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Growth of mathematical knowledge for teaching – the case of long division

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When university mathematicians teach mathematics courses for non-mathematicians, there may be a discrepancy between the mathematics they aim to teach and the mathematics their students aim to learn. In this paper, I analyze a lesson on long division taught by a mathematics Ph.D. student, where the learners were in-service elementary school teachers. Taking a Commognitive approach, I describe some crucial differences in the teachers' and the mathematician's discourse on mathematics and on teaching, which created opportunities for mutual learning. Uncovering the affordances and limitations of this teaching/learning situation is expected to help mathematicians become more effective teachers of non-mathematicians in general, and of pre-service and in-service teachers in particular.

Keywords: Elementary school mathematics, university mathematics, professional development, Commognition.

INTRODUCTION

Researchers such as Nardi and colleagues (2014) are coming to view university mathematics as a discourse, conceived as accepted modes of communication in mathematics departments. However, other communities engage in their own mathematical discourses (physicists, chemists, mathematics teachers), which may be quite different from the discourse of mathematicians. What happens when mathematicians teach courses for non-mathematicians? What is the nature of productive learning in such situations? These are the questions that guide my investigation of an extreme case – a mathematics Ph.D. student teaching in-service elementary-school teachers a lesson on the long division algorithm (LDA). This mathematician may be an expert on abstract algebra, but what can he possibly know about division in elementary school, how it's taught, or what kind of difficulties students typically encounter? I show how the differ-

ences in mathematical discourses of the two parties created opportunities for mutual learning, and how this meeting of two communities of mathematics educators brought a rich perspective to the teaching of LDA, where pedagogical considerations of teaching and learning interacted with mathematical considerations. Understanding how this came about may guide mathematicians in teaching pre-university mathematics to non-mathematicians, particularly in the teaching of school teachers.

SETTING

The professional development (PD) under investigation was the initiative of a university professor of mathematics, and was taught by mathematics graduate students. Its declared goal was to broaden and deepen the teachers' understanding of the mathematics they teach. Approximately 90 teachers enrolled in the 2011–12 program, which consisted of ten 3-hour sessions taught in six groups. The data collected in this research project consists of audio recordings of all the sessions, interviews with the instructors, and teacher questionnaires – expectations at the outset and feedback after each session. In this paper, I analyze a lesson on LDA in which 15 grade 3–6 teachers participated. The instructor was a mathematics doctoral candidate.

Here are some features of LDA that the instructor decided to attend to in this lesson:

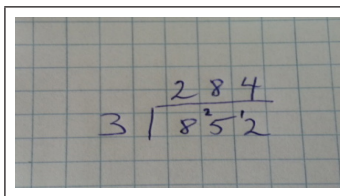
- LDA is opaque – the underlying mathematical ideas of number decomposition, distributive rule, place value and regrouping are not salient.
- Treating the dividend as a sequence of digits discourages estimation.

- Answering “how many times does the divisor go into...” is difficult, since there is no margin for error – we must find the *greatest* multiple that “goes in”.

The lesson proceeded as follows in Table 1.

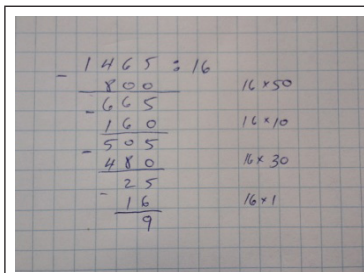
The Short Division Algorithm (SDA) discussed in lesson segment B is a variant of LDA for cases where the divisor has a single digit. Remainders are calculated mentally and written between the digits of the dividend (Figure 1). The sequence of LD problems (segment C) focused first on place-value decomposition: , then on decomposition induced by LDA: , connected with regrouping and the distributive property. Two alternative division algorithms were presented (seg-

ment D), neither of which requires the performance of division operations: Division by approximation: choose an *easy* multiple of the divisor, subtract it from the dividend, and repeat. This algorithm does not have a single correct implementation; you are free to choose any multiple of the divisor you are comfortable with (Figure 2). Division in parts: pre-calculate the divisor multiplied by 1, 2, 4 and 8 by repeatedly multiplying by 2, and use these multiples (possibly with added zeros) to *divide by approximation* as described above. This algorithm, unlike division by approximation, has a single correct implementation. You are not free to choose any multiple of the divisor, you should always subtract the largest one from the pre-calculated multiples (Figure 3).



