



HAL
open science

Proportionality problems in some mathematical texts prior to fourteenth century

Antonio M. Oller-Marcén, José María Gairín

► **To cite this version:**

Antonio M. Oller-Marcén, José María Gairín. Proportionality problems in some mathematical texts prior to fourteenth century. CERME 9 - Ninth Congress of the European Society for Research in Mathematics Education, Charles University in Prague, Faculty of Education; ERME, Feb 2015, Prague, Czech Republic. pp.1859-1865. hal-01288402

HAL Id: hal-01288402

<https://hal.science/hal-01288402>

Submitted on 15 Mar 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Proportionality problems in some mathematical texts prior to fourteenth century

Antonio M. Oller-Marcén¹ and José María Gairín²

1 Centro Universitario de la Defensa de Zaragoza, Zaragoza, Spain, oller@unizar.es

2 Universidad de Zaragoza, Departamento de Matemáticas, Zaragoza, Spain

Proportional reasoning seems to be one of the oldest mathematical problem solving tools. Problems involving proportionality have consistently appeared in mathematics texts over time. Moreover, proportionality is still today one of the main topics in school arithmetic and several studies point out difficulties shown by the students in this context. In this paper, we perform an analysis of historical proportionality problems in order to describe the main situations where proportionality naturally appeared. Knowledge of these situations might be of interest in order to design and develop well-contextualized teaching and learning activities.

Keywords: Proportionality, arithmetic, problems, history.

INTRODUCTION

Rhind papyrus is the oldest known work that can be considered, in some sense, a Mathematics textbook. It is dated by the copyist Ahmose to the sixteenth century BC, although he acknowledges to be copying older material from around nineteenth century BC. In this papyrus, we already find problems like the following one (Chace, 1979, p. 102):

“If 10 *hekat* of fat is given out for a year, what is the amount used in a day?”

This daily problem clearly points out the important applications that proportional reasoning had in several situations involving productive aspects, commerce or administration. This fact, together with the difficulties that many students show when they face concepts related to proportionality (Modestou, Elia, Gagatsis, & Spanoudis, 2008; Van Dooren, De Bock, Janssens, & Verschaffel, 2008) justify that proportionality and its applications are still important topics in Primary and Secondary school curricula in many countries.

THEORETICAL FRAMEWORK

Mathematics, just like any other human activity, takes place in a particular historic, social and cultural context that is, of course, changeable (Wussing, 1998). Consequently, the study of mathematical concepts and problems from a historical perspective must be an essential part of research in Mathematics Education. In this sense, the contributions of History of Mathematics to Mathematics Education are diverse and admit many different approaches (Fauvel & Van Maanen, 2000; Katz & Tzanakis, 2011).

Phenomenology can also be approached from a historical perspective. Freudenthal (1983) presents historical phenomenology as one of the four types of phenomenology which have interest from a didactical point of view. The idea is to pay attention to those phenomena lying in the origin of the concept that we want to study.

Since Mathematics can be conceived from a problem-solving point of view (Ernest, 1989), it seems interesting to focus a historical-phenomenological study on those problems arising in the origin of a particular concept. This interest is even clearer if we note that “problems have occupied a central place in the school mathematics curriculum since antiquity” (Stanic & Kilpatrick, 1989, p. 1). Moreover, Tzanakis and Arcavi (2000, p. 224) point out that historical problems that motivated and/or anticipated the development of a mathematical domain are interesting from a didactic perspective.

Many authors (De Lange, 1996; Van Reeuwijk, 1997) point out that contextualization is an important issue regarding mathematical work in the classroom. Obviously, the case of proportionality is not an exception. Thus, the knowledge of those situations in

which proportionality has historically appeared can be useful in order to design and develop well-contextualized teaching and learning activities that might help the students to overcome their difficulties.

OBJECTIVE AND METHOD

In this paper, we perform an analysis of historical proportionality problems. In particular we are interested in studying the contexts in which these problems appeared. Our main goal will be to give a survey of the most important situations in which proportional reasoning arises in a natural way.

