Unsolvable mathematical problems in kindergarten: Are they appropriate?

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Not all mathematical problems have a solution. This paper describes a decomposition problem, set in a real-life context, for which no mathematical solution exists. It describes the strategies children use and how the rea-life context impacted on children's solutions. Results indicated that young children accept the possibility that a problem may not have a solution and that some turn to the context in order to find a practical solution instead.

Keywords: Unsolvable problems, kindergarten, real-life context.

Problem solving is an inherent part of learning mathematics at every age. In the U.S., the Curriculum Focal Points suggests that the content and concepts learned in kindergarten should be addressed in a context that promotes problem solving, reasoning and critical thinking (NCTM, 2006). In England, the Statuary Framework for the Early Years Foundation Stage (DCSF, 2008) states that "Children must be supported in developing their understanding of problem solving, reasoning, and numeracy in a broad range of contexts…" (p. 14). In Israel, the preschool mathematics curriculum stresses that problem solving should be interweaved throughout all of the main content strands (INMPC, 2008). Some major, related questions are: What types of problems should we present to young children? Should we only present problems that have a solution? Are young children capable of handling mathematical problems that have no solution? The INMPC (2008) suggests that problems, especially for young children, be set in a real-life context. How might such a context impact on the way young children approach a mathematical problem that has no solution?

One of the major content strands of preschool mathematics education is to promote number conceptualization and skills, including number composition and decomposition. The INMPC (2008) states that between the ages of 4 and 6 years, children should learn that a given amount of objects may be divided into two or more groups and that those groups may or may not have equal amounts of objects. Being able to flexibly compose and decompose numbers allows children to apply various strategies when solving number problems in various contexts and assists in the development of number sense (Baroody, 2004). This study describes kindergarten children's engagement with a number decomposition problem set in a real-life context that has no mathematical solution. Specifically, it investigates: What strategies do children employ when faced with a composition and decomposition problem? Do young children recognize when a problem has no mathematical solution? How might the context of a problem impact on the way children approach the absence of a solution?

NUMBER DECOMPOSITION AND PROBLEM SOLVING

Children's ability to compose and decompose numbers is at the heart of this study. Two prerequisite skills for composing and decomposing numbers are verbal counting and object counting. Both of these skills are almost always mastered by age five (Baroody, 2004). Related to quantification, but distinct from counting, is the process of subitizing (Baroody, Lai, & Mix, 2006). Subitizing refers to the immediate recognition of the number of items in a collection without counting the objects. It involves recognizing that three dots in a row has the same number of dots as three dots set up in a triangular array and that both of these sets have exactly three dots. As the child develops, these conceptions may be mentally operated on, such as mentally decomposing a pattern of five into two and three and combining them back together.

In addition to subitizing, children develop and use various counting strategies when solving arithmetic problems. Initially, when children are requested to add two numbers or to combine sets of objects, they use a "counting-all" strategy. For example, when combining three items with four items, children will count 1, 2, 3 and then 1, 2, 3, 4 and then finally, 1, 2, 3, 4, 5, 6, 7. This strategy develops into several more sophisticated strategies such as "counting-on-from-first" and "counting-on-from-larger". Eventually, procedural counting knowledge develops into declarative knowledge in the form of number-facts, which are remembered and retrieved (Baroody, 2004). Children take advantage of number facts, along with their knowledge of decomposing numbers, to count in a more efficient manner. In the above example, children may decompose the four into three and one, use a known fact such as "three and three makes six", and then add the leftover one, to make seven (Verschaffel, Greer, & deCorte, 2007).

Not all arithmetic problems present the same cognitive challenge. The easiest problems to solve are those that the child can use and manipulate physical items in a step-by-step procedure set out by the story problem (Sarama & Clements, 2009). For example, if the child is requested to find out how many candies Danny has if he received three candies from his grandma and four candies from his grandpa, the child can physically act out this story using substitute items, in order to figure out how many candies Danny now has.