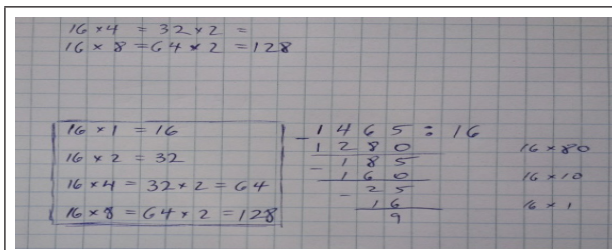
3 goes into 8 2 times. The remainder (2) is calculated mentally and written to the left of the 5, which is now read as 25. 3 goes into 25 8 times. The remainder is written to the left of the 2. Finally 3 goes into 12 4 times.

Figure 1: Short Division Algorithm (SDA)



“16×50=800 is easy for me. I subtract from 1465 to get 665. Subtracting 16×10 is convenient, leaving 505. I know 16×30=480. Subtracting leaves 25. Subtracting 16×1 leaves 9. 50+10+30+1=91. The solution is 91 (9)”.

Figure 2: Division by approximation



Four multiples of 16 are pre-calculated. The algorithm proceeds as in Division by Approximation, except that each stage is determined – we must use the *greatest* multiple from the table, adding zeros where appropriate.

Figure 3: Division by parts

	Utterances	Duration	What was going on
A	88–195	11 min.	Introducing the lesson’s topic and motivation
B	196–389	13 min.	LDA and SDA exemplified and compared
C	390–862	30 min.	Sequence of LD problems that focus on mathematical ideas one at a time.
D	863–1308	23 min.	Two alternate division algorithms

Table 1: Overview of transcript data

THEORETICAL FRAMEWORK AND RESEARCH QUESTIONS

The framework of Mathematical Discourse for Teaching (MDT) (Cooper, 2014) takes inspiration from Mathematical Knowledge for Teaching (MKT) (Ball, Thames, & Phelps, 2008) – in viewing mathematics *for teaching* as special and different from mathematics for other purposes. Epistemologically, MDT endorses tenets of the commognitive framework (Sfard, 2008), seeing fields of human knowledge, such as mathematical knowledge for teaching, as well defined forms of communication, and thus communication and cognition as aspects of a single entity termed discourse. Hence, MKT's distinction between Subject Matter Content Knowledge and Pedagogical Content Knowledge is mirrored as a distinction between Subject Matter Content *Discourse* (SMCD) and Pedagogical Content *Discourse* (PCD). Similarly, each of the six subcategories of MKT has its discursive counterpart. Discourses are associated with communities, thus university mathematicians' MDT and elementary school teachers' MDT are expected to be different. In the Commognitive framework, learning is conceived as changes in one's discourse, which typically start out as discourse-for-others – imitation, possibly thoughtful, of a leader's discourse – but may evolve into discourse-for-oneself – the discourse with which one communicates with oneself in problem solving (i.e. thinking).

Research questions

- 1) In what ways are the significance and the role of long division different for the instructor (in his university MDT) and for the teachers (in their elementary school MDT)?
- 2) What opportunities for learning emerged from these differences?
 - a) How did the meeting of two MDTs create opportunities for learning?
 - b) What learning actually took place in this lesson, on the part of the teachers and the instructor?

METHOD AND DATA

This paper draws on the instructor's lesson plan, a 77 minute audio recording of the lesson – fully tran-

scribed and selectively translated, and an unstructured interview with the instructor five days after the lesson.

