)$ be two vectors taken in a ball of radius $R > 0$ of \mathbb{B}_ρ :

$$\| U \|_\rho, \| V \|_\rho < R. \quad (5.7)$$

The main purpose is to obtain an estimate of the form

$$\| F(U) - F(V) \|_{\rho'} \leq \frac{C}{\rho - \rho'} \| U - V \|_\rho, \quad (5.8)$$

for all $0 < \rho' < \rho$, with $C > 0$ **independent of ρ and ρ'** . From now on, we will always note C such a constant.

To do so, we will need several basic estimates, that we enumerate below. For the sake of clarity, we will denote Ψ_f (instead of ψ) the lifting of f (see definition (2.14)), J_f the jacobian $1 + \partial_3 \Psi_f$, and N_f the vector $(-\partial_1 \Psi_f, -\partial_2 \Psi_f, 1)$; we proceed in the same way with g .

► **The term J_f^{-1} :**

The term J_f^{-1} is estimated using Corollary 2.3. Indeed, the condition $\|f\|_{\rho, \frac{7}{2}} \leq \eta_0$ is directly satisfied because of (5.4). Thus we have:

$$\left\| \frac{1}{J_f} \right\|_{\rho, 3, \sigma} \leq C. \quad (5.9)$$

The same result obviously holds for J_f^{-1} .

► **The term $v^\pm \cdot N_f$:**

To estimate $\|v^\pm \cdot N_f\|_{\rho, r, \sigma}^\pm$, it suffices to control the norms $\|v_i^\pm \partial_i \Psi_f\|_{\rho, 3, \sigma}^\pm$ for $i = 1, 2$:

$$\begin{aligned} \|v_i^\pm \partial_i \Psi_f\|_{\rho, 3, \sigma}^\pm &\leq C \|v_i^\pm\|_{\rho, 3, \sigma}^\pm \|\partial_i \Psi_f\|_{\rho, 3, \sigma} && \text{(algebra property)} \\ &\leq C R C \|f\|_{\rho, \frac{7}{2}} && \text{(assumption (5.7) and Proposition 2.1)} \\ &\leq C R C \eta_0. && \text{(using (5.4))} \end{aligned}$$

Finally,

$$\|v^\pm \cdot N_f\|_{\rho, 3, \sigma}^\pm \leq C. \quad (5.10)$$

Using the symmetry of the roles played by U and V , we will have the same estimate for the norm $\|w^\pm \cdot N_g\|_{\rho, 3, \sigma}$.

► **The terms $N_f - N_g$ and $J_f - J_g$:**

We also estimate $\|N_f - N_g\|_{\rho, 3, \sigma}$ using, for $i = 1, 2$:

$$\|\partial_i(\Psi_f - \Psi_g)\|_{\rho, 3, \sigma} \leq \|\Psi_f - \Psi_g\|_{\rho, 4, \sigma} \leq C \|f - g\|_{\rho, \frac{7}{2}} \leq C \|U - V\|_\rho.$$

Let us notice that we have used the linearity of the map $f \mapsto \Psi_f$ (see Proposition 2.1). We deduce that:

$$\|N_f - N_g\|_{\rho, 3, \sigma} \leq C \|U - V\|_\rho. \quad (5.11)$$

Likewise, we have the following straightforward estimate:

$$\|J_f - J_g\|_{\rho, 3, \sigma} \leq C \|U - V\|_\rho. \quad (5.12)$$

► **The term $\partial_t \Psi_f$:**

Now we detail the treatment of the norm $\|\partial_t \Psi_f\|_{\rho, 3, \sigma}$. Using the construction of the operator $f \mapsto \Psi_f$ given in the proof of Proposition 2.1, we have in particular $\partial_t \Psi_f = \Psi_{\partial_t f}$. Consequently, we can write:

$$\begin{aligned} \|\partial_t \Psi_f\|_{\rho, 3, \sigma} &= \|\Psi_{\partial_t f}\|_{\rho, 3, \sigma} \leq C \|\partial_t f\|_{\rho, \frac{5}{2}} \leq C \|(v^+ \cdot N_f)|_\Gamma\|_{\rho, \frac{5}{2}} \\ &\leq C \|(v_1^+ \partial_1 f + v_2^+ \partial_2 f - v_3^+)|_\Gamma\|_{\rho, \frac{5}{2}}. \end{aligned}$$

