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The history of the concept of a function and its teaching in Russia

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The history of teaching the concept of function in Russian school and university mathematical education is described. For this reason, in the first section the brief survey of the history of the function concept is given. Finally, we conclude that the opinion about the introduction of a modern rigorous and precise definition of a mapping has not been reached not only in school-based education, but also at the undergraduate level of mathematics education.

Keywords: Textbooks, history of mathematical education, function.

BRIEF SURVEY OF THE HISTORY OF THE FUNCTION CONCEPT

The function concept is one of most complicated concepts not only of school mathematics but also at undergraduate level. The students' difficulties with this notion have been reported for several decades (see, e.g., Tall, 1990; Sierpiska, 1992; Bardini et al., 2014). Therefore, a need to elaborate upon effective approaches to the teaching of this notion still exists.

This survey is a first step to the research project aimed at elaborating new ways of teaching the concept of function using the genetic approach. The application of genetic approach to the undergraduate mathematics teaching is described, e.g., by Safuanov (2005). The genetic approach could be combined with methods based on the APOS theory (Dubinsky, 1991).

In order to understand the problems of teaching the notion of function, first of all we will trace the history of its teaching, at least in Russia. The changes in the teaching the concept of function have been largely determined by the history of the concept itself. Thus, in order to trace the history of teaching the concept of function, we should first make a short survey of

the history of the concept itself. The other reason for surveying the history of the concept is our belief in the necessity of knowing the historical development of the concept for elaborating its teaching by genetic method.

We will briefly trace the history of the development of the concept of function following the fundamental monograph edited by Yushkevich (1970, pp. 139–148, and 1972, pp. 250–255), and also books by Klein (1977, pp. 286–292), Gleizer (1983, pp. 20–25) and the “Mathematical encyclopaedic dictionary” (1988, p. 617). We also trace the history of the development and teaching of the concept of function in Russia on the basis of the analysis of mathematical, didactical, educational and reference books and articles.

In the monograph edited by Yushkevich (1970) it is noted that “shoots of this concept though not yet realized or made explicit, were present in the Greek mathematics and natural sciences” (Yushkevich, 1970, p. 139).

The concept of function for the first time emerged in medieval Europe in connection with the renewal of attempts of mathematical studying of various natural phenomena...

For the subsequent development of the theory of functions crucial importance belongs to trigonometry and logarithms, on the one hand, and to the birth of symbolic algebra, on the other hand.

At the beginning of 17th century functions were still quite often defined verbally, graphically, kinetically or through tables, but already in the second half of the century their analytical expressions being to play a leading role.

The term ‘function’ for the first time appeared in Leibniz’s manuscripts of 1673, in particular in the

manuscript entitled “The inverse method of tangents, or on functions’ (Methodus tangentium inversa, seu de functionibus)... The terms ‘variable’ and ‘constant’ were introduced by Leibniz” (Yushkevich, 1970, p. 146).

The definition of function as analytical expression, for the first time was distinctly formulated by I. Bernoulli in the article published in “Memoires de l’Academie des Sciences de Paris” in 1718: “Definition. A function of a variable is a quantity composed in any way from this variable and constants”. The notation $f(x)$ for a function was introduced by Leonhard Euler in 1734.

The further development of the concept of function is connected first of all with Euler’s name. Yushkevich (1972) noted that “in the foreword to the ‘Introduction to the calculus of infinitesimals’ he (Euler. – I.S.) for the first time has distinctly expressed the idea that the analysis is the general science about functions, that the analysis of infinitesimals rotates around variable quantities and their functions” (Yushkevich, 1972, p. 250).

In the first chapter of the first volume L. Euler, correcting the definition of his teacher I. Bernoulli, “... has emphasized that functions are defined by formulas: ‘A function of a variable quantity is the analytical expression composed somehow by its variable and numbers or constant quantities’. Thus the important step forward was done; the independent variable is considered as the set of all real and imaginary numbers so that functions of a complex variable were introduced on the equal rights with functions of a real variable” (Yushkevich, 1972, p. 250).

To compose analytical expressions one could use four elementary operations, computation of a root, exponential and logarithmic operations and, furthermore, “uncountable others, provided by integral calculus”, meaning thus also the integration of differential equations.

