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Introduction to arithmetical expressions: A semiotic perspective

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An early introduction to arithmetical expressions is realized in a teaching experiment involving an artefact based on the rectangle model for multiplication. Children elaborate signs, strictly related to the activity with the artefact, which evolve to mathematical ones: the enchainment of different representations in many semiotic systems is described according to Theory of Semiotic Mediation. In particular the teacher role in selecting and elaborating specific personal signs, in order to make them pivot signs, results as crucial. Elaborated tasks reveal to be good triggers for a relational approach to arithmetical expressions.

Keywords: Arithmetical expression, multiplication, rectangle model, distributivity, semiotic mediation.

DISTRIBUTIVITY IN GRADE 2

In Italy there is a long tradition for which students are expected to know all the times-tables from 1 to 10 at the end of grade 2, often justified with sentences such as “if they do not learn times-tables in the second year, they will never learn them”. It is well known that recall of results of one-digit numbers multiplications is more difficult when the operands are closer to ten. Psychologists refer to this phenomenon as *operand size effect* (e.g., McCloskey, 1991). Operations’ properties can be used to reconstruct more difficult results relying on easier ones ($8 \times 2 = 2 \times 8$; $3 \times 8 = 3 \times 5 + 3 \times 3$), this kind of strategy could be particularly useful when times-tables have not yet been memorized completely. In other words an early introduction to operations’ properties may promote flexible calculation strategies instead of rote memorization.

The use of operations’ properties requires a relational approach to calculation in order to establish the equivalence between different calculation procedures;

establishing such an equivalence requires to grasp the relationship between two arithmetical expressions organizing the relationship between two operations (i.e. multiplication and addition) in a highly structured way. The classic symbolic representation seems hardly accessible to very young children, thus distributive property is often introduced by graphical representations (Ding & Li, 2014; Izsák, 2004): rectangles have been largely used as model for multiplication (for a large review see Izsák, 2005) from Euclid’s *Elements* since more recent western textbooks (Ding & Li, 2010).

The aim of this paper is to describe the emergence of symbolic arithmetical expressions as numerical representation of the distributive property: working with a specific artefact (a rectangular model for multiplication), pupils begin using the artefact, pass through an iconic representation and arrive to make sense of a structural relationship between different numerical expressions.

THEORETICAL FRAMEWORK AND METHODOLOGY

The work here presented is a part of a wider teaching experiment on multiplication, implemented in a grade 2 class in Italy. The general theoretical framework is the Theory of Semiotic Mediation (TSM) (Bartolini Bussi & Mariotti, 2008) and the teaching sequence is centered on the use of an artefact called “geometrical times-table”. In this table, rectangles are organized, increasingly, in columns and rows as showed in Figure 1.

According to the TSM approach, the didactic intervention was implemented in *didactical cycles*, comprehending individual activity with the artefact, small group work and whole class *mathematical discussion*.

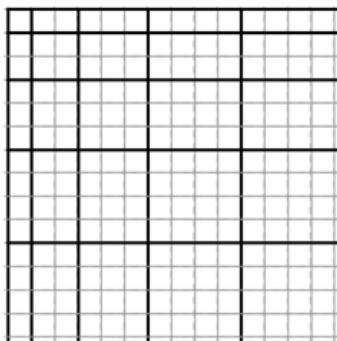


Figure 1: Geometrical Times-Table with five columns and rows

An artefact is defined as a *tool of semiotic mediation* when the teacher uses it intentionally to mediate a mathematical content to students (Bartolini Bussi & Mariotti, 2008). The artefact is both related to the personal meanings of its user while solving a particular task and to the mathematical knowledge underpinning the task and/or the construction of the artefact itself. This double relation is called *semiotic potential* of the artefact (ibid, p. 754). When students accomplish a task using the artefact, they produce signs strictly related to this activity; those signs, to which we refer as *artefact signs*, can be very different from the ones usually used by mathematicians while working with the mathematical knowledge related to the task, because pupils can be unaware of this knowledge. Anyhow students are expected to become able in using culturally determined *mathematical signs*. According to TSM we expect an evolution from artefact signs to mathematical ones. In order to describe such evolution we use the notion of *semiotic chain*, the set of dynamic relations among artefact signs and mathematical ones. In the description of the evolution through semiotic chain, a specific role is played by another type of signs named *pivot signs*. These signs are defined with respect to their function in promoting the relationship between the other two categories of signs (ibid, p. 756).