Setting mathematical problems in real-life contexts is another issue discussed by mathematics educators. The realistic mathematics education (RME) approach, emphasizes that context problems must be imaginable to the students (e.g., Streefland, 1990). Such problems can serve as an introduction to a mathematical topic as opposed to merely demonstrating the applications of a mathematical topic at the end of a learning sequence. Students, however, do not always take into account the real-life context of the problem. When posed with the following problem:

> 450 soldiers must be bussed to their training site. Each army bus can hold 36 soldiers. How many buses are needed?

a considerable amount of students wrote that 12.5 buses would be needed (Verschaffel, De Corte, & Lasure, 1994).

While the bus problem is solvable (e.g., hire 13 buses), not all problems have a solution. For example, De Corte and Verschaffel (1985) posed the following problem to first graders: Pete had some apples; he gave 4 apples to Ann; how many apples does Pete have now?" (p. 11). More than half of the children did not recognize that the problem was unsolvable. In other studies with similar unsolvable situations, students gave answers which were irrelevant (Reusser & Stebler, 1997). It was as if students felt that if a problem is posed, a solution must exist. When does this belief develop? Does it exist already in kindergarten? In this study, we pose a problem which has no solution and investigate children's strategies when attempting to solve this problem.

METHODOLOGY

The participants in this study were 19 kindergarten children, ages 5–6 years old, attending three different kindergarten classes. All three kindergartens were located in the same middle-low socio-economic neighbourhood. Their teachers had participated for two years in one of our professional development programmes (Tirosh, Tsamir, & Levenson, 2011). The children were scheduled to enter first grade the following term. The study was conducted in March, the second half of the school year.

Birthday parties are part of children's reality. Every child experiences birthday parties in the kindergarten class. For this task, four empty party plates were placed on the table. In addition, eight cards were placed on the table, with different amounts of candies on each card. One of the eight cards was left blank (See Figure 1). The child was told, "Four children are coming to your party and you want to give each child seven candies on their plate. Can you arrange it so that there are seven candies on each plate? You can arrange the candies however you wish, but there have to be seven candies on each plate." At this point, the interviewer let the child work out the problem, adding comments as needed, such as letting the child know that it was permissible to place more than one card on a plate. When it seemed that the child had finished, the interviewer asked, "Are you finished?" When the child acknowledged that he or she was finished, the interviewer asked, "So, does every child get seven candies?" The problem has no solution; twenty-seven is not divisible by four; seven candies cannot be placed on each of the four plates.

Figure 1: The *Birthday Party* task cards

Analysis of the results began by considering correctness. First, we noted if a child who claimed to have placed seven candies on a plate, did indeed place seven candies on that plate. Second, when the child completed the task, did he or she recognize that not all plates had seven candies or did the child claim that each plate had seven candies? Third, we noted what strategies were used when composing seven. Did the child use a counting strategy or were number facts recalled? Finally, we were interested in investigating how the context of the birthday party might influence the children's solutions, and how the children would deal with a problem that had no acceptable mathematical solution.

FINDINGS

This section begins by summarizing the final arrangements of the cards on each plate, highlighting the issue of the blank card. It then describes one child's strategies when engaging with the problem. Finally, we describe how children dealt with the fact that this problem had no mathematical solution.

Table 1 summarizes the frequency of the final arrangements of candy cards on individual plates. The order in which the plates were filled was not recorded as children went back and forth moving cards from one plate to another. From the results we see that most children (11 out of 19) attempted to place seven candies on as many plates as possible, leaving only one plate with six candies.