The continuity of the trace map from $H^3(\Omega^+)$ to $H^{\frac{5}{2}}(\Gamma)$ gives

$$\|v_i^+(\cdot, 0)\|_{\rho, \frac{5}{2}} \leq C \|v_i^+\|_{\rho, 3}^{0,+} \leq C \|v_i^+\|_{\rho, 3, \sigma}^+, \quad i = 1, 2.$$

Therefore,

$$\begin{aligned} \|v_i^+(\cdot, 0) \partial_i f\|_{\rho, \frac{5}{2}} &\leq C \|v_i^+(\cdot, 0)\|_{\rho, \frac{5}{2}} \|\partial_i f\|_{\rho, \frac{5}{2}} \\ &\leq C \|v_i^+\|_{\rho, 3, \sigma}^+ \|f\|_{\rho, \frac{7}{2}} \leq C R \eta_0. \end{aligned}$$

Thus we deduce that

$$\|\partial_t \Psi_f\|_{\rho', 3, \sigma} \leq C. \quad (5.13)$$

The same estimate also holds for $\partial_t \Psi_g$.

► **The term $\partial_t \Psi_f - \partial_t \Psi_g$:**

In the same way as (5.13), we have:

$$\|\partial_t \Psi_f - \partial_t \Psi_g\|_{\rho, 3, \sigma} = \|\Psi_{\partial_t(f-g)}\|_{\rho, 3, \sigma} \leq C \|(v^+ \cdot N_f)|_\Gamma - (w^+ \cdot N_g)|_\Gamma\|_{\rho, \frac{5}{2}}.$$

Then, we write

$$v^+ \cdot N_f - w^+ \cdot N_g = v^+ \cdot (N_f - N_g) + N_g \cdot (v^+ - w^+).$$

The terms v^+ and N_g are bounded by a constant $C > 0$ (**independent of ρ and ρ'**), and the terms $N_f - N_g$ and $v^+ - w^+$ are bounded from above by $\|U - V\|_\rho$ (we use in particular (5.11)). We get:

$$\|\partial_t \Psi_f - \partial_t \Psi_g\|_{\rho, r, \sigma} \leq C \|U - V\|_\rho. \quad (5.14)$$

► **The term** $\partial_3 v^\pm - \partial_3 w^\pm$:

We finish with a straightforward estimate due to (1.25) (see Theorem 1.8). It will be crucial to apply Cauchy-Kowalevskaya theorem:

$$\|\partial_3(v^\pm - w^\pm)\|_{\rho',3,\sigma}^\pm \leq \frac{C}{\rho - \rho'} \|v^\pm - w^\pm\|_{\rho,3,\sigma}^\pm \leq \frac{C}{\rho - \rho'} \| \|U - V\| \|_\rho. \quad (5.15)$$

Endowed with the estimates (5.9)-(5.15), we have the main arguments to estimate the convective terms contained in F .

Proposition 5.1. *Let $\rho \in (0, \rho_0]$ and $U, V \in \mathbb{B}_\rho$ satisfying (5.7) with f and g belonging to the ball given by (5.4). Then for all ρ' such that $0 < \rho' < \rho$, the following estimate holds:*

$$\| \|(\tilde{v}^\pm \cdot \nabla)v^\pm - (\tilde{w}^\pm \cdot \nabla)w^\pm\| \|_{\rho'} \leq \frac{C}{\rho - \rho'} \| \|U - V\| \|_\rho, \quad (5.16)$$

where $C > 0$ does not depend on ρ and ρ' .

Remark: the estimates of the other convective terms appearing in F are analogous.

Proof. Using the definitions of \tilde{v}^\pm and \tilde{w}^\pm given by (1.10), we write:

$$(\tilde{v}^\pm \cdot \nabla)v^\pm - (\tilde{w}^\pm \cdot \nabla)w^\pm = (v'^\pm \cdot \nabla')v^\pm - (w'^\pm \cdot \nabla')w^\pm \quad (5.17)$$

$$+ \left(\frac{v^\pm \cdot N_f - \partial_t \Psi_f}{J_f} \right) \partial_3 v^\pm - \left(\frac{w^\pm \cdot N_g - \partial_t \Psi_g}{J_g} \right) \partial_3 w^\pm. \quad (5.18)$$

We recall that the notation “'” refers to the tangential parts. We note E_1 the right term of (5.17) and E_2 the term given by (5.18).

