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On advanced mathematical methods and more elementary ideas met (or not) before

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Some mathematical problems, which could be solved by a general approach, also have different and often original solutions that appeal to less advanced mathematics. For example, drawing a tangent line to a parabola can be done by methods of differential calculus, and by exclusively using methods of Euclidean geometry. A project that I conducted with students who recently completed a Bachelor degree focusing in mathematics revealed that they were familiar with advanced methods but lacked more elementary views. I argue that unfamiliarity with alternative elementary solutions hinders for students the opportunities both to build mathematical connections and to appreciate the groundwork of related advanced methods.

Keywords: Problems with multiple solutions, connectivity of mathematics, parabola, Euclidean geometry, calculus.

INTRODUCTION

Some problems employed in teaching to illuminate the essence of a mathematical method appear to be universally useful in a variety of courses. In these cases students can compare how different ideas and techniques are applied to address the same mathematical question. The practice of identification of problems useful for systematic use in various university level courses is discussed in the literature. For example, Mingus (2002) referred to “calculation of n^{th} roots of unity” as a problem that “encourages students to see connections between geometry, vectors, group theory, algebra and long division” (p. 32). Further discussion revealed, “proving identities involving the Fibonacci numbers provide a solid connection between linear algebra, discrete mathematics, number theory and abstract algebra” (ibid, p. 32). Winsløw (2013) referred to several ways to approach constructions of a^x for $a > 0$ and x real, which are based on either “direct” extension of the domain from natural to real numbers,

or the inverse function to $\log_e(x) = \int_1^x dt/t$, or the initial value problem $dy/dx = y$, $y(0) = 1$, or the functional equation $f(x + y) = f(x) \cdot f(y)$, or the Maclaurin power series (p. 2481). Sun and Chan (2009) discussed nine proofs of the “Mid-Point theorem of triangles”. The fact that “the sum of the interior angles in a plane triangle is 180° ” can be shown in eight different ways (Tall et al., 2012, p. 35). In my view, these are examples of *interconnecting* problems, which have the following characteristics: they allow various solutions at more elementary and more advanced levels; they can be solved by various mathematical tools from different mathematical branches, which leads to finding multiple solutions; and they are used in different courses and can be understood in various contexts (Kondratieva, 2011a, 2011b).

When students familiar with a problem from their prior experiences use their intuition to support more elaborate techniques applicable to a problem, they also have a chance to perceive mathematics as a consistent subject (Kondratieva, 2011a). By means of investigation of such problems in different courses “students were able to review concepts from previous courses and improve their understanding of the old and new concepts” (Mingus, 2002, p. 32). Knowledge of multiple ways to treat a mathematical object can strengthen the relation of a learner to the object “in the sense of providing extensions or alternatives to standard presentations” (Winsløw, 2013, p. 2483). Leikin and her collaborators extensively studied “tasks that contain an explicit requirement for solving the problem in multiple ways” (Leikin & Levav-Waynberg, 2008, p. 234), particularly in the context of the development of mathematics teachers’ knowledge, and for an examination of mathematical creativity (Leikin & Lev, 2007). When used in mathematics teachers’ education, interconnecting problems foster teachers’ ability to link elementary ideas with advanced techniques (Kondratieva, 2013), which

might contribute to the construction of their horizon content knowledge (Ball, Thames, & Phelps, 2008), that is, an awareness of how mathematical topics are related in the span of the entire curriculum. While being familiar with some individual examples of the use of interconnecting problems in teaching mathematics, I do not know to what extent students in general are exposed to teaching practices that encourage them explicitly to make connections between advanced and more elementary mathematics. According to Winslow (2013), “there remains a practical need for systematic didactical research on how standard undergraduate mathematics is, or could be, developed in view of facilitating its use by students in inquiries related directly to high school mathematics.” (p. 2477)

My research question is to what extent students who have completed a Bachelor degree with a focus in mathematics are familiar with both advanced mathematical methods and more elementary ideas related to them. In order to address this question I developed a research instrument in the form of a handout and questionnaire, which is described in this paper. The handout includes an interconnecting problem, which is typical and familiar for students studying undergraduate mathematics. This problem can be solved by a standard method taught at the university level and also has a more elementary treatment, which reveals some insightful ideas. This research instrument, along with the theoretical framework and results of testing in a small group of students, are discussed in the following sections.