An interesting aspect of this problem is the blank card. The blank card was meant to represent zero candies. In an abstract context, seven may be decomposed into seven and zero. However, in the birthday party context, there is no added value to placing the blank card on any of the plates. After all, if we place the card with seven candies on one of the plates, then that plate now has seven candies. Why add the blank card? And yet, eight children added the blank card to one of the plates. It could be that for these children, there was

Plate	Plate	Plate	Plate	Final Status	Frequency (claims not every plate has 7 candies)
7+0	6+1	5+1	4+3	7, 7, 7, 6	5(-)
7	6+1	5+1	4+3	7, 7, 7, 6	4(4)
7	6	5+1+1	4+3	7, 7, 7, 6	2(1)
7+1	6+1	5	4+3	8, 7, 7, 5	2(1)
7	6+1	5+3+0	4+1	8, 7, 7, 5	1(1)
7	5+1+1	-	-	7, 7, -, -	1*(1)
7	6	5	4	7, 6, 5, 4	1(-)
7	6	5	3	7, 6, 5, 3	1(-)
7+1	6+0	5+4	3+1	9, 8, 6, 4	1(-)
7+1	6+5	4+1	3+0	11, 8, 5, 3	1(-)

* This child stopped in the middle realizing that the problem was unsolvable.

Table 1: Frequency of arrangements of candy cards per plate (acknowledged that there were not 7 candies on every plate)

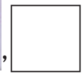
an implicit understanding that all of the cards had to be used, even if it did not make sense according to the story. If that was the case, then the blank card could have essentially been added to any of the plates. Yet, of the eight children, five placed it on the plate with the card with seven candies, representing the decomposition of seven into seven and zero. Perhaps they were trying to place two cards on each plate.

Regarding strategies for solving the problem, it was sometimes difficult to tell which strategy a child was employing. Counting was the most obvious, as children usually counted aloud or at least pointed and touched each candy, probably counting silently in their heads. Estimation and subitizing were more difficult to discern, although we gathered that many children used this strategy, especially those who placed seven candies on the first plate and six on the second, without counting the candies on either plate. Recalling number facts was most apparent for the decomposition of seven into four and three. For example, one girl looked at all the cards, chose the card with four candies on it and placed it on the first plate. Then she carefully looked over the rest of the cards and deliberately chose the card with three candies on it, placing it on the same plate as the four candies. She did not say anything. Because of the difficulty in accurately identifying which strategy was used, we do not quantify and summarize the number of times each strategy was used but instead illustrate in the following example how one child employed several different strategies when solving this problem and how the birthday party context impacted on her final answer.

Shula begins by counting the candies on the cards (lines 1, 3, and 4) and choosing which to place on plates. However, after she counts four candies (line 4) she adds a single candy without counting the total.

1 Shula: (Shula uses her finger to count the candies on the card:  .) 1, 2, 3, 4, 5.
2 Interviewer: You can place more than one card on the plate.
3 Shula: (Shula counts the seven candies on the card ) 1, 2, 3, 4, 5, 6, 7 (and places the card on the first plate).
4 Shula: (Shula counts the four candies on the card ) 1, 2, 3, 4 (and places it on the second plate. She then places the card with  on the same plate.)

She then seems to randomly place cards on the other plates (lines 5 and 6), without counting, perhaps just to fill up the plates with something. Later (line 10), when she does count them, we learn that perhaps she thought she was placing seven candies on each plate, possibly by estimating the amounts.

5 Shula: (Shula places  and  on the third plate.)
6 Shula: (Shula takes the three cards that are left: , ,  and puts them on the fourth plate.

Shula realizes that not every plate has seven candies (lines 8 and 12), but claims that it does not matter.

7 Interviewer: So, does each friend have seven candies?
8 Shula: Here (pointing to the second plate with four and one) there are 1, 2, 3, 4, 5. It's not important.
9 Interviewer: It's not important?
10 Shula: (Shula counts the candies on the other plates, miscounting the 5 and 3 and concluding that there are seven candies on that plate.)
11 Interviewer: So, does everyone have seven?
12 Shula: No.
13 Interviewer: Who doesn't have seven?
14 Shula: (Shula points to the second plate with 5 candies.)
15 Interviewer: So, what should we do?
16 Shula: Leave it.

Shula uses a combination of strategies to solve the problem. She counts, sometimes correctly and sometimes incorrectly. She may also have used estimation, placing the card with a lot of candies (six candies) on the same plate as the card with only one candy. Only when asked if each plate has seven candies, does she go back and count the candies on that plate. It is interesting to note that after Shula acknowledges that not all plates have the required seven candies, she does not try to rearrange the cards and work it out. Nor does she complain that there are missing candies. Instead, she says that it is not important and we should leave it.