Euler’s definition of a function given in the first volume of the “Introduction to the analysis of infinitesimals”, appeared to be too narrow for the calculus as a whole. Euler formulated the new definition of a function in his foreword to the “Differential calculus” (Euler, 1755): “When quantities depend on others in such a way that at the change of the last they are also changed the first are called functions of the second ones. This term has extremely wide character;

it covers all the ways by which one quantity can be determined by means of others”. In the quoted definition nothing is spoken about the way of calculation of values of a function.

Close to the modern definition of a function is Lobachevsky’s definition: “...the general concept of function demands, that one should name a function of x a number that is given for every x and that gradually changes together with x . A value of a function can be given either by analytical expression or by a condition which gives means to test all numbers and to choose one of them, or, at last, dependence may be unknown” (Mathematical encyclopaedic dictionary, p. 617).

Thus, “... the classical definitions of a function given by Lobachevsky in 1834 and by Dirichlet in 1837 the second of which has passed to the latest textbooks (“If in some interval to each separate value x a unique value of the variable y corresponds then the variable y is called a function of x ”), are hereditarily connected with a definition belonging to Euler” (Yushkevich, 1972, p. 254).

Felix Klein (1977, p. 291) on the other hand noted that, beginning with the development of Cantor’s set theory “also functions defined for values of x from any set (not necessarily numerical) are considered...” In van der Waerden’s classical textbook, we already see the quite modern definition of a mapping: “If to each element from some set M by any rule a unique (generally speaking, new) object $?(x)$ is put in a correspondence then this correspondence $?$ is called a function. If all objects $?(x)$ belong to some set N , the correspondence $x??(x)$ is called also a mapping from M into N ” (van der Waerden, 1930).

In the post-war decades (1945–1960), in connection with fast development of topology and abstract algebra, the most formal definition of a function (=mapping) was introduced into the world of mathematics: a function is a correspondence from one set into another (i.e., a subset of their direct product) where for any element x from the first set there is a unique element y from the second set such that the pair (x, y) belongs to this correspondence.

Klein (1977, p. 292) complained that “...the school mostly ignores all the development of a science which took place after Euler” and offered: “... we wish that

the general concept of a function ... has entered as the enzyme into all the teaching of mathematics at school; but it should be introduced not in the form of abstract definition but rather on concrete examples ... in order to make this concept a living property of a pupil". He noted that "it would be desirable that among numerous teachers there was at least a small number of independently working people who would be familiar also with newest concepts of the set theory".

Despite Klein's appeal, formulated by him (and attributed by him to Euler) and more precisely by van der Waerden definitions of a function did not soon find the way to the educational practice and literature not only in secondary school but also in undergraduate mathematical education in our country. Up to the beginning of 21th century only the first part of Klein's appeal has been in essence executed: Euler's definition of a function has taken a strong place in school and university mathematical curricula. The second part – taking into account the development of mathematics after Euler and use of set-theoretic concepts has not been actually used until now at the either school or undergraduate level.

HISTORY OF TEACHING THE CONCEPT OF FUNCTION IN RUSSIA

In this preliminary survey of teaching the function concept we restricted the scope mainly by textbooks. Practices of teaching the notion of function have been largely determined by presentation of the subject in textbooks. These practices (of the last decades of 20-th century) have been partly described, e.g., by Dorofeev (1991).

Most probably, the first to mention the term "function" in Russian mathematical textbooks was Kotel'nikov who published in 1771 a book "On variable quantities" that was essentially the concise translation of Euler's "Introduction to the analysis of infinitesimals" (Prudnikov, 1956, p. 76). Note that the first German translation of Euler's "Introduction to the analysis of infinitesimals" appeared later, in 1788. Thus, Kotel'nikov's textbook was essentially the first translation of Euler's book into modern language. In this book, the definition of function was given where functions should be represented by analytical expressions. The similar definitions were given in the 3-rd volume of the textbook "Fundamentals of pure mathematics" published in 1812 by Fuss who was the disci-

ple and grandson-in-law of Euler (Fuss, 1812, p. 278), and also in the 3-rd volume of the textbook "Course of Mathematics" by Osipovsky (Osipovsky, 1823, p. 2).

