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# Abstract algebra, mathematical structuralism and semiotics

Thomas Hausberger

University of Montpellier, Montpellier, France, [thomas.hausberger@univ-montp2.fr](mailto:thomas.hausberger@univ-montp2.fr)

*I report in this paper on my attempt to help students reflect on the axiomatic method and structuralist thinking in mathematics through a didactically-engineered activity (the theory of banquets, an invented structure simpler than group theory, but still quite rich semantically), as a lever to tackle the issue of the learning of abstract algebra. It sheds light into the cognitive processes involved in the conceptualization of an abstract algebraic structure, which are discussed within a semiotics framework. Empirical data show an insufficient syntax-semantic dialectic and mental processes based on the recognition of (visual) patterns.*

**Keywords:** Abstract algebra, mathematical structuralism, didactics and epistemology of mathematics, semiotics, syntax and semantics.

## INTRODUCTION

This article focuses on the teaching and learning of abstract algebra (the discipline dedicated to the study of algebraic structures, that is, the investigation of logical consequences of specific systems of axioms involving composition laws, and the relationships among them) which is taught at Montpellier University at the third-year university level. The difficulties are acknowledged by several authors (Leron & Dubinsky, 1995; Nardi, 2000; Hausberger, 2013) and reflect a “transition problem” (Gueudet, 2008) which, in the present case, occurs inside the university curriculum.

The epistemological analysis presented in Hausberger (2013) allowed a connection with the following epistemological transitions: “the systematization of the axiomatic method, after Hilbert, and the transition, after Noether, from thinking about operations on elements to thinking in terms of selected subsets and homomorphisms”. Indeed, as emphasized by Cory (2007):

This image of the discipline turned the conceptual hierarchy of classical algebra upside-down. Groups, fields, rings and other related concepts, appeared now at the main focus of interest, based on the implicit realization that all these concepts are, in fact, instances of a more general, underlying idea: the idea of an algebraic structure.

In other words, this epistemological gap leads to the *vanishing of concrete mathematical objects in favor of abstract structures*. This induces the following didactical problems: the teaching of abstract algebra tends to present a *semantic deficiency* regarding mathematical structures, which are defined by abstract axiomatic systems and whose syntactic aspects prevail. How does the learner build an “abstract group concept”? Indeed, what kind of representations can he rely on to do so when the purpose is to discard the particular nature of elements, in other words the mathematical context? Moreover, the investigation of the didactic transposition of the notion of structure shows that it is a *meta-concept* that is never mathematically defined in any course or textbook (and cannot be so):

As a consequence, students are supposed to learn by themselves and by the examples what is meant by a structure whereas sentences like “a homomorphism is a structure-preserving function” is supposed to help them make sense of a homomorphism (Hausberger, 2013).

As announced in loc. cit., I have engineered an activity for students to reflect on the axiomatic method and structuralist thinking in a simple context (simpler than group theory): the *theory of banquets*, an invented structure. It aims at operating the fundamental *concrete-abstract and syntax-semantic dialectics* (see below) and at clarifying the concept of mathematical structure using the meta lever (Dorier et al., 2000), that is “the use, in teaching, of information or knowl-

edge *about* mathematics. [...]. This information can lead students to reflect, consciously or otherwise, both on their own learning activity in mathematics and the very nature of mathematics”.

The purpose of this article is to present a few results that were obtained as I experimented with this activity. It tackles the following questions: what kind of cognitive processes and reasoning do students use to make sense of an axiomatically-presented structure such as the banquet structure? How do they engage in the task of *classifying models* of the axiomatic system (and interpret the task: for instance, what kind of representations do they use, and do they formalize a concept of isomorphism of banquets)? What kind of abstract banquet structure concept do they build through the completion of such a task? Similarly as in the context of classical algebra in secondary education, semiotics will give interesting tools to answer these questions. Still, some adaptation needs to be made to reflect the context of abstract algebra, since structures represent a higher level of organization compared to the classical mathematical objects that they formalize, generalize and unify.

## EPISTEMOLOGICAL AND DIDACTICAL FRAMEWORKS

### Abstraction

The French verb *abstraire* has three different meanings: 1. to discard (“faire abstraction de”) 2. to isolate (from a context) 3. to construct (a concept). Although these are three different actions, they may take place in order to reach a common goal as is the case in abstract algebra: mathematicians disregard the particular nature of elements and isolate relations to build the structure as an abstract concept.