THEORETICAL CONSIDERATIONS

Modern curriculum at all levels is moving from a formal approach to a more inquiry-based study of mathematics, focusing on genuine understanding and connecting various concepts and methods. House and Coxford (1995) argued that presenting mathematics as a “woven fabric rather than a patchwork of discrete topics” is one of the most important outcomes of mathematics education. The goal of mathematical instruction consists of helping a dedicated learner go beyond instrumental understanding secured by knowing mathematical procedures, and achieve relational understanding between different mathematical topics (Skemp, 1987), which assumes connections of various mathematical ideas. “An ability to establish and use a wide range of connections offers students alternative paths to the solution” (Hodgson,

1995, p. 19). While making connections and multiple representations of ideas are recognized among the primary processes in learning mathematics (NCTM, 2000), there is also a need for teaching strategies “for engaging students in exploring the connectedness of mathematics” (House & Coxford, 1995, p. vii).

One possible way to address this need is to use problems which allow multiple solutions, or specifically, interconnecting problems. In the latter approach (Kondratieva, 2011a), students encounter an interconnecting problem several times as they progress throughout their education, each time learning a new aspect of the same problem and building their understanding on “supportive met-befores” (Tall, 2013, p.15). Rephrasing Watson and Mason’s (2005) description of reference examples, an interconnecting problem is “the one that becomes extremely familiar and is used to test out conjectures, to illustrate the meaning of theorems” (p. 7). Indeed, problems that have a range of solutions not only can help learners to move from elementary to advanced understanding, but they also may be used to exemplify advanced methods in elementary terms or to come up with an alternative and more elementary explanation of results found in a different way (Kondratieva, 2013).

In order to collect and analyze results presented in this paper, I employ Tall’s notion of crystalline concept. Formation of a crystalline concept in a learner’s mind refers to a phenomenon where an object of mathematical study “which originally was a single gestalt with many simultaneous properties, and was then defined using a single specific definition – now matures into a fully unified concept, with many properties linked together by a network of relationships based on deductions” (Tall et al., 2012, p. 20). The crystalline concept of an object combines all prior experiences of a learner in relation to this object, which include perceiving, acting upon, describing in natural or symbolic language, further formalizing, theorizing, and organizing knowledge about the object in a compressed way. Perceiving the object in various contexts, recognizing its multiple representations, and establishing equivalence relations between its various properties are important steps in the cognitive development of a learner towards building a corresponding crystalline concept. Eventually, “equivalent concepts may be grasped as a single crystalline concept that has all the requisite properties blended together within a single entity. Powerful mathematical thinking at the highest

level involves the external relationship between, and the internal relationship within, crystalline concepts” (Tall, 2013, p. 403).

The notion of crystallization comprises various frameworks and theories of knowledge compression including the Structure of Observed Learning Outcomes (SOLO) Taxonomy (Biggs & Collis, 1982). The SOLO Taxonomy describes the progression of a learner’s development using the following stages: (0) pre-structural, when the learner demonstrates a very limited understanding of a problem; (1) uni-structural, when the learner uses only one aspect of a concept and follows a single procedure to solve a problem; (2) multi-structural, when the learner refers to several aspects and is able to carry out several procedures to solve a problem; (3) relational, when the learner relates several aspects together or sees the equivalence of different procedures; (4) extended abstract, when the learner grasps the concept so well that they can apply it outside of the problem’s domain. Thus, crystallization requires both the familiarity with multiple aspects of a concept and relational unification of them.

Further analysis of the development of mathematical thoughts at both the historical and individual level suggests that while mathematical arguments become more sophisticated and formal, “true mathematical thinking should become not only more powerful but more simple” (Tall, 2013, p. 19). According to Atiyah, “not only mathematics but science as a whole, only progresses if you can understand things... Its aim is to produce ideas and explain things in simple terms” (see Tall, 2013, p. 400). Similarly, Polya (1945/2004) suggested to always look back at your solution and ask yourself “Can you derive your result differently? Can you see it at a glance?” However, the simplicity and transparency of mathematical arguments produced by students often depends upon their prior exposure to basic but enlightening ideas related to more advanced methods (Kondratieva, 2014), as well as on whether they possess proper crystalline concepts. The aim of this paper is to delve on this issue using an example from undergraduate mathematics.