Generally, throughout the 19-th century the definition of function as analytical expression representing a dependence of variables prevailed (see. e.g., Scheglov, 1853, p. 310; Chebyshev, 1936, p. 8). The situation was slightly improved in the beginning of 20-th century when, following to the Klein's appeal, prominent mathematicians called to reform the mathematics teaching and, in particular, "to permeate the entire exposition of elementary algebra, beginning with elementary grades, by the concept of functional dependence" (Grave, 1915, p. 1). The textbooks with definitions not reducing functions to analytical expressions appeared (e.g., Shaposhnikov, 1908, p. 92)

In Soviet schools, up to the middle of 60-th the textbook of algebra written by Kiselev (first published before the October Revolution) was used with the following definition of a function: "That variable whose numerical values change depending on numerical values of another one is called a dependent variable or a function of that other variable" (Kiselev, 1964, p. 25). In the textbook there was a talk about tabulated and graphical presentations of functions; however the emphasis was made on the analytical expression of a functional dependence.

Similar definition was contained in the textbook of Barsukov for grades 6–8 used from 1956 to 1967: "If two variables are connected in such a way that to each value of one of them a unique value of another one corresponds, one speaks that there is a functional dependence between these variables. ... If two variables are in a functional dependence, the variable that can accept any (admissible) values is called an independent variable... Other variable, whose values depend on values of the former one, is called a dependent variable or a function..." (Barsukov, 1967, p. 250).

Still in 1970, in the algebra textbook of Kochetkova and Kochetkov for grade 10 that replaced Kiselev's textbook, the similar definition is given (in the slightly more precise form), essentially ascending to Euler's one: "If to every value of a variable x somehow a certain value of another variable y is put in a correspondence one says that a function is defined" (Kochetkova & Kochetkov, p. 127).

The same situation is observed in undergraduate textbooks. In the textbook on higher algebra by Shapiro (1935, p. 5) also the functional dependence is stressed: “If two variables x and y are connected in such a way that to each value x a certain value of the variable y corresponds, then the variable y is a function of x : $y = f(x)$ ”.

Similarly, in Sushkevich’s textbook (1941) it is supposed by default that a function is an expression $f(x)$, where x is a variable (p. 86). At the same time in his textbook there is (in a little archaic language) the quite modern definition of a group homomorphism with the requirement that to each element of the first group one has put in a correspondence a unique element of the second group, i.e. with the requirement to a homomorphism to be a mapping in the modern sense (p. 353).

Uspensky (2002, p. 163) noted that in Great Soviet encyclopaedia in 1956 a function was defined as a dependence of variables on other ones. He also mentions similar definitions of a function in authoritative undergraduate textbooks on calculus.

We explored a number of undergraduate textbooks on mathematics. Here are some results.

In the textbook of Stepanov (1953) on differential equations the definition of a function is absent. On the other hand, in the second edition of the textbook of Pontryagin (1965) the special appendix was added containing the modern definition of function as a mapping (pp. 292–293).

Note that the greatest mathematicians of the level of Luzin, Kolmogorov, Aleksandrov, Pontryagin, were apparently, first to realize the necessity of the introduction of the modern definition of a mapping into the scientific and educational literature. Such definition of a mapping is used in books of Luzin (1948), Aleksandrov and Kolmogorov (1948), Kolmogorov and Fomin (1954). Note that all these descriptions characterizing a function as a rule of correspondence were not strict definitions and left the concept of a function (mapping) undefined.

At the same time some great scientists still did not introduce the general concept of a function and its definition, limiting themselves to special cases. So did Mal’tsev (1956) and Gel’fand (1971) in their textbooks

on linear algebra. Apparently, it was implicitly supposed, that mastering special cases of the concept of a mapping is enough for mastering the appropriate themes of mathematics, and it is not necessary “to multiply entities”. Dorofeev (1978, p. 21) expressed similar educational ideas when he in a discussion article even protected a thesis about uselessness of the definition of a function: “Pupils have, basically, the correct substantial view of a function as a mathematical object, but experience significant difficulties when they encounter the definition of this object... This situation, namely the possession of a concept without knowledge of its exact definition is not strange at all... it is typical in the majority of kinds of human activity...”.