First, we easily estimate E_1 as follows. We use properties (1.24) and (1.25) holding in the spaces $\mathbb{B}_{\rho,3,\sigma}$. The i -th component of the vector $(u \cdot \nabla)H - (\tilde{u} \cdot \nabla)\tilde{H}$ is given by

$$u_j \partial_j H_i - \tilde{u}_j \partial_j \tilde{H}_i,$$

using Einstein's summation convention over j . Then, we compute:

$$\begin{aligned} \|u_j \partial_j H_i - \tilde{u}_j \partial_j \tilde{H}_i\|_{\rho',r,\sigma} &= \|u_j \partial_j (H_i - \tilde{H}_i) + (u_j - \tilde{u}_j) \partial_j \tilde{H}_i\|_{\rho',r,\sigma} \\ &\leq \frac{C_r R \sigma^{-1}}{\rho - \rho'} \|H_i - \tilde{H}_i\|_{\rho,r,\sigma} + \frac{C_r R \sigma^{-1}}{\rho - \rho'} \|u_j - \tilde{u}_j\|_{\rho,r,\sigma} \\ &\leq \frac{C_r R \sigma^{-1}}{\rho - \rho'} \left(\| \|u - \tilde{u}\| \|_{\rho,r,\sigma} + \| \|H - \tilde{H}\| \|_{\rho,r,\sigma} \right). \end{aligned}$$

Therefore we obtain for E_1 :

$$\|E_1\|_{\rho'} \leq \frac{C}{\rho - \rho'} \| \|U - V\| \|_\rho,$$

where $C = C(R) > 0$ depends on R (recall that $\| \|U\| \|_\rho, \| \|V\| \|_\rho < R$), but **not** on ρ and ρ' . Now we split E_2 as follows:

$$\begin{aligned} E_2 &= \left[\left(\frac{v^\pm \cdot N_f}{J_f} \right) \partial_3 v^\pm - \left(\frac{w^\pm \cdot N_g}{J_g} \right) \partial_3 w^\pm \right] - \left[\frac{\partial_t \Psi_f}{J_f} \partial_3 v^\pm - \frac{\partial_t \Psi_g}{J_g} \partial_3 w^\pm \right] \\ &=: E_{21} - E_{22}. \end{aligned}$$

We go on with the following equality:

$$\begin{aligned} E_{21} &= \left(\frac{v^\pm \cdot N_f}{J_f} \right) \partial_3 (v^\pm - w^\pm) + \left(\frac{v^\pm \cdot N_f}{J_f} - \frac{w^\pm \cdot N_g}{J_g} \right) \partial_3 w^\pm \\ &=: E_{211} + E_{212}. \end{aligned}$$

Using (5.9), (5.10) and (5.15), we get the estimate:

$$\| \|E_{211}\| \|_{\rho'} \leq \frac{C}{\rho - \rho'} \| \|U - V\| \|_\rho. \quad (5.19)$$

Let us now treat the term E_{212} . We expand it this way:

$$\begin{aligned} E_{212} &= \left[\frac{1}{J_f} (N_f - N_g) \cdot v^\pm \right] \partial_3 w^\pm + \left[\frac{J_g - J_f}{J_f J_g} v^\pm \cdot N_g \right] \partial_3 w^\pm + \left[\frac{1}{J_g} N_g \cdot (v^\pm - w^\pm) \right] \partial_3 w^\pm \\ &=: E_{2121} + E_{2122} + E_{2123}. \end{aligned}$$