In order to do so, we will focus on the study of ancient texts following the steps of content analysis (Krippendorff, 1990).

It is difficult to make a selection of texts. First of all we must determine a temporal frame. We decide that the most recent work in our study would be the *Liber abaci*, edited twice during the thirteenth century, because it presents proportionality in a relatively closed form. Later medieval and renaissance texts on Arithmetic will not include any essentially new problems regarding proportionality [1].

On the other hand, since proportionality admits different conceptual approaches (Oller & Gairín, 2013), it is necessary to gather works coming from different traditions. Moreover, to ensure the representativeness of the sample, it seems interesting to search for texts that were *encyclopedic* in the sense that they tried to summarize the mathematical knowledge at the moment and place where they were written.

As a consequence of the previous considerations the following main sources were selected:

- 1) *Rhind papyrus*, ca. 1650 BC, from Egypt (Chace, 1979).
- 2) *Jiu zhang suan shu*, an ancient Chinese text commented in 263 by Liu Hui but containing older material (Kangshen, Crossley, & Lun, 1999).
- 3) *Lilavati*, written in 1150 by the Indian mathematician Bhāskara II (Patwardan, Naimpally, & Singh, 2001).

- 4) *Liber abaci*, written in 1202 by Leonardo of Pisa, *Fibonacci* (Sigler, 2002).

Quite often, ancient texts only present the statement of the problems or, even if they provide a solution, it is only of numerical nature. In these cases we do not have a clue about the actual solving method. Thus, if we want to avoid the mistakes [2] pointed out by Grattan-Guinness (2004) we cannot classify the problems according to their (modern) method of resolution. Instead, we decide to classify the problems according to the particular context and situation where they appear. This criterion is coherent both with the practical nature of proportional reasoning and with the didactical usefulness of the study.

CONTEXTS WHERE PROPORTIONALITY PROBLEMS ARISE

The practical and pedagogical importance of proportionality is made clear by the great amount of related problems that appear in the consulted texts. In the *Jiu zhang suan shu*, for instance, three out of the nine chapters are solely devoted to this topic. Also, in Fibonacci's text we find a chapter exclusively devoted to companies presenting thirteen problems with their detailed solutions.

This huge number of problems also represent a variety of problems ranging from distribution of food to purely mathematical and abstract situations. Hence, it is difficult to give an exhaustive and detailed classification of proportionality problems according to their contexts. Nevertheless, we have identified four categories that include most of the problems:

- 1) Exchange problems.
 - a) Exchange of merchandise (barter).
 - b) Buying and selling.
 - c) Exchange of currency.
- 2) Distribution problems.
- 3) Loan problems.
- 4) Mixing problems.

EXCHANGE PROBLEMS

It is quite possible that the exchange and distribution of merchandise were the situations where proportional reasoning firstly appeared. In fact, texts like the *Jiu zhang suan shu* or the *Liber Abaci* use barter situations as an introduction to this topic. In cultures where currency existed buying and selling made sense but these situations can be seen as an Exchange of merchandise for Money. Finally, when different cultures, towns or countries interacted, the need to compare and interchange their respective currencies arose. We also consider these cases as exchange problems.

Exchange of merchandise

In order to perform an interchange, some criterion is needed to guarantee its fairness. Throughout our study we have found different criteria that we now exemplify.

- 1) Sometimes a table or some additional information giving exchange rates for different merchandise is given. That is the case, for instance, in the *Jiu zhang suan shu*, where a table indicating the exchange rates for different cereals is presented and used to solve problems such as the next one (Kangshen et al., 1999, p. 146):

“Now 5 *dou* $\frac{2}{3}$ *sheng* of millet is required as sesame seed. Tell: How much is obtained?”