The “principle of abstraction” as a process to create concepts has been used by Frege (1884) to define cardinal numbers. To introduce the reader to this revolutionary idea, Frege gives the enlightening example of the direction of a line which is defined as the class of all lines that are parallel to the given line. The principle was formalized later on by Russell (1903): to say that “things are equal because they have some property in common” and reduce a class to a single element, the relation that traduces this property should be symmetric and transitive (an equivalence relation).

### Semiotics

Just as language is compulsory to express any idea, mathematical objects are accessed through mathematical signs. Frege’s semiotics will be used in this paper, thus making the distinction between the sense and the denotation of a sign (Frege, 1892). The denotation of the sign is the object it refers to whereas the sense is related to the “mode of presentation” of the object. Mathematical signs are often polysemic but the context is meant to determine the reference uniquely. Conversely, different signs may represent the same object, thus having a different sense but a common denotation. In this way, different representations may bring to light different aspects of an object; they are acknowledged as denoting one and the same object through the realization that a particular processing or conversion of semiotic register of representation (Duval, 1995) allows to transform one representation into the other, and reciprocally. In other words, as stated and illustrated by Winslów (2004, p. 4), one may “think of objects as *signs modulo object preserving transformations* (OPT)”:

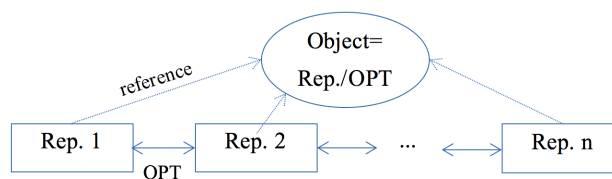


Figure 1

### Syntax and semantic

Mathematical signs are organized within sentences and formulae that are built according to strict syntactic rules. From a logical point of view, a definition is an “open sentence” that may be satisfied or not when the variables of the sentence are assigned in a suitable universe of discourse: this is the semantic conception of truth introduced by Tarski (1944); see also Durand-Guerrier (2003) for a more detailed account and didactical applications. In this respect, a piece of data that satisfies the definition of a mathematical structure (which involves a set of axioms that forms its syntactic content) may be called a *model* of the structure (in the given universe of discourse or hosting theory). The

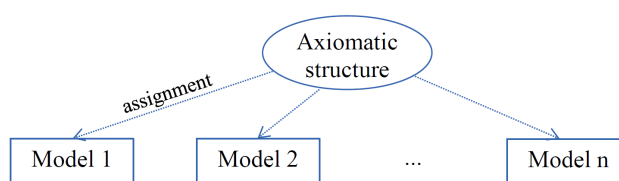


Figure 2

models will be regarded as the semantic content of the axiomatically-defined mathematical structure, its extension as a concept.

Tarski also defined the notion of logical consequence from a semantic point of view. It will be used below in order to show that a given axiom  $A_1$  cannot be deduced from other axioms  $A_i$ : it amounts to showing the existence of a model satisfying the  $A_i$ 's but not  $A_1$ . This contrasts with syntactic methods which consist of deduction by application of valid rules of inference.

### Structural objects

In a famous dispute with Hilbert, Frege argues against the legitimacy of abstract definitions by systems of axioms. One argument concerns the intrinsic polysemy of such definitions: in semiotic terms, an axiomatic definition as a sign has multiple references, the models of the axiomatic system. In abstract definitions, the context doesn't inform on the denotation simply because it is abstracted (in meaning 1 of the verb).

In order to build an abstract structure (group, ring, banquet, etc.) concept, and therefore give a more adequate (still polysemic) semantic meaning to the set of axioms as a sign, one needs to use structure preserving applications (SPT), the so-called isomorphisms, which are defined as relation-preserving bijections (all models may be viewed as sets endowed with additional data which define relations and satisfy the axioms). This allows us to associate to an axiomatic structure its "structural objects" (our terminology), the isomorphism classes of models or models modulo SPT, in the same manner as mathematical objects were built from representations modulo OPT, by means of the principle of abstraction.

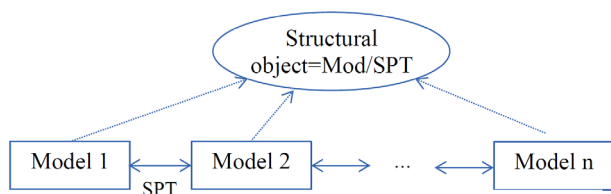


Figure 3

It should be pointed out that, compared to Winslow's diagram, dotted arrows do not represent the denotation of a sign but only "quotient maps". Since models are accessed through signs, the preceding diagram should in fact be reprinted, to reflect semiotic views, replacing each model by one of its representation and SPT by its semiotic version SOPT (structural object