To estimate E_{2121} , we use (5.9), (5.11), (5.7) and bound from above $\|\partial_3 w^\pm\|_{\rho',3,\sigma}^\pm$ by $\frac{CR}{\rho-\rho'}$. We end up with the desired estimate for E_{2121} . We use similar estimates to treat the terms E_{2122} and E_{2123} . Therefore we can write

$$\| \| E_{212} \| \|_{\rho'} \leq \frac{C}{\rho - \rho'} \| \| U - V \| \|_{\rho}, \quad (5.20)$$

which ends the estimate of E_{21} . We briefly finish with the case of E_{22} . We split it as follows:

$$\begin{aligned} E_{22} &= \left[\frac{J_g - J_f}{J_f J_g} \partial_t \Psi_f \right] \partial_3 v^\pm + \left[\frac{1}{J_g} (\partial_t \Psi_f - \partial_t \Psi_g) \right] \partial_3 v^\pm + \frac{1}{J_g} \partial_t \Psi_g \partial_3 (v^\pm - w^\pm) \\ &=: E_{221} + E_{222} + E_{223}. \end{aligned}$$

In order to estimate E_{221} , we use (5.9), (5.12), (5.13) and we bound from above $\|\partial_3 v^\pm\|_{\rho',3,\sigma}^\pm$ by $\frac{CR}{\rho-\rho'}$. Likewise, we use the same type of arguments to handle E_{222} and E_{223} , which gives the desired estimate for E_{22} . \square

Remark: the estimate of the third component of $F(U) - F(V)$ is analogous. However, we use norms only on Γ instead of Ω^\pm , together with trace estimates. It suffices to follow the same previous computations leading to (5.13). To conclude, all the ‘‘convective’’ terms of $F(U) - F(V)$ provide the desired estimate (5.8). From now on, it remains to deal with the ‘‘pressure part’’ of F , in order to apply Cauchy-Kowlevskaya theorem.

5.2. Estimate of the pressure. Let us denote A^f (resp. A^g) the jacobian matrix associated with f (resp. g), already defined by (1.8). The total pressure associated to U (resp. V) will be noted Q^\pm (resp. P^\pm). In order to complete the estimate of $\| \| F(U) - F(V) \| \|_{\rho'}$, it remains to obtain the following result:

$$\| \| (A^f)^T \nabla Q^\pm - (A^g)^T \nabla P^\pm \| \|_{\rho',3,\sigma}^\pm \leq \frac{C}{\rho - \rho'} \| \| U - V \| \|_{\rho}, \quad (5.21)$$

where $C > 0$ may depend on R , but not on ρ and ρ' . Let us write the i -th component of $(A^f)^T \nabla Q^\pm - (A^g)^T \nabla P^\pm$ as follows:

$$\begin{aligned} A_{ji}^f \partial_j Q^\pm - A_{ji}^g \partial_j P^\pm &= \left(A_{ji}^f - A_{ji}^g \right) \partial_j Q^\pm + A_{ji}^g \partial_j (Q^\pm - P^\pm) \\ &=: E_1 + E_2. \end{aligned}$$

We write $A^h = I_3 - \tilde{A}^h$, where h stands for either f or g , and \tilde{A}^h is defined by

$$\tilde{A}^h := \frac{1}{J_h} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_1 \Psi_h & \partial_2 \Psi_h & \partial_3 \Psi_h \end{pmatrix}. \quad (5.22)$$

We get $E_1 = \left(\tilde{A}_{ji}^f - \tilde{A}_{ji}^g \right) \partial_j Q^\pm$. To estimate this term, we use the algebra property of the space $B_{\rho',3,\sigma}(\Omega^\pm)$:

$$\| \| E_1 \| \|_{\rho',3,\sigma}^\pm \leq C \| \| \tilde{A}_{ji}^f - \tilde{A}_{ji}^g \| \|_{\rho',3,\sigma} \| \| \partial_j Q^\pm \| \|_{\rho',3,\sigma}.$$

On the one hand, we have:

$$\| \| \tilde{A}_{ji}^f - \tilde{A}_{ji}^g \| \|_{\rho',3,\sigma} = \left\| \left\| \frac{\partial_i \Psi_f}{J_f} - \frac{\partial_i \Psi_g}{J_g} \right\| \right\|_{\rho',3,\sigma}.$$