- 2) In other cases, the monetary value of both merchandises is known and this information is used to deduce the fair way to interchange them. These kind of situations appear, for instance, in the *Liber abaci*, where we find problems like the following (Sigler, 2002, p. 181):

“A hundredpound of pepper is worth 13 pounds, and a hundredweight of cinnamon is worth 3 pounds; it is sought how many rolls of cinnamon are had for 342 pounds of pepper.”

- 3) Finally, we can find a unit that measures, in some sense, the quality of a certain good. This happens remarkably in the *Rhind papyrus*, where a unit called *pefsu* (Robbins & Shute, 1987, p. 51) measures the lack of quality. This gives rise to problems like this (Chace, 1986, p. 108):

Example of exchanging loaves for other loaves. Suppose it is said to thee, 100 loaves of *pefsu* 10 are to be exchanged for a number of loaves of *pefsu* 45. How many of these will there be?

Buying and selling

Buying and selling processes can give rise to several different problematic situations, but the two main possibilities that we have found are:

- 1) The price of a certain quantity of merchandise is known and the price of some other quantity is sought. For instance (Sigler, 2002, p. 134):

“Also a hundredweight of some merchandise is worth 14 pounds and 7 soldi; how much are 37 rolls of the same merchandise worth?”

- 2) The *inverse* situation; i.e., the price of a certain quantity of merchandise is known and the quantity of merchandise that can be bought with other quantity of money is sought. For instance (Patwardan et al., 2001, p. 78):

“If $\frac{3}{2}$ *pala* of saffron costs $\frac{3}{7}$ *niskas* O you expert bussinessman, tell me quickly what quantity of saffron can be bought for 9 *niskas*.”

Exchange of currency

The exchange of currency is a relatively recent commercial phenomenon. Among the revised texts, only the *Liber abaci* presents these problems. This is not surprising considering the historical and geographical context in which it was written. In fact, these problems will be popular in renaissance arithmetics due to the active commerce around the Mediterranean Sea.

Just like in the case of barter, there can be different pieces of information available at the moment of the exchange. The main possibilities are:

- 1) In the simplest case, the amount of one currency that is obtained in exchange of a certain quantity of the other is known (Sigler, 2002, p. 157):

“Also a Genoese soldo is sold for $2\frac{1}{2}$ Pisan denari, and it is sought how much 7 Genoese soldi and 5 denari are worth.”

- 2) In some cases a third currency is involved that acts as a measure for the value of the two exchanged currencies (Sigler, 2002, p. 186):

“Also it is proposed that one Imperial soldo is worth 31 Pisan denari, and one Genoese soldo is worth 22 Pisan denari, and it is sought how many genoese denari 7 Imperial denari are worth.”

- 3) The previous situation can be generalized so more than one currency acts as mediator (Sigler, 2002, p. 195):

“Twelve Imperial denari are worth 31 Pisan denari, and one Genoese soldo is worth 23 Pisan denari; and one Turin soldo is worth 13 Genoese denari, and one Barcelona soldo is worth 11 Turin denari; it is sought how many Barcelona denari are 15 Imperial denari worth.”

- 4) Finally, the exchanged can be performed according to the actual value of the silver (for instance) contained in each coin (Sigler, 2002, p. 199):

“Again one indeed has $26/3$ pounds of some common coin that has $2\frac{1}{4}$ ounces of silver, an one pound of silver is worth $149/20$ Pisan pounds, and it is sought how many Pisan pounds will be had for the $26/3$ pound of the coin.”

DISTRIBUTION PROBLEMS

As we already pointed out, distribution situations are among the oldest situations where proportional reasoning arises. *Rhind papyrus* already presents this type of problems.

The “fair” way to perform the distribution is often implicitly assumed and the context determines if the distribution must be direct or inverse. We now give some examples.

- 1) Sometimes there is a benefit to distribute, coming from either work or investment. In these cases the distribution is made directly proportional to the individual contributions (Chace, 1979, p. 104):

“Suppose a scribe says to thee, four overseers have drawn 100 great quadruple *hekat* of grain, their gangs consisting, respectively, of 12, 8, 6 and 4 men. How much does each overseer receive?”