RESEARCH INSTRUMENT AND ITS PRELIMINARY TESTING

A group of 16 students who had completed their undergraduate degree participated in the project. These students were enrolled in my methods course for

pre-service teachers whose teachable areas included mathematics and another subject (most commonly science). Each of these students had taken at least eight mathematical courses including at least three courses at the 3rd or 4th undergraduate level (with average mark above 70%). While the project was conducted during a regular class time, the participation was optional and no mark was assigned to this work. The students could have chosen to perform an alternative practice assignment; however, everyone in class agreed to participate in the project. The students worked in pairs. They were asked to develop a theoretical solution and implement it using technology. They were presented with the problem and questionnaire.

Problem: Use dynamic geometry software to draw a parabola and its tangent line at a point using tools of your preference.

Through other assignments in my course the students were familiar with dynamic geometry software such as GeoGebra capable of drawing points, segments, parallel and perpendicular lines, circles etc., as well as graphs of functions given by their equations in the form $y = f(x)$, and manipulating these objects. The tool of drawing a tangent line to a given curve was not amongst the available options.

Questionnaire:

- (1) Have you seen this or similar problems before?
- (2) If yes, identify the course title or corresponding mathematical context appropriate to treat this problem (e.g. linear algebra, geometry, combinatorics, pre-calculus, differential or integral calculus).
- (3) List all methods you know to approach the problem. Do you see any connections between them?
- (4) If you have never seen this problem before, state another problem of which this problem reminds you, and which you can solve.

When students had completed the first four questions (normally within 20 minutes or less) the last question was given to them.

- (5) Read the following sample solutions and choose the most appropriate description: (a) I have found the same solution; (b) This approach is familiar but I

forgot the details; (c) I have never seen this method before but I can understand and explain the ideas used in this approach; (d) I have never seen anything like that and I do not understand it.

The students who chose answer (b) or (c) were asked to explain the corresponding solution in writing. Then, we had a whole class discussion where students could reflect on the solutions and connections between them.

Sample solution 1: Use the standard equation of the parabola in the Cartesian coordinates and Differential Calculus.

Let the parabola be given by the equation $y = ax^2$. The derivative of the function $f(x) = ax^2$ is $f'(x) = 2ax$. The tangent line at a point (x_0, ax_0^2) has the form $(y - ax_0^2) = 2ax_0(x - x_0)$ or after a simplification $y = (2ax_0)x - ax_0^2$. Now we can draw the parabola and the line in the coordinate plane using the tool “graphing functions given by their equations $y = f(x)$ ” for some values of $a \neq 0$ and x_0 .

Sample solution 2: Produce this construction by means of Euclidean geometry given the focus F and directrix l of the parabola.

Geometrically, a parabola is the locus of points equidistant from the given point F and the given line l not passing through F . This definition enables the following construction (see Figure 1, left): (1) Drop the perpendicular from F onto l with foot at A ; the midpoint V of the segment FA is the vertex of the parabola; (2) Pick any point B on l ; join F and B and draw the perpendicular bisector l' to FB ; draw l'' perpendicular to l at B ; let l' and l'' meet at C . As point B moves along line l , point C traces a parabola (Figure 1, right). Then the line l' is tangent to the parabola with C being the point of tangency.