In the case of geometrical times-table, each one of the rectangles can represent a multiplication: the sides are the factors and the area is the result. Rectangles can be cut, moved and pasted and, in particular, some of results of these actions can be related to specific operations' properties. For instance, commutative property can be related to observation of different positions of rectangles with sizes of the same length. In the first part of the teaching sequence, children explored the artefact and were asked to explain how numbers and operations could relate to it. Finally, after a couple of months, a didactic cycle was released

aimed at introducing pupils to the distributive property.

In the first task, students are asked to cut two pieces of paper with the same dimensions of two rectangles of the same row (both sides smaller than five). They also have to paste the two pieces of papers along one of the sides in order to form a new rectangle, then they are asked to look for a rectangle with the same dimensions into the table. Children are expected to use the obtained rectangle dragging and rotating it on the table, trying to fit it inside the borders of one of the rectangle in the table. We imagine a relation between the pasting of two rectangles in a bigger one as the sum of two multiplications to obtain another one, i.e. it is a transformation of an arithmetical expression according to the distributive property: $a \times b + a \times c \rightarrow a \times (b + c)$. In our a priori analysis we expected that pupils notice that all the rectangles have the same height and that the final rectangle has a width that is the sum of the two of the others, this constitute the germ of the mathematical meaning of the distributive property. In order to work on the other direction of the transformation ($a \times b + a \times c \leftarrow a \times (b + c)$), a second task has been implemented: each student received a copy of a letter by Giovanni, an imaginary child who lives in another city. In his letter, Giovanni explains that he has to calculate 3×7 but he only remembers multiplications with factors smaller than five. In our a priori analysis of the task we expected children to use the signs produced during the past activities to decompose 3×7 in two smaller multiplications, eventually using the table to find the right ones.

Using TSM lens, we analyse the activity of children in the progression of the didactic cycle, while facing the two tasks involving the use of the geometrical times-table and in the following collective discussions. According to TSM, we will look for semiotic chains, identifying the three different types of signs and their mutual relations, in order to describe the development of the semiotic mediation process. Thus our research questions can be articulated as follows:

- (1) which semiotic chains can describe the semiotic mediation process from the use of geometrical times-table to symbolic representations of the distributive property?
- (2) What is the role of the teacher in triggering this process?

The two tasks have been implemented in a grade 2 class of 20 students, in Tuscany in a period of two weeks. All lessons have been videotaped by the first author who, even if he was not the teacher, interacted with the pupils during the different work phases. Following the analytical model for studying videotape data by Powell, Francisco and Maher (2003), all videos have been viewed several times and a description of the events has been written. According to a priori analysis of the semiotic potential of the artefact, we selected critical events, then transcribed and coded. The different signs produced by teacher and students have been classified according to TSM (artefact signs, pivot signs or mathematical signs) and semiotic chains identified. Videos and written productions made us able to reconstruct the storyline interpreted in the next section.

DATA ANALYSIS

Combining tiles of the same row: the emergence of a semiotic bundle

After a previous activity, pupils became familiar with the artefact, with cutting and moving rectangles and recognizing them on the table. A shared system of artefact signs was established around the key word 'tile' that, at this point, has a complex meaning: it refers to one of rectangles (either on the table or cut on the paper), but also to the multiplication between the two numbers that represents the dimensions of the rectangle, specifically it refers to both the multiplication and the result. The first task of a new cycle asked to select two tiles in the same row of the table, to cut them and to combine them into a new tile and to identify on the table a rectangle that corresponds to this new tile. Working on this task children engaged in finding where their rectangles (to which they refer as "tiles") were inside the table. When all the students completed this task, the teacher asked for comments. A child made an intervention:

Lor: That...when two tiles are far (he points at the table) you can calculate the result and then you know it.