- 2) In other cases, there is a loss to be distributed (like a tax payment, for instance) and the distribution is inversely proportional to the “rank” of the payers (Kangshen et al., 1999, p. 166):

“Now given five officials of different ranks: *Dafu*, *Bugeng*, *Zanniao*, *Shangzao* and *Gongshi*. They should pay a total of 100 coins. If the payment is to be shared in accordance with ranks, the higher pays the less, and the lower pays the more. Tell: how much should each pay?”

- 3) Finally, there can be several factors involved in the distribution which will be made directly proportional to some of them and inversely proportional to the others (Kangshen et al., 1999, p. 315):

“Now given a problem of fair distribution of tax millet. The tax bureau is at County A, which has 20520 households and where millet costs 20 coins a *hu*. County B, 200 *li* away from the bureau, has 12313 households and millet costs 10 coins a *hu* there. [...] County E, 150 *li* away, has 5130 households, and millet costs 13 coins a *hu*. The total tax millet for the five counties is 10000 *hu*. A cart carries 25 *hu*; the transport cost is 1 coin a *li*. Assume the payment by each household is equal in cash and labour. Tell: how much millet should each county pay?”

These distribution problems will continue to appear consistently over time, leading to the so-called problems of companies that received much attention from thirteenth century on (Lamassé, 2001).

Other interesting and important situation where distribution problems can be found is that of inheritance [2]. This is particularly the case in Muslim tradition because of the strict and somewhat complicated rules determined by the Islam. In fact, the whole third chapter of al-Khwarizmi’s *Algebra* (Rosen, 1986) is devoted to inheritance problems. A typical example is given by the following problem coming from the twelfth century Andalusian mathematician al-Hufi (Laabid, 2001, p. 321):

“Une femme a laissé après sa mort, son mari, sa mère, sa sœur germain, sa sœur consanguine, sa sœur utérine. Et une succession constituée d’un esclave et de 15 dinars. La sœur germaine a pris pour sa quote-part l’esclave et a remboursé pour

les autres héritiers 5 dinars. (Il s'agit de calculer le montant global de la succession et la valeur de l'esclave, la quote-part de chaque héritier.)”

This type of problems do not appear in the sources used for this work and, strictly speaking, they could be considered as an example of benefit distribution problems. Nevertheless, their context is specific enough to study them separately.

LOAN PROBLEMS

In spite of the bad reputation that the loan of money at a certain interest had in several cultures, it seems that this was a quite ancient and common practice. Moreover, simple interest is still a topic in textbooks; although this is more due to tradition than to its practical use (compound interest is far more common).

The problems that we have found in this context are very similar, if not identical, to problems that appear nowadays. For example (Patwardan et al., 2001, p. 84):

“If the interest on 100 for $\frac{4}{3}$ months is $\frac{26}{5}$, what will be the interest on $\frac{125}{2}$ for $\frac{16}{5}$ months?”

In most ancient problems the reader is asked to find the interest obtained by a certain amount of money after a certain time. In some cases the unknown quantity corresponds to other magnitude. The following example from the *Liber abaci* shows such a situation, although it is not strictly a proportionality problem since it involves an affine function (Sigler, 2002, p. 384):

“A certain man placed 100 pounds at a certain house for IIII denari per pound per month of interest, and he took back in each year a payment of 30 pounds; one must compute in each year the 30 pounds reduction of capital and the profit on the said 100 pounds. It is sought how many years, months, days and hours he will hold money in the house.”

MIXING PROBLEMS

The idea of mixing products of different qualities and prices naturally arises when working with raw materials. In this way one can obtain a product of a predetermined price or improve or worsen its quality.

A very particular case is that of alloys. These problems were important in Europe during the thirteenth century (Williams, 1995). In fact, chapter 11 from the *Liber abaci* is entirely devoted to this topic (Sigler, 2002, pp. 227–257).

Again, a huge variety of different situations arise in this context. Nevertheless there are essentially two types of problems.