preserving transformations). Dotted arrows may then represent denotation when the context indicates a structural perspective: for instance,  $Z/2Z \times Z/2Z$  and "the symmetry group of a rectangle" may both refer to the Klein 4-group  $V_4$ , as an abstract group concept. One may also write  $V_4 = \langle a, b; a^2=b^2=(ab)^2=1 \rangle$  for a more syntactical description. Nevertheless, since mathematicians take care in emphasizing the difference between a class and one representative, many authors would prefer to use the sign  $V_4$  to denote the group  $Z/2Z \times Z/2Z$  and say that it is isomorphic to the symmetry group of a rectangle or to the quotient of the free group on two generators by the relations  $a^2=b^2=(ab)^2=1$ . The idea behind structural objects is, following Sfard, that some kind of reification must occur for concept building: "Reification is defined as an ontological shift – a sudden ability to see something familiar in a totally new light" (Sfard, 1991). For this to happen, a plurality of models should be needed, borrowed from different mathematical domains and represented in different semiotic registers. Similarly as in Winslow's context, the coordination of these representations (through SOPT) should be crucial to obtain a conceptual schema of the structural object. It should open the possibility to abstract from "templates" a "pattern" (Resnik, 1997). Nevertheless, unlike in Winslow's context, a representation of a model as a sign may now refer to both the model and the structural object (in a context where both appear), whereas a mathematical distinction must be kept. Solving this issue would require a more direct mediation of the structural object by a new adequate (to be specified) sign.

### THE THEORY OF BANQUETS

As a piece of didactical engineering, the theory of banquets was built on the basis of an epistemological analysis previously presented (Hausberger, 2013) in order to cover the three usage contexts of the meta-concept "structure": 1. the structure as defined by a system of axioms 2. the abstract structure (of a given group or banquet) 3. a 'structure-theorem' (which describes the way an object can be reconstructed from simpler objects of the same type). It is filled with meta-discourse, as is already visible in the worksheet title: "The theory of banquets: a mini-theory to reflect on structuralist thinking". The interested reader is requested to email the author for a copy of the complete worksheet.

The activity is divided into three parts: 1. logical investigation of the axiomatic system and classification of

models 2. elaboration of an abstract theory of tables (this is the other way round: students are asked to formalize the disposition of guests around a round table) and structure-theorem for banquets (a banquet is the disjoint union of tables) 3. connection with the theory of permutations (a reinterpretation of the banquet theory that permits you to see the structure-theorem as a direct consequence of the well-known theorem of canonical cycle-decomposition of a permutation).

Part 1 and 2 clearly bring-in a concrete-abstract dialectic. A top-bottom approach has been chosen in part 1 for two reasons: as this is the standard strategy in textbooks, it is interesting to inquire how students will make sense of such a definition; moreover, part 1 will suitably enrich the didactical *milieu* for students to be able to model the situation given in part 2. Nevertheless, part 1 is already dialectical in itself: it amounts to making learners move on from the still abstract and syntactical conception of a structure exemplified by Figure 2 to the more concrete and semantic conception of Figure 3 (with several structural objects). The expected result of the abstract-concrete and syntax-semantic dialectics is a formulation of an abstract and syntactic characterization of structural objects (question 2 d of the worksheet, see below).

The definition of a banquet is as follows: it is a set  $E$  (the objects) endowed with a binary relation  $R$  (encoding the relations between objects) which satisfies the following axioms: A1. No object fulfills  $xRx$  A2. If  $xRy$  and  $xRz$  then  $y=z$  A3. If  $yRx$  and  $zRx$  then  $y=z$ . A4. For all  $x$ , there exists at least one  $y$  such that  $xRy$ .

In part 1, students were asked the following questions:

- 1 a. Coherence: is it a valid mathematical theory, that is, are the axioms non-contradictory? In other words, does there exist a model?
- 1 b. Independence: is one of the axioms a logical consequence of others or are all axioms mutually independent?
- 2 a. Classify all banquets of order  $n \leq 3$
- 2 b. Classify banquets of order 4
- 2 c. What can you say about  $Z/4Z$  endowed with?

2 d. How to characterize abstractly the preceding banquet (meaning its abstract structure of banquet among all the different classes of banquet, in fact how to characterize its class)?