Using the same techniques as in Paragraph 5.1, we obtain the following estimate without loss of $\frac{1}{\rho-\rho'}$:

$$\| \| \tilde{A}_{ji}^f - \tilde{A}_{ji}^g \| \|_{\rho',3,\sigma} \leq C \| \| U - V \| \|_{\rho}.$$

On the other hand, to estimate the term $\| \| \partial_j Q^\pm \| \|_{\rho',3,\sigma}$, we write:

$$\| \| \partial_j Q^\pm \| \|_{\rho',3,\sigma} \leq \frac{C}{\rho - \rho'} \| \| Q^\pm \| \|_{\rho,3,\sigma},$$

and we use the result (4.66) of Theorem 4.2 to handle the case of $\| \| Q^\pm \| \|_{\rho,3,\sigma}$:

$$\| \| Q^\pm \| \|_{\rho,3,\sigma}^\pm \leq C \left(\| \| \mathcal{F}^\pm \| \|_{\rho,1,\sigma}^\pm + \| \| \mathcal{G} \| \|_{\rho,\frac{3}{2}} \right),$$

where from now on \mathcal{F}^\pm and \mathcal{G} are defined by (3.2) and (3.3).

In order to estimate \mathcal{F}^\pm in $B_{\rho,1,\sigma}(\Omega^\pm)$, we feel free to directly use the inequality $\|\cdot\|_{\rho,1,\sigma}^\pm \leq \|\cdot\|_{\rho,2,\sigma}^\pm$, because \mathcal{F}^\pm only contains products of 1st-order derivative terms. Indeed, for instance let us estimate the last term appearing in \mathcal{F}^\pm (see (3.2)), namely $\tilde{B}^\pm \cdot \nabla A_{ki} \partial_k B_i^\pm$. Using definition (5.22), we first write:

$$\tilde{B}^\pm \cdot \nabla A_{ki} \partial_k B_i^\pm = \tilde{B}_j^\pm \partial_j \left(\frac{\partial_i \Psi_f}{J_f} \right) \partial_3 B_i^\pm.$$

By algebra property of the spaces $B_{\rho,2,\sigma}(\Omega^\pm)$, we get:

$$\left\| \tilde{B}_j^\pm \partial_j \left(\frac{\partial_i \Psi_f}{J_f} \right) \partial_3 B_i^\pm \right\|_{\rho,1,\sigma}^\pm \leq C \|\tilde{B}_j^\pm\|_{\rho,2,\sigma}^\pm \left\| \partial_j \left(\frac{\partial_i \Psi_f}{J_f} \right) \right\|_{\rho,2,\sigma} \|\partial_3 B_i^\pm\|_{\rho,2,\sigma}^\pm. \quad (5.23)$$

If $j \in \{1, 2\}$, we have $\tilde{B}_j^\pm = B_j^\pm$ therefore $\|\tilde{B}_j^\pm\|_{\rho,2,\sigma}^\pm \leq \|B_j^\pm\|_{\rho,3,\sigma}^\pm \leq R$. However, if $j = 3$, we have $\tilde{B}_3^\pm = \frac{B^\pm \cdot N_f}{J_f}$, so:

$$\|\tilde{B}_3^\pm\|_{\rho,2,\sigma}^\pm \leq C \left\| \frac{1}{J_f} \right\|_{\rho,2,\sigma} \|B^\pm \cdot N_f\|_{\rho,2,\sigma}^\pm \leq C.$$

Then,

$$\left\| \partial_j \left(\frac{\partial_i \Psi_f}{J_f} \right) \right\|_{\rho,2,\sigma} \leq \left\| \frac{\partial_i \Psi_f}{J_f} \right\|_{\rho,3,\sigma} \leq C \left\| \frac{1}{J_f} \right\|_{\rho,3,\sigma} \|\partial_i \Psi_f\|_{\rho,3,\sigma} \leq C \left\| \frac{1}{J_f} \right\|_{\rho,3,\sigma} \|f\|_{\rho, \frac{7}{2}} \leq C.$$

Finally, the last term of (5.23) is estimated as follows:

$$\|\partial_3 B_i^\pm\|_{\rho,2,\sigma}^\pm \leq \|B_i^\pm\|_{\rho,3,\sigma} \leq R.$$

Consequently, we deduce that

$$\|\tilde{B}^\pm \cdot \nabla A_{ki} \partial_k B_i^\pm\|_{\rho,1,\sigma}^\pm \leq C,$$

where $C > 0$ depends on R , but not on ρ and ρ' . All the other terms appearing in the definition (3.2) of \mathcal{F}^\pm are similarly estimated. Therefore we conclude that $\|\mathcal{F}^\pm\|_{\rho,1,\sigma}^\pm \leq C$.