- 1) Knowing the weight and quality of the ingredients, the weight and quality of the resulting mixture is sought (Patwardan et al., 2001, p. 98):

“O golden mathematician, four types of gold 10 *masas* of 13 carats, 4 *masas* of 12 carats, 2 *masas* of 11 carats and 4 *masas* of 10 carats are melted together to form a new one. Find its fineness. Is this is purified and 16 *masas* gold is obtained, what is its fineness? If the mixed gold when purified has 16 carats fineness, what is its weight?”

- 2) Inversely, knowing the weight and quality of the mixture some information about the ingredient is required (Sigler, 2002, p. 255):

“A certain man wishes to make a bell with five metals, of which a hundredweight of one metal is worth 16 pounds, another truly 18 pounds, an another 20 pounds, another truly 27 pounds, and another indeed 31 pounds; he therefore makes a bell from them that weights 775 rolls and costs $\frac{651}{4}$ pounds; it is sought how much he puts in of each metal.”

SOME DIDACTIC IMPLICATIONS

As we already pointed out, the previous study might be useful in order to design teaching and learning activities involving proportionality and, in particular, to decide which situations can promote that students significantly construct the main concepts regarding proportionality.

From this point of view, we think that the situations that are more adequate to introduce the basic techniques of proportionality in the classroom are exchange problems. This choice is not only motivated by the historical importance of those situations but also by the fact that students are very familiar to them.

Exchange problems are particularly suitable to introduce outer ratios and what Cramer and Post (1993) call unit rate strategies to solve missing-value problems. In addition, these situations can be used to present the idea of co-variation that may lead to a functional view of proportionality (Bosch, 1994).

Distribution problems are as old as exchange problems and they can also lead to meaningful activities that help to present important aspects of proportionality.

For example, distribution situations are suitable to work at the same time with inner and outer ratios and discuss their relations and different meanings. Moreover, the idea of fair distribution can help to clarify the concept of proportional magnitudes and a distribution situation is a natural context in which inverse proportionality appears.

We find loan and mixing problems less useful, mainly because students are less familiar to them. Nevertheless, they are also interesting and can also be used under certain circumstances. For instance, mixture problems can be a nice context in which progressively start the difficult transition (Wagner & Kieran, 1989) between arithmetic and algebra.