Commognitive methods of analysis focus on four interrelated characteristic features of discourse: *keywords*, *visual mediators*, *distinctive routines*, and generally *endorsed narratives*. Differences in interlocutors' discourse (e.g. differences in the ways they use keywords, in their attitudes to visual mediators, in the routines they typically engage in, or in the narratives they endorse), may present opportunities for learning. *Crisis points* in communication – “signs indicating that something has gone wrong in the interaction” (Jorgensen & Phillips, 2002, p. 125) – are a natural place to look for differences in interlocutors' discourse, and for learning taking place.

LONG DIVISION IN SUBJECT MATTER CONTENT DISCOURSE (SMCD)

Understanding LDA as process, concept and visual mediator

The teachers and the instructor agreed that understanding LDA is important, but the keyword *understanding* is used differently in their respective MDTs. For the teachers, understanding is strongly linked to the algorithm process, as is evident in a teacher's comment after a detailed review of a LD problem: *Here there's awareness of the process, it's not automatic. This is your understanding.* For the instructor, understanding LDA has to do with making connections between the related mathematical ideas – decomposition, the distributive rule and place value: *using the distributive rule and convenient decompositions (of the dividend) can be done without the algorithm... is very helpful for understanding the algorithm... sharpens the understanding of the distributive property in the context of division... understanding something we already know how to do.* Here are some additional examples of the procedural nature of the teachers' MDT versus the instructor's more conceptual nature: The keyword DMSB (acronym for divide, multiply, subtract, bring down) is central in teachers' discourse. Where the teachers speak of *remainder* in LDA (“in the LDA for $693 \div 3$ there's no *remainder*”), the instructor replies “you mean there's no *regrouping*”. Both keywords refer to the LDA procedure, but *regrouping* is the conceptual counterpart of *bring down*.

Another difference in the MDT of the two parties is in the significance attributed to the visual mediation of processes. Comparing SDA to LDA, the instructor notes: *we perform exactly the same operations in the same order... the difference is... [in LDA] we write a lot, and we've agreed that it's a big mess.* This does not do justice to the visual aspects of the algorithms. There are rules regarding how LDA and SDA are arranged on the page (see Figure 1), and these rules interact with the process and with the conceptual underpinnings, as is evident in some teachers' comments: *The bringing down is difficult... especially without grid paper* (a difficulty that is avoided in SDA where digits are not brought down); *[in SD the kids] immediately notice the remainder ... [because] they need to write it;* There are other examples of the teachers attending to visual mediation where the instructor seems to see it as secondary to the underlying mathematics: as the instructor walks through an alternate procedure, a teacher asks for a visual mediator: *don't you write down the [interim] result?* His response – *I'm trying to tell you how I'm thinking about it, not how I'd explain it to students* – implies that he views visual mediation as a teaching tool, having little to do with cognition. However, issues of visual mediation may be closely related to mathematical concepts. For example, working through using the approximation algorithm, the instructor mediates his actions by writing . A teacher says she would have written . Other teachers justify the division notation in many ways: *it's easier; it's confusing otherwise; in a classroom with students struggling with division we need to get them accustomed to thinking division.* For this last teacher, the visual mediation is entwined with a mathematical issue concerning the relationship between division and multiplication. In university mathematics there is no independent division operation, instead there is multiplication by inverse. This attitude to division is particularly evident in the alternate division algorithms, where division problems are solved without performing any division operations, only multiplications and subtractions! According to the instructor, the benefit is in avoiding the most difficult aspect of LDA – finding the *greatest* multiple of the divisor that goes into the dividend with no margin for error, but it is conceivable that he was influenced by his university conception of division as an unprivileged operation.