The case of \mathcal{G} is analogous, because using the continuity of the trace from $H^2(\Omega^\pm)$ to $H^{\frac{3}{2}}(\Gamma)$, all the 1st-order terms contained in \mathcal{G} will be estimated in $B_{\rho,2,\sigma}(\Omega^\pm)$, exactly as above for \mathcal{F}^\pm .

To conclude about the estimate of Q^\pm , we can write that

$$\|Q^\pm\|_{\rho,3,\sigma} \leq C.$$

Eventually, we end up with the desired estimate for E_1 , that is:

$$\|E_1\|_{\rho',3,\sigma} \leq \frac{C}{\rho - \rho'} \|U - V\|_{\rho}. \quad (5.24)$$

Let us now explain how to handle the case of E_2 . We begin with the estimate

$$\|E_2\|_{\rho',3,\sigma} = \|A_{ji}^g \partial_j (Q^\pm - P^\pm)\|_{\rho',3,\sigma}^\pm \leq \frac{C}{\rho - \rho'} \|A_{ji}^g\|_{\rho',3,\sigma} \|Q^\pm - P^\pm\|_{\rho,3,\sigma}.$$

Still using the same arguments as in Paragraph 5.1, we can write:

$$\|A_{ji}^g\|_{\rho',3,\sigma} \leq C.$$

Besides, let us denote \mathcal{F}_U^\pm (resp. \mathcal{F}_V^\pm) and \mathcal{G}_U (resp. \mathcal{G}_V) the source terms (3.2) and (3.3) respectively related to U and V , and let us set $\Pi^\pm := Q^\pm - P^\pm$. In order to estimate $\|\Pi^\pm\|_{\rho,3,\sigma}^\pm$, let us write the elliptic problem satisfied by (Π^+, Π^-) down, with the coefficients $A_{ji} = A_{ji}^{f-g}$ coming from the function $f - g$ (we adapt the definition (5.22)):

$$A^{f-g} := I_3 - \tilde{A}^{f-g} \quad \text{with} \quad \tilde{A}^{f-g} := \frac{1}{J_{f-g}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \partial_1 \Psi_f - \partial_1 \Psi_g & \partial_2 \Psi_f - \partial_2 \Psi_g & \partial_3 \Psi_f - \partial_3 \Psi_g \end{pmatrix}.$$

Above, we have used the linearity of the map Ψ . However the map $f \mapsto J_f$ is not linear: we only have the identity $J_{f-g} = 1 + \partial_3 \Psi_{f-g} = J_f - J_g + 1$. Up to decrease the constant ε_0 in the definition (5.4), this ‘‘new’’ jacobian remains positive. The elliptic problem satisfied by (Π^+, Π^-) is similar to (4.1), and reads as follows:

$$\begin{cases} -A_{ji}^{f-g} \partial_j (A_{ki}^{f-g} \partial_k \Pi^\pm) = \mathbb{F}^\pm & \text{in } (0, T) \times \Omega^\pm, \\ [\Pi] = 0 & \text{on } (0, T) \times \Gamma, \\ (1 + |\nabla'(f-g)|^2) [\partial_3 \Pi] = \mathbb{G} & \text{on } (0, T) \times \Gamma, \\ \partial_3 \Pi^\pm = 0 & \text{on } (0, T) \times \Gamma_\pm. \end{cases} \quad (5.25)$$

Applying estimate (4.66) to Π^\pm , we thus have

$$\|\Pi^\pm\|_{\rho,3,\sigma}^\pm \leq C \left(\|\mathbb{F}^\pm\|_{\rho,1,\sigma}^\pm + \|\mathbb{G}\|_{\rho,\frac{3}{2}} \right). \quad (5.26)$$