Researcher: And how do you calculate the result?

Lor: Between these two (he points two rectangles in the fourth row) you do nine times four, it is thirty-six (he points the 4×9 rectangle) plus twenty (he points the 4×5) it is fifty-six.

Researcher: [...] Well, Lor gave an example and he said that when the results of two tiles are known, it is possible to discover the result of another tile. I have understood this way, you have to say to me if I understood correctly. You said that if I know the result of two tiles (he does the gesture in Figure 2a), I can do the addition (gesture in Figure 2b). Isn't it?

Lor: No. It is that, if you do these far two, you calculate them!

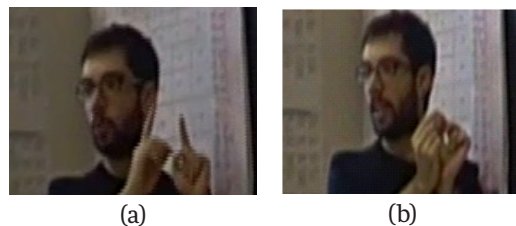


Figure 2: Researcher gestures during discussion

The researcher elaborates the statement of Lor generalizing it, and also passing to the interpretation of the calculation – addition of multiplications – as the combination of tiles. The semiotic process of interpretation is accomplished by enchainment of words, graphical representations, mathematical symbols and gestures. The word “addition” is combined with the gesture of joining the fingers, with the intention of relating the combination of tiles (gesture) and the operation of adding numbers. The pupil seems not to follow the researcher comment and repeats his statement stressing the fact that he is referring to tiles which are far one from the other. Maybe this child looks at the pragmatic scope of the activity as to find the results of the union of two tiles which cannot be compared directly (because they are far). He is able to produce a new personal example without cutting new pieces of paper and using numbers which lead to a multiplication out of the table. The researcher continues the discussion trying to change signs and to work on another, simpler, example.

Researcher: What do you obtain?

Lor: Fifty-six.

Researcher: And what do you need this number for? What does it represent?

Lor: A tile.

Researcher: It is another tile, it is what I was saying: if you put together two tiles, then you find another one. Let's do an example [...] if I, for instance, take the tile two

times three (he draws a 2×3 rectangle on the squared blackboard, he writes “ 2×3 ” inside it). Do you all agree that this is the two times three tile?

Chorus: Yes!

Researcher: And I put together the four times three tile (he draws a second rectangle juxtaposed to the previous one, Figure 3a) this is big as which tile? Putting all together?

The sign “addition” has been replaced with the artefact sign “put together” which refers directly to the activity done (i.e., the pasting of the pieces of paper) with the aim of bridging the gap between the idea of combining the tiles and that of adding multiplications. It is also introduced a graphical icon (Figure 3a) together with written and oral mathematical signs. The researcher refers to the graphical representation using the word “tile”, an artefact sign. Arzarello (2006) defines a *semiotic bundle* as a collection of semiotic sets (set of signs, modes of producing, relationships with meanings) and relationships among them. In the last excerpt there is a system of different related signs involving the ones used by students and new ones introduced by the researcher, it can be described as an example of semiotic bundle. In particular we observe a *genetic conversion* (ibid, p. 281), namely the oral and gestural signs, produced in the previous part of the discussion, are converted in new graphical ones enlarging the bundle.