In real life, 27 hard candies cannot be distributed equally among four plates. Of the 19 children who participated in this study, eight children answered correctly at the end that not all plates had seven candies. Four of those children, like Shula in the vignette above, acknowledged this fact without further comment. They did not seem bothered by not having seven candies on each plate; they made no comments or gestures that might be interpreted as frustration or exasperation; nor did they intimate that the problem was unsolvable. They seemed satisfied that the task was completed. One of those children placed the card with seven candies on the first plate, the card with five candies and the cards with one candy each on the second plate (i.e., 5+1+1), and then sat back and said, "There aren't enough." When the interviewer asked her how come, she pointed and replied, "Because here this is zero and six and here four and three." This child did not count and instead seemed to use a combination of subitizing (she recognizes without counting that there are three candies on the one card and four on the other) and number facts (e.g., 3+4=7). When the interviewer asks, "so only two friends will get candies?", she does not respond further and at her own initiative, she stops the activity, leaving the other two plates empty.

Of the four other children, three made it clear that one candy was missing. One child, trying over and over to solve the problem said, "Something is not right," before explicitly stating that one candy was missing. Another child turned over the blank card to make sure that there wasn't a picture on the other side. When asked what he was looking for, he pointed to the card with one candy on it as if to say that he was missing one candy. The third child said that a card was missing and stated "like that one" pointing to the card with one candy on it. The interviewer responded, "You are right, so, what do you say we go to the store and buy candies?" The boy nodded his head in agreement and laughed. The fourth child suggested removing the plate that had less than seven candies, leaving the remaining three plates with exactly seven candies.

Of the 11 children who incorrectly claimed that each plate had seven candies, many simply did not count the candies as they distributed the candy cards. Some seem to have relied on the strategy of estimation. That is, the card with seven candies and the card with six candies both looked like a lot of candies and thus, they did not feel the necessity for counting. One boy began by placing the card with seven candies on one plate, the card with six candies on the next, and the card with five candies on the third. Only at that point did he stop and realize that the five candies were not enough. After he thought about it and added two extra cards, with one candy on each to that plate, he placed the card with four candies and the card with three candies on the fourth plate, counting the candies on that plate to make sure. He did not go back and check the cards on the first two plates (the second plate had six candies) and so claimed that each plate had seven candies. Other children counted incorrectly. Incorrect counting most often occurred for the cards with seven and five candies, possibly because the candies were not pictured in a row, making it more difficult to keep track of the counted candies. In general, children engaged enthusiastically with this problem, used various strategies to solve it, and displayed various levels of competency when it came to counting.

DISCUSSION

The mathematical context of the problem posed to children in this study was the decomposition of seven. As described above, many children did not recognize that the problem was unsolvable, possibly because they estimated and did not count or because they miscounted. One factor which contributed to the difficulty of counting was the physical setup of the task. If children had been given actual candies and could physically move the items, they might have found it easier to count (Sarama & Celements, 2009). However, if they had been given candies, the problem would have lost the focus on decomposing seven. Furthermore, with physical candies, children might have distributed the candies by placing one candy at a time on each plate until all the candies were used up. It would then be quite obvious that one candy was missing and thus the impact of having an unsolvable problem might have been lost.

When children did recognize that every plate could not be filled with exactly seven candies, they did not seem very disturbed by the insolvability of the problem. None claimed that every problem must have a solution. Only one child tried over and over to move the cards around and solve the problem. In addition, unlike the students in the studies reviewed in the background (e.g., De Corte & Verschaffel, 1985), none of the children in this study gave a non-relevant solution.