LONG DIVISION IN PEDAGOGICAL CONTENT DISCOURSE (PCD)

Teaching and learning long division

The instructor's Pedagogical Content Discourse pertains both to teacher education and to elementary school pedagogy. For him they are connected; in his words he aimed to *give the angle that will connect [the PD] to what goes on in the classroom.* One such connection would be the teachers using PD activities in their classrooms, possibly with minor modifications. This is something the teachers also hoped for, based on questionnaires that explored their expectations. However, the teachers deemed much of the LD lesson unteachable. Regarding the alternate division algorithms responses included: *[Division by parts is] not a method you can teach a class; [it's] explanations for good students.* The instructor appealed to other modes of relevance, suggesting the method as an aid for struggling students, but this too was rejected by a chorus of teachers: *it would confuse them so badly; it's difficult; no way.* Finally the instructor suggested yet another role for these algorithms, as a means for independent checking of standard LDA results. The teachers were not explicit about why they rejected methods that the instructor considered useful, but I offer some speculations, supported by what I have shown regarding the MDT of the two parties:

In the teachers' MDT, an alternate algorithm is yet another procedure that would need to be mastered, i.e. memorized. For the instructor these algorithms make so much sense that they should not need to be memorized.

The instructor considers the flexibility of the approximation algorithm a strength: *By using approximations and working with multiples that we're comfortable with, we're converting the problem to an easier problem.* However teachers may be wondering how to teach an idiosyncratic algorithm, which each student may solve differently. And what about students who are not comfortable with *any* multiples of the divisor?

Visual mediation may also be an issue. The instructor's focus was on the mathematics involved in each of the algorithms, but a teacher commented that *[we] need to remember [the multiplier] at each stage,* apparently attending to the lack of well-defined rules for organizing the solution visually. This is backed by teacher comments throughout the activities such as:

I'd have organized [the decomposition] in a [place value] chart; why don't you do [by parts algorithm] in the table?

WHAT ARE THE PARTIES LEARNING?

In commognitive terms, the differences in the MDTs of the teachers and the instructor present opportunities for mutual learning. Although the goal of PD was for *teachers* to learn – primarily mathematical content – learning on the part of instructor is crucial for the design and implementation of activities that will be relevant for teachers. In this section I show some examples of both kinds of learning taking place.

The teachers, in their active engagement in the various division algorithms – LDA, SDA, by approximation and by parts – were exploring connections between mathematical objects such as division, multiplication, decomposition, place value, estimation. They had accepted the goal of understanding LD and its entailments, as is evident in this teacher's comment, which followed the decomposition activity: *When we eventually reach long division, everything we did in this activity [decompositions of the dividend], the understandings, they disappear... The question is how to achieve understanding [for our students].* I now analyze some transcript excerpts to show the kind of learning that was taking place.

Excerpt 1: Developing Specialized Content Discourse, decomposing $852 \div 3$

- 574 I: What I think can help prepare for LDA is doing the division first without mentioning the algorithm. Let's write it as a word problem. What do we get from dividing 8 hundreds, 5 tens and 2 units into 3 equal groups?
- 578 I: First we divide what we can...six-hundred... I'm left with 2 hundreds... not 2.
- 585 T1: Which are in fact 20 tens
- 596 I: [We now have 25 tens.] How many tens can be divided?
- 597 T1: 240
- 598 I: 24 tens...
- 599 T2: You can do 240. Why 24?
- 601 I: Oh, here I wrote six-hundred...
- 610 T3: Why convert to hundreds? ... 6 instead of six-hundred... in Hebrew six...
- 613 I: Exactly. When I said six hundreds, I automatically thought six-hundred

Perhaps the most salient aspect of this excerpt is that teachers engaged in explorative discourse around a division procedure which is not part of the curriculum. I would like to draw attention to what may appear to be a rather trivial slip on the part of the instructor, saying and writing *six-hundred* (600) instead of *six hundreds*. This is not at all trivial. Three related but very different division algorithms are being considered, each with its own language usage. In LDA the symbol 6 in the hundreds place represents 600, in the precursor algorithm under discussion in the excerpt 6 hundreds need to be divided equally (explicitly compared by the instructor to the problem of equally dividing 6 melons), and in the alternate algorithms (by approximation and in parts) the number six-hundred needs to be divided as a number, not a quantity. These subtle differences are at the heart of the instructor's design. The alternate algorithms, in referring to the number, support estimation strategies. LDA makes sophisticated but opaque use of the principle of place value. The procedure in the excerpt subtly bridges the two; procedurally it follows LDA (hundreds, tens, units) while keeping track of the dividend as a quantity and not just a sequence of symbols. T1 and T2 are not yet fully aware of these subtleties, but T3, in catching the instructor's slip, appears to be on the way to making these distinctions part of her own discourse.