After these transcribed episodes, the researcher changes the example and asks children to develop the interpretation process, focusing on the dimensions of the rectangles, then on the height and width of the resulting rectangle. When they agree on the answer, he synthesizes their intervention saying that the two tiles, together, equal [1] the two times seven

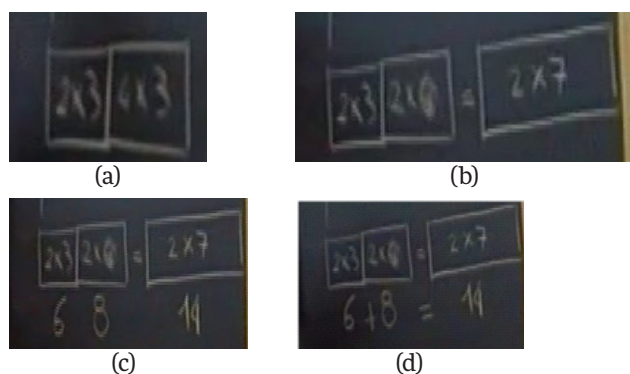


Figure 3: Graphical signs on the blackboard

tile and he draws it (Figure 3b). Finally, the researcher asks the children to say how many squares there are inside each one of the rectangles and he writes these numbers under the drawing, obtaining the signs represented in Figure 3c.

Some students begin to notice that six plus eight is fourteen. The researcher decides to rephrase one of his previous sentences.

Researcher: So, when I put together the squares inside this tile (he points the 2×3 tile) with the squares of this tile (he points the 2×4) I obtain the squares of this entire tile (pointing 2×7). Isn't it?

Non: It is true!

Researcher: Which operation does “put together” correspond to?

Chorus: Six plus eight!

Mab: Equals fourteen.

Researcher: (writes the symbol + and = between the numbers, Figure 3d) So, what does this mean? If I know the results of two little tiles (he points 2×3 and 2×4), I can put them together (he points the numbers 6 and 8) and what do I find? (he points 14) The result...

Chorus: Of a tile!

Researcher: And how do we find this tile? It has the same height, and this? (he points the base of the rectangle)

Sim: It is large as the two together.

Researcher: It is large as the two together. Do you all agree?

Chorus: Yes!

“Put together” becomes a pivot sign that is explicitly related to the operation of addition, but in the same time it is related to the combination of tiles and more specifically to the addition of the width-dimension of these tiles. As suggested by the children, the researcher also introduces the symbol + in the discourse, it is not between the operations of multiplications but between the respective results. At the end of this discussion, the children are asked to produce some personal examples. Two children cannot find a way to accomplish this final task, another child just copies the example on the blackboard. The others give one or even more original examples imitating the pivot sign of Figure 3c to represent the distributive property (Figure 4a). Four children transform the

drawing adding the sign + between the rectangles (Figure 4b). All these texts are made of two lines: the first line is composed of pivot signs and the second one of mathematical signs, as each text were a kind of Rosetta Stone: establishing an explicit relationship between the two lines they solicit the translation of artefact signs into mathematical ones, and so they may function as a resource for the unfolding of the semiotic potential of the artefact. The different texts produced by the pupils show interesting differences; for instance, the kind of text in Figure 4b (where the first line text includes both artefact and mathematical signs) presents an hybridisation between the different semiotic systems that provides evidence of the movement from personal meanings, strictly related to

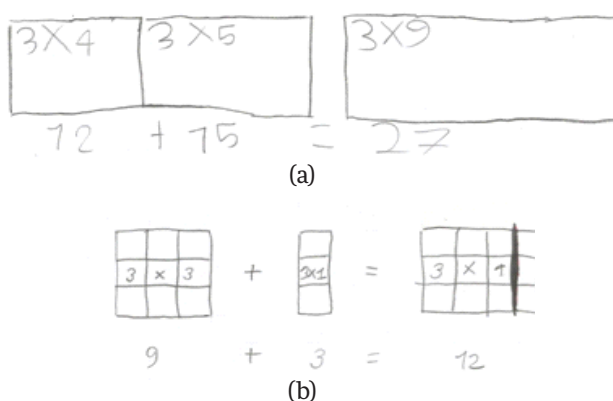


Figure 4: Fra's (a) and Mal's (b) productions after the class discussion

the use of tiles, to mathematical meanings expressing the relationship between operations. In the following days the teacher asked the students to give examples using the version with the + between the rectangles (Figure 4b). The teacher recalls this bundle, used by some students, and decides to foster the sharing of this type of signs that has the potential of linking the text composed of artefact signs with the text made of mathematical signs.