REFERENCES

- Bosch, M. (1994). *La dimensión ostensiva en la actividad matemática: El caso de la proporcionalidad* (Unpublished doctoral dissertation). Barcelona, Spain: Universitat Autònoma de Barcelona.
- Chace, A. B. (1979). *The Rhind mathematical papyrus*. Reston, VA: National Council of Teachers of Mathematics.
- Cramer, K., & Post, T. (1993). Connecting Research to Teaching Proportional Reasoning. *Mathematics Teacher*, 86(5), 404–407.
- De Lange, J. (1996). Using and applying mathematics in education. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education* (pp. 49–98). Dordrecht, The Netherlands: Kluwer.
- Ernest, P. (1989). The impact of beliefs on the teaching of mathematics. In P. Ernest (Ed.), *Mathematics teaching: The state of the art* (pp. 249–254). London, UK: Falmer Press.
- Fauvel, J., & Van Maanen, J. (Eds.). (2000). *History in mathematics education*. Dordrecht, The Netherlands: Kluwer.
- Freudenthal, H. (1983). *Didactical phenomenology of mathematical structures*. Dordrecht, The Netherlands: D. Reidel Publishing Company.
- Grattan-Guinness, I. (2004). The mathematics of the past: distinguishing its history from our heritage. *Historia Mathematica*, 31(2), 163–185.
- Kangshen, S., Crossley, J. N., & Lun, A. W.-C. (1999). *The nine chapters on the mathematical art*. Beijing, China: Oxford University Press.
- Katz, V., & Tzanakis, K. (Eds.) (2011). *Recent developments on introducing a historical dimension in Mathematics Education*. Washington, D. C.: MAA.
- Krippendorff, K. (1990). *Metodología de análisis de contenido. Teoría y práctica*. Barcelona, Spain: Paidós.
- Laabid, E. (2001). Le partage proportionnel dans la tradition mathématique maghrébine. In *Commerce et mathématiques du Moyen Age à la Renaissance, autour de la Méditerranée. Actes du Colloque international du Centre International d'Histoire des Sciences Occitanes* (pp. 315–326). Toulouse, France: Editions du C.I.H.S.O.
- Lamassé, S. (2001). Les exercices de compagnie entre commerce et mathématiques. In *Commerce et mathématiques du Moyen Age à la Renaissance, autour de la Méditerranée. Actes du Colloque international du Centre International d'Histoire des Sciences Occitanes* (pp. 285–300). Toulouse, France: Editions du C.I.H.S.O.
- Modestou, M., Elia, I., Gagatsis, A., & Spanoudis, G. (2008). Behind the scenes of pseudo-proportionality. *International Journal of Mathematics Education in Science and Technology*, 39(3), 313–324.
- Oller, A. M., & Gairín, J. M. (2013). La génesis histórica de los conceptos de razón y proporción y su posterior aritmetización. *Revista Latinoamericana de Investigación en Matemática Educativa*, 16(3), 317–338.
- Patwardan, K. S., Naimpally, S. A., & Singh, A. L. (2001). *Lilavati of Bhaskaracarya. A treatise of mathematics of vedic tradition*. Delhi, India: Motilal Banarsidass Publishers.
- Robins, G., & Shute, C. (1987). *The Rhind mathematical papyrus. An ancient Egyptian text*. London, UK: British Museum Publications.
- Rosen, F. (1986). *The algebra of Mohammed ben Musa*. Hildesheim-Zürich-New York: Georg Olms Verlag.
- Sigler, L.E. (2002). *Fibonacci's Liber abaci. A translation into modern English of Leonardo Pisano's Book of Calculation*. New York, NY: Springer Verlag.
- Stanic, G., & Kilpatrick, J. (1989). Historical perspectives on problem solving in the mathematics curriculum. In R. Charles & E. Silver (Eds.), *The teaching and assessing of mathematical problem solving* (pp. 1–22). Reston, VA: National Council of Teachers of Mathematics.
- Tzanakis, C., & Arcavi, A. (2000). Integrating history of mathematics in the classroom: an analytic survey. In J. Fauvel & J. Van Maanen (Eds.), *History in mathematics education* (pp. 201–240). Dordrecht, The Netherlands: Kluwer.

- Van Dooren, W., De Bock, D., Janssens, D., & Verschaffel, L. (2008). The linear imperative: An inventory and conceptual analysis of students' over-use of linearity. *Journal for Research in Mathematics Education*, 39(3), 311–342.
- Van Reewnijk, M. (1997). Las matemáticas en la vida cotidiana y la vida cotidiana en las matemáticas. *UNO. Revista de didáctica de las matemáticas*, 12, 9–16.
- Wagner, S. & Kieran, C. (Eds.). (1989). *Research issues in the Learning and Teaching of Algebra*. Reston, VA: National Council of Teachers of Mathematics.
- Williams, J. (1995). Mathematics and the alloying of coinage 1202–1700: Part I. *Annals of Science*, 52(3), 213–234.
- Wussing, H. (1998). *Lecciones de historia de las matemáticas*. Madrid, Spain: Siglo XXI.

ENDNOTES

1. We must point out that we focus on arithmetic proportionality. Of course, we find proportionality in several geometric problems of practical nature and the theoretical frame can be traced back to Euclid and before. Nevertheless, we think that arithmetic situations are more useful in order to introduce proportionality in the classroom.

2. Namely, the confusion of history and heritage. History describes what happened (and did not happen) in the past and tries to explain it, while heritage focus on the question “how did we get here?” In our case, it would be a mistake to classify problems as ‘direct proportion problems’, ‘inverse proportion problems’, etc. since these are modern categories.