Now it remains to write both source terms \mathbb{F}^\pm and \mathbb{G} down. To do so, we will make a link with the previous elliptic problems satisfied first by (Q^+, Q^-) (i.e. with the coefficients A_{ji}^f), then by (P^+, P^-) (i.e. with the coefficients A_{ji}^g). We thus write

$$A^{f-g} = A^f - A^g + R(f, g), \quad (5.27)$$

where the matrix $R(f, g)$ is a remaining term defined by

$$R(f, g) := I_3 - \tilde{R}(f, g) \quad \text{with} \quad \tilde{R}(f, g) := \frac{1}{J_{f-g}} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{R}_{31} & \tilde{R}_{32} & \tilde{R}_{33} \end{pmatrix},$$

and

$$\tilde{R}_{3i} := \frac{J_g - 1}{J_f(J_f - J_g + 1)} \partial_i \Psi_f - \frac{2J_g - J_f - 1}{J_g(J_f - J_g + 1)} \partial_i \Psi_g, \quad i = 1, 2, 3.$$

Using (5.27) into the first equation of (5.25), we get the expression of the source terms \mathbb{F}^\pm :

$$\begin{aligned} \mathbb{F}^\pm &:= \mathcal{F}_U^\pm - \mathcal{F}_V^\pm + A_{ji}^f \partial_j (A_{ki}^f \partial_k P^\pm) - A_{ji}^g \partial_j (A_{ki}^g \partial_k Q^\pm) \\ &\quad + A_{ji}^f \partial_j (A_{ki}^g \partial_k \Pi^\pm) + A_{ji}^g \partial_j (A_{ki}^f \partial_k \Pi^\pm) - (A_{ji}^f - A_{ji}^g) \partial_j (R_{ki} \partial_k \Pi^\pm) \\ &\quad + R_{ji} \partial_j ((A_{ki}^f - A_{ki}^g) \partial_k \Pi^\pm) + R_{ji} \partial_j (R_{ki} \partial_k \Pi^\pm). \end{aligned}$$

The boundary source term \mathbb{G} can be obtained by expanding $|\nabla'(f-g)|^2$ in the third equation of (5.25):

$$\mathbb{G} := \mathcal{G}_U - \mathcal{G}_V - (1 + |\nabla'f|^2) [\partial_3 P] + (1 + |\nabla'g|^2) [\partial_3 Q] - (1 + 2\nabla'f \cdot \nabla'g) [\partial_3 \Pi].$$

Then, it remains to estimate the appearing source terms in (5.26). For the sake of clarity, we choose not to detail these computations, since the method is rigorously identical to Paragraph 5.1. We end up with the following estimate:

$$\|\mathbb{F}^\pm\|_{\rho,1,\sigma}^\pm + \|\mathbb{G}\|_{\rho,\frac{3}{2}} \leq C \|\|U - V\|\|_\rho,$$

where $C > 0$ depends on R but not on ρ and ρ' . This achieves the case of E_2 :

$$\|E_2\|_{\rho',3,\sigma} \leq \frac{C}{\rho - \rho'} \|\|U - V\|\|_\rho. \quad (5.28)$$

Combining (5.24) together with (5.28), we eventually get the desired estimate (5.21).

5.3. Application of Cauchy-Kowalevskaya theorem: proof of Theorem 1.10. In the previous paragraph, we have shown that the “vector field” $F(U)$ satisfies the main assumption of Cauchy-Kowalevskaya theorem, namely the following “Lipschitz” estimate holding for $\|\|U\|\|_\rho, \|\|V\|\|_\rho < R$:

$$\|\|F(U) - F(V)\|\|_{\rho'} \leq \frac{C}{\rho - \rho'} \|\|U - V\|\|, \quad \forall 0 < \rho' < \rho \leq \rho_0.$$

The map $U \mapsto F(U)$ also has to be continuous from the ball $B_\rho(0, R) \subset \mathbb{B}_\rho$ to $\mathbb{B}_{\rho'}$ for all $\rho' < \rho$. This is obviously satisfied, using in particular the continuity of the differentiation from \mathbb{B}_ρ to $\mathbb{B}_{\rho'}$ (see Proposition 1.6 and Theorem 1.8).