Excerpt 2: Specialized Content Discourse – endorsing an algorithm

Please refer to Figure 3 to make sense of this excerpt. T3 makes a "mistake", subtracting 128 from 185, the instructor goes along with it, T4 catches the mistake.

- 1125 T1: I'd start with 128
- 1127 T2: 1280
- 1128 I: Alright? 128, but times 10 is 1280. 1465 less 1280 is ... 185
- 1175 T3: Less 128
- 1177 I: 128
- 1178 T4: But why did you do 128? You can do 160... It's much easier
- 1180 T3: Yes, 160 is preferable
- 1189 I: It's in the table, I forgot. I have 160 here, you're right.
- 1190 T5: Because in your table, instead of 8 you can do 10... 10 times 16.

Here again a number of teachers are actively exploring a division algorithm, and again are correcting the instructor's authentic error. An important aspect of

this algorithm is that it has a unique “correct” move at each stage. By contrast, division by approximation is idiosyncratic – looking for “convenient” multiples of the divisor and subtracting them from the dividend. In division by parts we look for the unique *greatest* multiple from the table, where the instructor, using a procedural discourse, stated that *obviously we can add zeros*. T5 did not adopt this procedural language, preferring the more conceptual *10 times*. Furthermore, T3 and T4 seem to have appropriated something from the approximation algorithm, since they do not consider the instructor’s slip an error, rather 160 would have been *easier* or *preferable*. This is not a minor point. LDA has a single correct flow; procedures that can correctly proceed in different paths are from quite a different discourse.

Excerpt 3: Specialized Content Discourse – mediating division and multiplication

The instructor commented that *how many times does 3 go into 8* is an instance of the measurement model of division. But *times* suggests multiplication at least as much as it suggests division. With this in mind, consider the following exchange:

- 1000 I: I think about it this way: I want to know how many times 16 goes into 1220.
 1001 T1: So, times...
 1002 T2: times...
 1003 T1: 50 times
 1004 I: Yes. Ok, times. So we found 50 times 16. It sounds a bit strange doesn't it?
 1009 T3: Because you're asking how many times 16 there are in...
 1011 T4: goes into 1220? It goes in 50 times.

The instructor invested some effort in mapping out connections between division and multiplication prior to the lesson. The discussion evolved in ways he could not have anticipated or prepared for. The teachers’ testimonies regarding students’ strategies for answering *how many times it goes in* (skip counting and repeated subtraction) revealed some such connections, but in this excerpt we see something different – the parties are listening to each other carefully and jointly exploring the role of the word *times* in mediating meanings of division and multiplication. We have “how many times 16 goes in” (division), “50 times 16” (multiplication), and “how many times 16 there are in 1220”, which can be seen as bridging the two preceding meanings. This is a case of a joint ob-

ject-level learning in the realm of Specialized Content Discourse, with obvious implication for Discourse of Content and Teaching.

Excerpt 4: Discourse of Content and Teaching – is the algorithm teachable?

From the teachers’ participation throughout the activities it is clear that by and large they mastered the suggested algorithms, yet they were convinced that their students would not, whereas the instructor believed that they would. This discrepancy is due to the rules by which the parties endorse narratives about students and teaching – the teachers based on their experience and the instructor based on an analysis of the mathematics. The teachers’ experience is a valuable resource and should not be taken lightly, and indeed I have shown why the alternate division algorithms might be difficult to teach, however, the teachers’ don’t have any direct experience regarding what they have *not* taught. Four teachers came to realize this towards the end of the lesson, where there were a total of 9 utterances to this effect, for example:

- 1237 T1: Could be, I haven't tried it. Could be that if you do one or two lessons this way... they'd understand the meaning of decomposition.
 1240 T2: Exactly.
 1244 T1: Not necessarily after LDA. I'm saying this cautiously since I haven't tried...
 1248 T3: Not in order to know *how* to do it, rather to understand the meaning
 1264 T4: Theoretically. We should try it some time.