Translating rectangles into arithmetical expressions

After a couple of days, the teacher proposed a variation to the task: she gave them an example writing it on the blackboard, saying them to copy it on the notebook (Figure 5a) and then to invent some personal examples. In this way the teacher enriched the text introducing a new line of mathematical text, in parallel with the previous ones. Such a text is provided as an alternative 'translation' of both the artefact text and the mathematical text. Though at this moment just few students create new text including this new

expressions (Figure 5b), the availability of a translation key from a semiotic system to another will play a crucial role in the further activity.

A week after, the second task was given. The students were asked to read the letter, discuss some possible answers in small groups and then to report their solutions to the whole class. A proposed solution consisted in decomposing the rectangle 3×7 in 3×4 and 3×3 or in 3×5 and 3×2 using the graphical pivot signs developed in the previous activities to represent this process. It is interesting to notice that, while reporting this second solution, Fra comes to the blackboard and begins from the expression $3 \times 5 + 3 \times 2$ (written just in symbolic mathematical signs) and only after he completes it drawing rectangles (with the same height) around the two multiplications, without caring about their lengths.

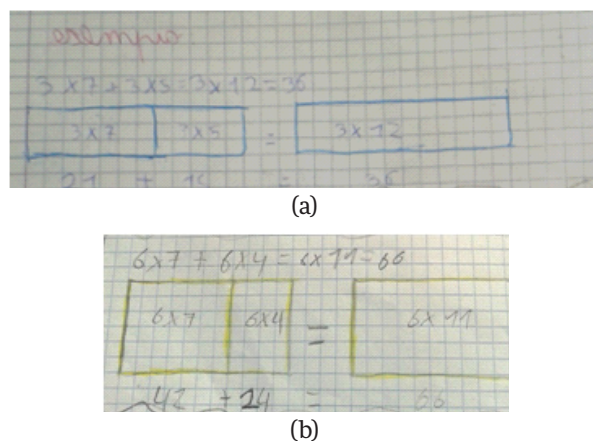


Figure 5: Mir's copy of the teacher example (a) and Lor's personal production (b)

A while after, Mir asks to go to the blackboard saying that he would like to try a new different solution (Figure 6a). As matter of fact, from the mathematical point of view, what Mir writes does not differ from Fra's proposal; the difference is only in its representation: Mir's representation eliminates any reference to the artefact using genuine symbolic register, as it is confirmed by the following exchange.

Researcher: Ok, Mir can you explain me a thing. What do all that equal signs mean?

Mir: Three times five equals fifteen. Three times five plus three times two equals fifteen plus six.

The new signs are strictly related to the signs used till this moment, as shown by the activity of Fra, but

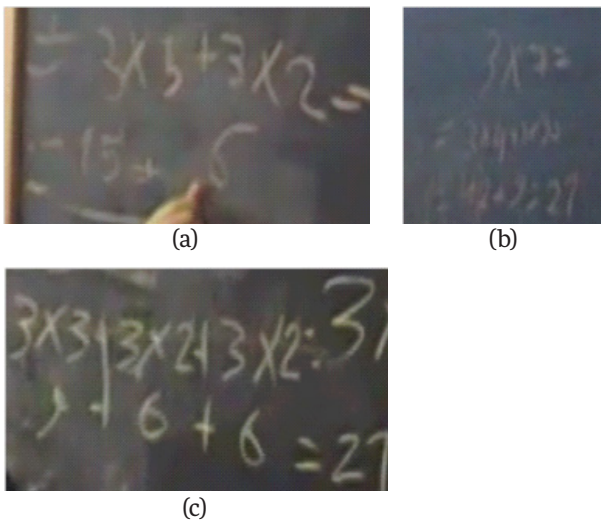


Figure 6: Arithmetical expressions are used to represent the proposed calculations

have lost all the graphical qualities shared with the artefact. The text now relies completely on mathematical culturally determined signs, its meaning is not explained but the direct linkage with previous signs, condensed in our Rosetta Stone, allows students to interpret it. When Mir receives a good feedback for his representation, other students ask to come to the blackboard to use the same representation (Figure 6b), among them, at the very end of the discussion, Lor comes to the blackboard and suggests another solution (Figure 6c).