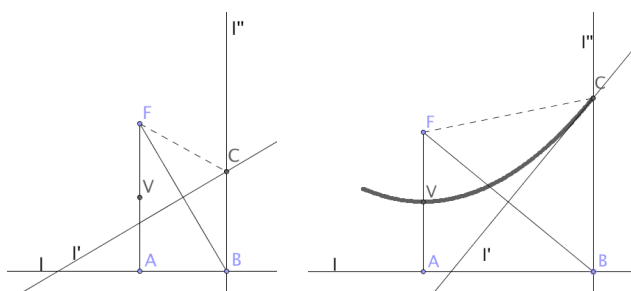


Figure 1: Construction of a parabola given its focus F and directrix l

RESULTS AND DISCUSSION

The problem that was given to the students is a typical problem studied in Calculus courses, where students are taught that the derivative $f'(x_0)$ represents the slope of the line tangent to the curve $y = f(x)$ at a given point $x = x_0$. They are also supposed to know that the equation of a straight line passing through a given point (a, b) with a given slope k is given by $(y - b) = k(x - a)$. Therefore, Sample solution 1 is a universal approach to this type of problem and all of the 16 participants recognized it as a familiar method to apply. However, none of them mentioned the second method as a possibility to solve this problem and only one participant recognized familiar ideas in the method. The majority of participants (15) answered “I have never seen this method before but I can understand and explain the ideas used in this approach”. The problem was that the majority of students did not know the definition of a parabola in terms of its directrix and focus.

Nevertheless, once students read the geometric definition of the parabola all of them were able to understand and explain the construction presented in Figure 1: since C lies on the perpendicular bisector l' of FB , the triangle FCB is isosceles and hence C is equidistant from F and l . In terms of the SOLO Taxonomy described above this signifies passage to the second stage, namely to the multi-structural understanding of the problem by the students.

I refer to the second solution as being more elementary because it uses mathematical ideas less advanced compared to those needed to develop calculus. However, students may have a different point of view. They may perceive the algorithmic approach studied in Calculus as being easier than using Euclidean geometry, which they experienced to a lesser degree.

Knowing two approaches to solve the problem is obviously progress. However, in order to develop the crystalline concept it is also necessary for students to see a relation between the two approaches. Indeed, the students were able to establish a connection to the formulas given in the Sample solution 1 by introducing Cartesian coordinates so that l has equation $y = -p$, and the coordinates of points are $F(0, p)$, $V(0, 0)$ and $C(x, y)$. Without loss of generality we assume that $p > 0$. Then, by the Pythagorean theorem, $\overline{FC}^2 = x^2 + (y - p)^2$. But, $\overline{FC} = \overline{CB} = y + p$, and after simplifications one gets the equation $y = \frac{x^2}{4p}$. At the same time the slope

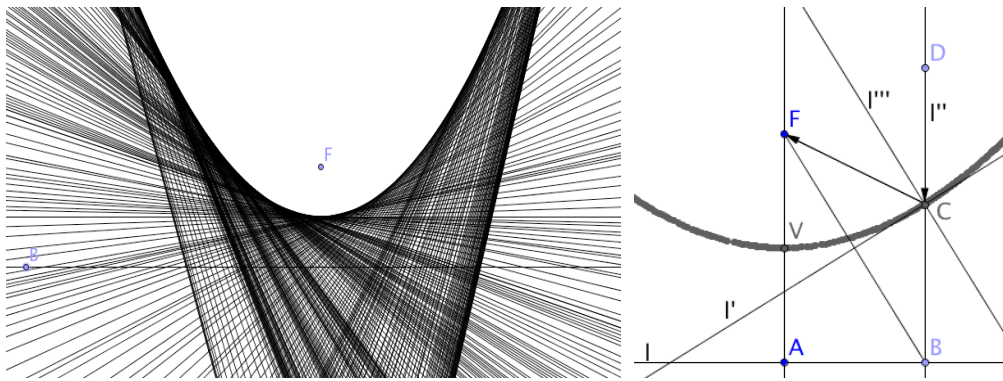


Figure 2: A family of tangent lines to the parabola (left) and reflection property (right)

of FB is $-\frac{p}{x}$, so the slope of l' is $\frac{x}{2p}$. This result agrees with the Calculus approach, by which the derivative is $\frac{d}{dx}\left(\frac{x^2}{4p}\right) = \frac{x}{2p}$. Finally, the equation of the tangent line at point $(x_0, \frac{x_0^2}{4p})$ is $y = f(x) = \frac{x_0}{2p}x - \frac{x_0^2}{4p}$. One can also see that $f(x_0) = \frac{x_0^2}{4p}$ and for any $x_1 \neq x_0$ we have $f(x_1) = \frac{2x_0x_1 - x_0^2}{4p} < \frac{x_1^2}{4p}$ since $x_0^2 - 2x_0x_1 + x_1^2 = (x_0 - x_1)^2 > 0$ and $p > 0$. The inequality demonstrates that the points of the parabola $y = \frac{x^2}{4p}$, $p > 0$, lie above any tangent line so that the family of tangent lines form the envelope of the curve (Figure 2, left).