Even as these teachers entertained the thought of teaching this algorithm, the principle by which it will (or not) be endorsed remains reliant on teaching experience.

Specialized Content Discourse – appreciating the role of visual mediation

I have shown that there is less attention to visual mediation in the instructor’s discourse than in the teachers’, however, the instructor was attentive to the teachers’ comments and suggestions. He accepted two suggestions: mediating division by approximation in terms of division instead of multiplication; and, keeping track of the stages of division by parts in a table. Furthermore, in the discussion about SDA he realized that writing the remainder between the dividend’s digits addresses the common error of skipping

digits in the *bring down* stage. He also noted that the alternate algorithms generate solutions that *look completely different* than the LDA solution, even though they make use of similar mathematics.

SUMMARY

I have shown that the teachers tended to see LDA as a procedure that needs to be mastered and understood, whereas the mathematician saw it as an opportunity for deepening understanding by connecting a number of different topics. This is just one of the many ways in which their MDTs differed. In the face of such differences, the lesson could have followed various different paths: The instructor could have insisted on his agenda, alienating the teachers, or he could have adopted the teachers' point of view, setting aside his own agenda. Neither of these is what actually took place. The instructor was true to his mathematical agenda, but made a genuine attempt to appropriate the teachers' discourse. This can be seen both in his preparation of the lesson and in the way it played out. Why did he present two similar algorithms – approximation and in parts? The underlying mathematical principle is the same – incrementally decomposing the dividend into multiples of the divisor. I suggest that the instructor was developing sensitivity to the teachers' Discourse of Content and Teaching, and, realizing how difficult it would be to teach an idiosyncratic algorithm, suggested the deterministic version as an alternative. Furthermore, the motivation he gave for these algorithms was from the teachers' Discourse of Content and Students – they avoid the aspect of LDA that he considers most difficult for children (division).

The lesson on LDA may be considered productive in the sense that both the teachers and the instructor were enriching their MDT. It is interesting to note that this learning, when it involved changes in the rules of the discourse, did not follow the pattern described by Ben Zvi & Sfard (2007); there was no agreement on the leading discourse, the roles of the interlocutors, or the course of discursive change. Expertise was shared by teachers and mathematicians, who were all in the position of learners.

My aim in this paper was to point out *opportunities* for learning in the meeting of two MDTs. I have claimed that, in some cases, learning was in fact taking place, in the sense that the parties – teachers and instructor – were not superficially adopting aspects of an

unfamiliar discourse. Rather they were in the process of transforming this discourse into *discourse-for-one-self*, that is, into the type of communication in which the person is likely to engage of her own accord, while trying to solve her own problems (Sfard, 2008, p. 285). This was evident on the part of the teachers; the mathematical activities stretched their Specialized Content Discourse, yet all the while they were considering implications for their teaching. In this sense the teachers were *constructing* new knowledge not through experience (teaching) but rather through discursive interactions, transforming mathematical ideas into ideas for teaching. How these discursive shifts subsequently influenced their teaching (if at all) is an important question that will be addressed in future research.

The instructor extended his own SCD while unpacking LDA in preparation for the lesson. The lesson itself presented opportunities for learning, but it is difficult to make claims regarding the nature of the instructor's learning based on the lesson transcript. For example, when some teachers suggested $30 \div 3 = 10$ instead of $3 \times 10 = 30$, he responded: *Ok, but I'll tell you why I did the multiplication*. If he is thoughtfully considering the teachers' discourse and is on the way to transforming it to discourse-for-himself, there are no indications of it in the transcript. Nonetheless, such learning is crucial for mathematicians to be relevant for the education of teachers. Based on an interview following the lesson, the instructor was thoughtfully exploring ways in which his teaching might be *relevant* for the teachers. It is not clear if the instructor's learning would have been as productive in the absence of a researcher. However I believe that exposing mathematicians to these research findings is a crucial step for supporting their sensitivity towards teachers and their learning in similar situations.

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