Kolmogorov (1978, p. 29) in his reaction, however, indicated: “...Dorofeev ... at school in general ... allocates to any version of the set-theoretical definition of function a modest place (basically only for optional lessons). I think, however, that for school textbooks ... rules (composing the definition of the concepts of a function. – I.S)... should be given to pupils early enough and should be coordinated with some certain final definition”.

Beginning with 1960s, as many researchers observe, owing to “bourbakization” of mathematics, set-theoretical concepts and, in particular, the general concept of a mapping (function) entered into curricula of secondary and tertiary school.

Kolmogorov supervised the reform of school mathematics teaching at the end of 60s. However, in the textbook for upper secondary school edited by Kolmogorov (first editions appeared in 1960s) a concrete definition of function is used, and authors consider only numerical functions: “A correspondence with a domain D where to each number x from the set D a unique number y corresponds by some law, is called a numerical function” (Kolmogorov et al., 1990, p. 20).

Thus, “Kolmogorov” reform did not aim at giving to teaching of mathematics abstract and formal character of what it was severely accused by opponents. The purposes were to eliminate archaic language and character of teaching, to correct the scientific level of mathematical education. The great attention was given to the didactical maintenance of the reform.

The general definition of a mapping has been introduced into the first textbooks corresponding to reformed curricula and published under the edition of Markushevich (1975) – the prominent mathematician and educator, the ally of Kolmogorov in the reforming of school mathematics. In 1960–70-s didacticists have developed also methods of teaching the concept of function at school (Kolyagin et al., 1977), and concluded that it is expedient to study the concept of function at school, consistently introducing concepts of sets, ordered pairs, direct products of sets, and correspondences.

The “Kolmogorov” reform met strong resistance in the school environment. Many of innovations were not accepted. It seems that “Kolmogorov” reforms were doomed to failure in conditions of inflexible, uniform and authoritarian education system. Some prominent mathematicians such as Pontryagin in 1980, attacked Kolmogorov reforms. They used as their mouth-piece the official magazine “Communist”. The article in that magazine (as well as in the newspaper “Pravda”) was equivalent to the denunciation. As a result, the general concept of a mapping, as well as other general set-theoretic concepts, has been expelled from school curricula, and Euler’s definition of a function occupied a strong position in school mathematics.

In our opinion, this was a mistake, because the absence of the strict definition often complicates mastering the concept of a mapping by students in their further study in universities (Kolmogorov, 1991; Carlson, 1998; Bardini et al., 2014). We believe that, whilst not demanding from pupils the faultless possession of strict definitions, it is still necessary for them to attain an awareness of the modern definition of function.

Further development of methods of introduction and teaching of the concept of a function became possible in 1990-s when it was allowed to use alternative textbooks at school.

So, in the set of textbooks of Mordkovich for grades 7–11 the dialectic approach to the introduction of mathematical concepts is applied: “...the concept of a function ... should not, in a deep belief of the author, be introduced strictly from the very beginning, it should grow” (Mordkovich, 1998, p. 6). Strict definition of a function is introduced only in grade 9. Nevertheless,

A. Mordkovich also considers only numerical functions, defining them in the language of variables.

We see a similar picture in the textbook of Bashmakov (1992, p. 11): “The variable y is a function of a variable x if such dependence between these variables is defined that for each value x uniquely determines the value of y ”.

Thus, the most widespread textbooks of mathematics for the senior grades contained definitions of functions which are similar to Euler’s definition, as mentioned by Klein. We note also that Klein’s advice was to introduce these definitions gradually, through examples. We see, that the majority of mathematical and educational community in Russia admitted that it is inexpedient to study the modern set-theoretical definitions of a mapping at school.

In undergraduate textbooks, since 60–70-s, the strict definition of a mapping basically has gained a strong position: one may mention textbooks of Skorniyakov and Kostrikin on algebra, of Arnold on differential equations, of Zorich on calculus. In some textbooks for pedagogical institutes (for example, in the textbook of Kulikov on algebra) the strictness and formalism have got excessive character, complicating the learning of a subject. Nevertheless, in some undergraduate textbooks of the prominent scientists of the senior generation (for example, of Faddeev on algebra, of Gel’fand on linear algebra) authors still tried to avoid the introduction of the general definition of a mapping.

Thus, the final opinion about the introduction of modern strict definition of a mapping is not reached concerning not only school, but also the undergraduate textbooks.

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