The above derivation, along with reflection upon its results, is an example of constructing the crystalline concept of the parabola. In this project, I was able to observe that my students had adequate knowledge in order to move from the first to the third stage of SOLO Taxonomy in understanding and conceptualization of the parabola within only one lesson period. I did not conduct any activities aiming at identification of the fourth, extended abstract, stage. Nevertheless, I will comment on some opportunities emerging from this lesson. Clearly, a learner familiar with the geometric definition of parabola may find it satisfactory to obtain the same result by the more universal method learned in Calculus. By using dynamic geometry software and moving the point B along the line l , one can see that the point C traces a curve to which the line l' is tangent (Figure 1, right). A learner equipped with such experiences will develop a more comprehensive understanding of the object called parabola and might be able to use alternative representations and properties of a parabola and its tangent line depending on the problems they need to solve. By developing representational flexibility of an object (e.g., the parabola) students become better prepared for solving non-routine problems (see also Bergsten, 2015).

The geometrical view allows students to make closer connections of mathematics with physical phenome-

na, such as reflection of light from a parabolic mirror. Indeed, since $\angle FCB$ and $\angle FCD$ are supplementary angles, their angular bisectors l' and l'' are orthogonal to each other (Figure 2, right). The parabolic mirror near point C can be 'replaced' by the tangent line l' . This implies that the ray DC parallel to the axis FA of the parabola will be reflected at the point C towards the focus F . Such observations and insights are especially important for future teachers of mathematics and science because it will allow them to enrich discussions in their classrooms.

Students' unfamiliarity with the geometric definition of the parabola prior to my lesson can be explained by recent changes in the mathematical curriculum. Bergsten (2015) observed that in Sweden, while in the 1960s the study of the parabola was embedded in a local mathematical organization of analytic and Euclidean geometry, since the 1980s it became embedded in a local mathematical organization of functions. Similarly, in Canada the geometrical definition of conics no longer has a place in the secondary school mathematics. At the university level, this definition is supposed to appear in the Calculus stream, for example, when equations of conics in polar coordinates are introduced. But at that point there is no time to study the geometry of the parabola in any detail because the focus of the course is on different methods. Thus, university graduates with a mathematical degree may actually never have seen a discussion of a parabola as a geometrical object even though they may know properties of geometrical objects (isosceles triangle, perpendicular bisector, etc.) necessary for understanding how and why the Sample solution 2 works, as was the case in my project.

CONCLUSION

The fact that completion of a bachelor degree in Mathematics does not automatically ensure *deep* knowledge of elementary mathematics is not new (see discussion in Winsløw & Grønabæk, 2014) and references there). Indeed, even if students can apply an advanced approach or formula familiar to them, they are not always able to elaborate or explain why that formula works through connecting more advanced ideas with elementary facts. I conjecture that in some cases the students are simply not aware of relevant elementary ideas, definitions, and interpretations.

To verify my conjecture I developed an instrument that can identify the degree to which students are familiar with various solutions of a standard problem in Calculus. A preliminary study revealed that while the students felt comfortable applying the standard method, most of them were not familiar with the ideas which are more elementary compared to the considered standard method. I am planning to conduct a larger study in order to validate these findings.

Meantime, I suggest that while it is important for students to be exposed to standard and advanced approaches through university courses, their education should also include experiences highlighting other approaches and more elementary mathematical ideas related to these advanced methods (especially if the students already have necessary background to understand and elaborate on these ideas). Otherwise, they will tend to stick to procedural and formal methods and will not be able to fully appreciate the results of advanced approaches because the connection to more basic but enlightening mathematics will be lost.

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