Solving these questions amount to solving the following tasks and sub-tasks:

- T1. Construct a model by suitable assignment of variables
- T2. Classify banquets of a given order:
  - ST2a. Define a notion of isomorphism
  - ST2b. Give a list that covers representatives of all possible classes
  - ST2c. Show that two elements of the list are non-isomorphic
- T3. Show that 2 models are isomorphic by explicit construction of an isomorphism
- T4. Characterize abstractly an isomorphism class

Note that answering question 1 b amounts to solving T1 from the semantic point of view of logical consequence (see above) and negation of an axiom. In doing so, the boundaries of the banquet concept will be marked out. In the sequel, it will be necessary to focus first on T1 and give a list of available domains of interpretation for the axiomatic system and corresponding semiotic registers, since available representations greatly impact the other tasks.

*Empirical interpretation:* the name banquet may suggest by itself (or by reading the entire worksheet) guests around tables, so one defines  $xRy$  if and only if  $x$  sits on the right of  $y$ . Note that proving that this universe of discourse can serve to interpret the whole banquet theory reduces to proving the structure-theorem. One could also imagine a rectangle table and pick up guest sitting face to face, as a particular model.

*Set theory:* the set  $E$  is described by naming its elements and the binary relation is represented by its graph inside  $E^2$ . This straightforward representation is not very interesting since it doesn't "encode" much structure.

*Matrix theory:* a binary relation may be seen as a function from  $E^2$  to  $\{0,1\}$  (true/false), and therefore be represented by a double-entry table, in other words a matrix. In this interpretation, the axioms say that the diagonal contains only zeros, that there is exactly one '1' in each row and at least one in each column. In finite dimension, one can easily prove that there is exactly one '1' per row and column, hence it is a permutation matrix.

*Graph theory:*  $xRy$  if and only if vertices  $x$  and  $y$  are connected by an edge directed from  $x$  to  $y$ . The axioms say that from a vertex there originates exactly one edge and terminates at most one; therefore, unlike in general graph theory, is it easy to see when two drawings define the same graph.

*Function theory:* According to axioms A2 and A4, defines a function  $f$  and the other axioms say that it is injective and has no fixed point. When the set  $E$  is finite, then  $f$  is a permutation without fixed points and one may use the standard semiotic representations for these (including cycle-decomposition).

It is in fact quite amazing to see the diversity of interpretations and models, which certainly reflects the unity and creativity of mathematics. Models may be represented in a mixed or purely symbolic register. When the graphical register is used, it may be a personal idealization of people around tables or an institutional representation borrowed from graph theory. Of course, one cannot expect students to connect to all these theories: for, instance, we bet that students won't translate the problem fully in the function setting and identify the connection with permutation theory (which would ruin part 3 of the activity). But one can wonder about the importance they may give to representations connected to everyday life (empirical setting). Moreover, models may be more or less "generic": compare  $(E, f)$ ,  $E = \mathbb{N}$  and  $f(x) = x + 1$ , with a matrix that may serve to represent any binary operation. Students may also think that a model should be given by a mathematical formula (like the example given in question 2 c), and restrict themselves to concrete examples in function theory, whereas a generic representation of  $R$  is necessary to complete the other tasks.

To give an idea about processes and conversions from one setting to the other, one should notice that a representation such as points marked on circles (empirical) is easily transformed into a graph by adding arrows

clockwise between points; one may then associate to the graph its adjacency matrix, from which the function is soon reconstructed by reading the positions of the ones. When  $E$  is finite, the algorithm of cycle-decomposition of the permutation gives the tables (one per cycle) and the length of the cycle gives the number of people around each table, thus coming back to the empirical setting.

The pertinence of the setting (choice of a domain of interpretation) depends on the task: graph theory may easily suggest a model that verifies all the axioms except A2; matrix theory is quite pertinent for ST2b (a complete list of all possible relations), still, graph theory again (or even better, a cyclic representation as obtained in the empirical setting) is best to decide if two banquets are isomorphic, as it gives a visual representation that makes common pattern visible and illustrates the etymology of isomorphism as a form-preserving mapping.

I will now present some students' productions. Tasks 2 to 4 will be discussed in greater detail while analyzing these.

## EMPIRICAL DATA

The full banquet activity has been tried out during the academic year 2013–14 with third year university students with a background in group theory before teaching ring and field theory. They worked in small groups of 4–5 students and were asked to keep a research notebook that was collected before each phase of institutionalization (Brousseau, 1997). In parallel, in laboratory sessions, I videotaped two pairs of more advanced students (having a master's degree). The interviewer (myself) intervened only at the end of part 1 in order to discuss with the students their answers and their conceptions regarding the classification task. Due to lack of space, I will only give an account of the laboratory session with one of the pairs. Nevertheless, this will already be enough to give an idea of some interesting phenomena that could be observed by using our theoretical framework and in particular Duval's *semiosis*, the apprehension or production of semiotic representations (Duval, 1995).