After this discussion, students are asked to write individually a letter to answer to Giovanni. The majority of the given examples about the order in which Giovanni has to perform his calculation are expressed by arithmetical expressions, only three students just use a representation with rectangles. In many productions the two kinds of representation appear together (Figure 7), showing evidence of the process of appropriation of a relational meaning of arithmetical expressions and its relation with the use of the artefact.

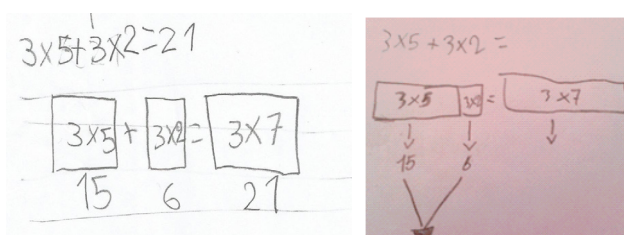


Figure 7: Sob's and Sim's combined usage of expressions and rectangles

DISCUSSION

The usage of artefacts to introduce the distributive property is a diffused approach: in some studies it is possible to find pre-constructed paths implemented in textbooks (Ding & Li, 2014) or in instructional materials (Izsák, 2004). However “how students might be supported to make transitions from concrete to abstract representations remains largely unknown” (Ding & Li, 2014). Our study shades light on such a complex process, showing the crucial role played by the semiotic dimension as it is modeled by the TSM. In our experiment we introduced an artefact with strong representative features based on the rectangular model for multiplication, with different tasks specifically designed for this experiment. The analysis of the data shows the expected process of semiotic mediation. The transition from artefact signs to arithmetical expressions is guided by the teacher and the researcher choosing tasks and orchestrating the mathematical discussions.

From a theoretical point of view, TSM gave suitable aids to design the activity with the artefact and provided useful analytical tools. In particular the distinction of different kind of signs gives many insights on the evolution of students' productions and on teacher's interventions. As clearly shown in the previous analysis, a sensible handling of the pivot signs in the collective discussion allows the teacher to foster the development from personal meanings to mathematical ones. It has to be noticed that many different kinds of signs go under the label “pivot signs”. These signs are more or less related to the artefact or to mathematical symbols and they belong to different semiotic systems, sometimes used in parallel or generated one from the other, briefly they constitute a semiotic bundle (Arzarello, 2006). This construct grasps this semiotic richness that may explain the potential move from representing the combination of rectangles (tiles) towards representing the relationship between arithmetical expressions. Moreover, the use of semiotic bundles as pivot signs aimed at relating the activity with the rectangles to mathematical signs, was intentionally exploited by the teacher through the production of hybrid texts explicitly relating – as in a Rosetta Stone – the two different systems of signs.

It is also meaningful to remark how asking for individual productions after the first discussion allowed the teacher to observe emerging personal represen-

tations produced and shared by the children; these signs (specifically the introduction of the sign + between the rectangles), appropriated by the teacher, became fundamental to foster the evolution towards mathematical signs.

The proposed kind of tasks seems to be very promising for introducing the usage of expressions in very first grades. The selection of appropriate signs led to a quite natural introduction of this mathematical representation in a context in which it is useful and meaningful. Moreover, the need of conventions about the order of operations computing appeared as natural in this context and children showed a good structural control of the expressions. We have also to stress that, even if it was a minority, there were children who had difficulties in following this approach. It is needed further research and more cases to give more relevance to these findings.

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ENDNOTE

1. Actually, the word used by the researcher is “è uguale” which, in Italian, means both “equal” and “look the same as”.