In analyzing the reasons behind the results, we offer a few possibilities. First, it could simply be that this problem was not abstract and children were able to actually see the pictures of the candies and see that there were not enough. However, in a previous study of students' handling of unsolvable problems, Reusser and Stebler (1997) found that when students were told to explicitly make a sketch of a real life problem, the results hardly improved. What did improve the results were students' past experiences with other unsolvable problems. Reusser and Stebler (1997) surmised that when students are made to solve "so-called" real- life problems that are essentially routine mathematical exercises disguised as word problems, they become immune to the actual contextual limitations of the problem. The students in that study were fourth and fifth graders, old enough to have absorbed implicit norms which indicate that every mathematical problem given in school, must have a solution. The children in our study were quite young, not yet enculturated to school mathematics problems. They were most likely young enough to rely on their natural problem solving sensibilities, sensibilities we wish to preserve and strengthen. We suggest that engaging young children with unsolvable mathematics problems is one way to provide experiences which children may reference in the future.

However, not every unsolvable problem may encourage children's insight into or acceptance of the insolvability of that problem. The context, and how close it is to the child's real world, may also impact on the child's ability to see that a specific problem is unsolvable. The solution to is not a whole number. The problem presented in this study cannot be solved. However, there are also cultural norms related to the practice of birthday parties in kindergarten. Four children accepted the final status of the plates without further comment. For them, it could be that as long as everyone received some candies, the problem had a satisfactory ending. Or perhaps they felt that the child with fewer candies might get an additional piece of cake. One child stated that one plate should be removed. Perhaps for that child, equal sharing is more important than filling up all the plates. Perhaps, from that child's perspective, it would be better to invite only three friends than have four friends who cannot get an equal amount of candies. Three children stated that one candy was missing; one of those three playfully agreed to go with the interviewer to buy more candies. That is a solution which is quite realistic. In answer to the the title of this paper, kindergarten children can engage with unsolvable mathematics problems.

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REFERENCES

Baroody, A. J. (2004). The developmental bases for early childhood number and operations standards. *Engaging young children in mathematics: Standards for early childhood mathematics education*, 173–219.

Baroody, A. J., Lai, M. L., & Mix, K. S. (2006). The Development of Young Children's Early Number and Operation Sense and its Implications for Early Childhood Education. In B. Spodek & O. Saracho (Eds.), *Handbook of research on the education of young children* (Vol. 2, pp. 187–221). Mahwah, NJ: Erlbaum.

Department for Children, Schools and Families (2008a). *Statutory Framework for the Early Years Foundation Stage*. London, UK: DCSF.

De Corte, E., & Verschaffel, L. (1985). Beginning first graders' initial representation of arithmetic word problems. *Journal of Mathematical Behavior*, 4, 3–21.

Israel national mathematics preschool curriculum (INMPC) (2008). Retrieved April 7, 2009, from http://meyda.education.gov.il/files/Tochniyot_Limudim/KdamYesodi/Math1.pdf

National Council of Teachers of Mathematics. (2006). *Curriculum focal points for Prekindergarten through Grade 8 Mathematics: A quest for coherence*. Reston, VA: Author.

Reusser, K., & Stebler, R. (1997). Every word problem has a solution – the social rationality of mathematical modeling in schools. *Learning and instruction*, *7*(4), 309–327.

Sarama, J., & Clements, D. H. (2009). *Early childhood mathematics education research: Learning trajectories for young children*. London, UK: Routledge.

Streefland, L. (1990). *Fractions in Realistic Mathematics Education, a Paradigm of Developmental Research*. Dordrecht, The Netherlands: Kluwer.

Tirosh, D., Tsamir, P., & Levenson, E. (2011). Using theories to build kindergarten teachers' mathematical knowledge for teaching. In K. Ruthven & T. Rowland (Eds.), *Mathematical Knowledge in Teaching* (pp. 231–250). Dordrecht, The Netherlands: Springer.

Verschaffel, L., De Corte, E., & Lasure, S. (1994). Realistic considerations in mathematical modeling of school arithmetic word problems. *Learning and Instruction*, *4*(4), 273–294.

Verschaffel, L., Greer, B., & De Corte, E. (2007). Whole number concepts and operations. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 557–628). Greenwich, CT: Information Age Publishing.