The pair of students tried to recognize the banquet structure as a *pattern* in known mathematical objects and theories ("what is it, what's this structure?"). Unlike in the classroom experiment, they didn't bring

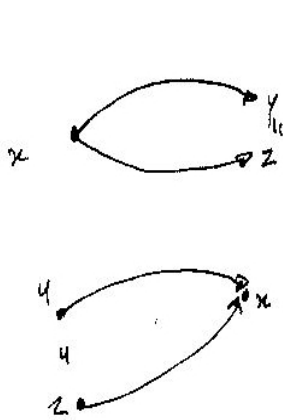


Figure 4a

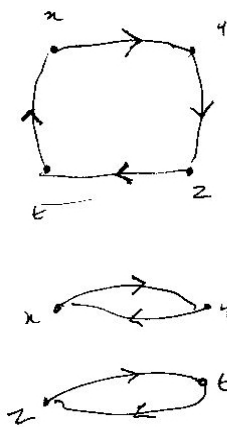


Figure 4b

$(xyzt)$        $(xtzy)$   
 $(xzyt)$        $(xtyz)$   
 $(xzyt)$   
 $(xytz)$

$(xy)(zt)$   
 $(xz)(yt)$   
 $(xt)(zy)$

in wedding banquets; they first thought about the order relation, then analyzed the example  $B_4=(\mathbb{Z}/4\mathbb{Z},R)$  of question 2c as a “kind of a shift” and generalized it ( $E=\mathbb{Z}, f(x)=x+1$  or  $x-1$ ). *Semiotic representations* of the semantic meaning of axioms A2 and A3 in the *graphical register* (Figure 4a) led them to build *models in graph theory* which they used for tasks T1b and T2. Recognition of *cyclic patterns suggested permutations*, as a common representation: they performed conversions of registers (but didn’t connect to the function setting), producing the following classification which comprises 9 banquets of order 4 (Figure 4b).

- Student A: Here, we are doing with what we know, but we speak about a structure
- Student B: Wait, we can always number the elements [...]
- Interviewer: For you, this is an abstract classification because you didn’t consider particular relations and you can always rename elements  $x,y,z,t$
- A: So there would be 2 classes up to isomorphism?
- B: Here  $\mathbb{Z}/4\mathbb{Z}$  and there  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
- I: You are thinking about the classification of groups [...] So there are 2 types of objects and  $(x y z t)$  and  $(x z y t)$  would be the same?
- B: Not the same, of the same type

The student B couldn’t define what he meant by a type, he just made a connection between the word used by the interviewer and the notion of type of a permutation. The word *bijection* finally appeared but students found it difficult to define what “structure-preserving” meant. They drew the graph for  $(x y z t)$  but obtained crossing edges which confused them even more

(both are identical as graphs but not as drawings). On the contrary, converting to a graph the example  $B_4$  allowed connection to  $(x y z t)$  (obvious congruence of drawings). They didn’t realize that abstracting the nature of elements simply meant forgetting letters, a mental process that makes the recognition of isomorphism classes in the representation as cycle products automatic.

### CONCLUSIONS AND PERSPECTIVES

This study, through its theoretical framework and the analysis of the presented data, contributes to the recognition of the influence of semiotic representations in cognitive activities dedicated to the learning of abstract algebra. I have discussed the hypothesis that a logical investigation of an axiomatic system and the classification of its models up to isomorphism, in the paradigm of the “theory of banquets” which connects to group-theory, is a cognitive activity that could bring good conditions for learners to develop an appropriate conceptualization of an abstract structure, and in particular access what I called “structural objects”. Empirical data show mental processes based on the recognition of (visual) patterns. Conversions of registers were operated on by the two students in order to realize that two objects are isomorphic, a strategy which is successful when the congruence of representations is obvious, but the students couldn’t handle the treatments inside a register since they couldn’t rely on a formal definition of an isomorphism or make this definition functional, which is evidence of insufficient syntax-semantic dialectic. This also suggests an incomplete understanding of abstraction as a process that leads to structural objects. Finally, the two students tried to work out the analogy with group theory from which they borrowed directly or tried

to adapt representations and concepts (see transcript above). As stated by Winsløw (2004), “mathematical concepts are not learned one by one but as coherent patterns or structures”, and this also happens at the level of structures themselves, thus gaining access to what I called level-2 unification (Hausberger, 2012).

The analysis of empirical data will be pursued in greater details in an expanded version of this article. It is expected that these investigations and refinements of the semiotic tools will lead to a better understanding of students’ difficulties in abstract algebra which are inherent in structuralist thinking, from a cognitive point of view.

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