Eventually, applying Cauchy-Kowalevskaya theorem [Nis77], we end the proof of Theorem 1.10.

Some remarks about Theorem 1.10:

We observe that the total pressure Q^\pm has the same regularity as v^\pm and B^\pm in a Sobolev scale. Together with the estimate (4.66), it explains why the gradient of pressure plays the role of a *quasilinear* term. Unlike the case of a fixed domain (i.e. without the sheet), Chemin [Che98] proved that for the Euler equations, the gradient of pressure plays the role of a *semilinear* term.

In (5.4), the appearing smallness condition for the front f turns out to be sufficient, but it might not be necessary. We recall that we used this condition in order to estimate the inverse of the jacobian of the change of variables in the analytic spaces $B_{\rho,r,\sigma}(\Omega)$ (see Corollary 2.3). This condition, which seems to be restrictive, could be neglected or, at least, improved. To do so, we could estimate $\|J^{-1}\|_{\rho,r,\sigma}$ in an other way, e.g. using Faà di Bruno’s identity to estimate the derivatives $\partial^\alpha \left(\frac{1}{1+\partial_3 \psi} \right)$ in $H^r(\Omega)$.

The proof of Cauchy-Kowalevskaya theorem given by Nishida [Nis77] does not exactly lead to the result stated in Theorem 1.10. But we can easily adapt the proof of Nishida, in order to use a fixed point argument in a product of open balls of the form $B_\rho(0, R)^2 \times B_\rho(0, \eta_0)$, and not in a ball $\mathcal{B}_\rho(0, R) \subset \mathbb{B}_\rho$. The first notation $B_\rho(0, R)$ stands for the open ball of radius R centred at 0 in $\mathbb{B}_{\rho, 3, \sigma}(\Omega^\pm)$, and concerns the unknowns (v^\pm, B^\pm) . The second notation $B_\rho(0, \eta_0)$ stands for the open ball of radius η_0 centred at 0 in $B_{\rho, \frac{7}{2}}(\mathbb{T}^2)$, and concerns the unknown f . In particular, it allows to require a smallness condition *only* on the front f .

6. CONCLUSION

The analytic solution we obtain is thus defined at least until the time $a\rho_0 > 0$. However, the quantity $a > 0$ depends on R , namely on the size of the initial data in the analytic scale. Therefore the lifespan can become smaller as the initial data get bigger in the analytic spaces $\mathbb{B}_{\rho_0, 3, \sigma}$.

The *a priori* estimate established in [CMST12] would be a starting point to prove, by a compactness argument, the existence and uniqueness of solutions to the current-vortex sheet problem, taking initial data in some Sobolev space (typically H^3 for the velocity v_0 and the magnetic field B_0 , and $H^{3.5}$ for the front f_0).

To do so, we would approximate the Sobolev initial data (v_0^\pm, B_0^\pm, f_0) by a sequence of analytic initial data $(v_0^{\pm, n}, B_0^{\pm, n}, f_0^n)_{n \geq 0}$, using the density of the spaces $\mathbb{B}_{\rho, r, \sigma}$ in H^r mentioned in Section 1.4. The unique solution $(v^{\pm, n}, B^{\pm, n}, Q^{\pm, n}, f^n)$ associated with this initial data thus has a radius of analyticity $\rho^n = \rho^n(t)$, that possibly tends to 0 as n goes to $+\infty$. To address such an issue, it would suffice to exhibit a lower bound on $\rho^n(t)$, that depends only on a Sobolev norm of the solution. Therefore we could propagate the analyticity of the solutions to a time interval depending **only** on the Sobolev norm of the solutions. The latter does not blow up as n goes to $+\infty$, and would allow to get a positive lower bound on the lifespan of the solutions.

For results about propagation of analyticity of the solutions to the incompressible Euler equations, we can refer to [BB77], [AM86] and more recently to [LO97], [KV09], [KV11a], [KV11b].

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LABORATOIRE DE MATHÉMATIQUES JEAN LERAY, UNIVERSITÉ DE NANTES, UMR CNRS 6629,
 2, RUE DE LA HOUSSINIÈRE, 44322 NANTES CEDEX 03, FRANCE
E-mail address: `olivier.pierre@univ-nantes.fr`