Why is it difficult to learn from history?

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This paper is of a methodological nature. First, it aims at spelling out several structural differences which may stand in the way when researchers in mathematics education endeavor to derive didactically relevant information from the history of mathematics as written by today’s historians. Second, the core of the paper aims at illustrating what historians actually do, with a methodological focus on the notions of “agency” and “puzzle-solving”.

Keywords: History, epistemology, methodology, Euclid, Descartes.

In a paper of 1990 entitled Epistémologie et didactique (Artigue, 1990), Michèle Artigue reflected on 10 years of practice within the French mathematics education community, while stressing the need of epistemology for the working researcher. First, she underlined the need for epistemological awareness as an experience for the researcher, enabling him to distance himself from his personal mathematical culture; second, she pointed out that some knowledge of the history of mathematics was of a key component of didactical research, either to understand the historical development of some mathematical concept, or to understand the shaping of mathematics as a ruled cultural activity.

This paper will, to a large extent, directly echo this Artigue paper, and is hence somewhat dependent on the French context [1]. For these reasons, we will here use the adjective “epistemological” to denote the endeavor to derive from knowledge/awareness of the history of mathematics some insight that is relevant from a mathematics education research perspective. As Artigue did, we will focus on issues of method, although with a shift of emphasis. Instead of focusing directly on didactical concepts, we will mainly discuss research practices at the intersection of two autonomous fields of knowledge: math education research on the one hand and history of mathematics on the other hand; in this context, “history of mathematics” will denote the outcome of the work of historians of mathematics.

After spelling out some basic structural differences (and some similarities) between the two fields of inquiry, we wish to present several examples in order to illustrate what historians actually do. These examples will enable us to highlight two methodological aspects: the focus on agency (how actors engage with mathematics), and the riddle-solving (or puzzle-solving) aspect of historical research (which we would like to contrast against the “erudite description” view some may have of historical research). We wish to provide a basis for further methodological discussion, and for ever more fruitful interactions.

STRUCTURAL DIFFERENCES BETWEEN TWO FIELDS OF RESEARCH

RME (Research/researchers in mathematics education) and HM (history/historians of mathematics) are two different and autonomous disciplines: each one has its own empirical field of investigation, its own set of legitimate questions, its own way of validating claims, its own reference works etc. That fact may be self-evident; however, we feel this fact should be taken into account seriously in order to pave the way for fruitful collaborations. It is also a fact that RME and HM have often been speaking at cross purposes: when HM read what RME say about the history of mathematics, the typical reaction goes: “this is not history, but a sketchy reconstruction of history framed within a-historical categories; what really happened is really much more complicated than that, you know ...”; which RME are usually fully willing to acknowledge while wondering why historians would deny them the right to make heuristic use of HM in a preliminary phase to their main investigation. For them, learning about history (which one of the things historians do) is a means to learn something from history (which is not what historians do). Reciprocally, RME are sometimes surprised by the lack of theoretical frameworks in the
work of historians, since such frameworks provide the main tools to describe and analyze specific issues; and enable researchers to integrate their particular study to a growing and soundly-structured body of knowledge about the learning of mathematics in current educational contexts. Historians do not usually rely on explicit theoretical frameworks.

The purpose of this paper is not to claim that these usual misunderstandings are only the result of the relative isolation of the two communities, and that they would soon fade away if everyone decided to work together with an open mind. Quite the contrary, we think these misunderstandings point to differences which are structural, and our purpose is to sketch ways of living with this fact.

Since the intended audience of this paper is that of RME, we would like to point to some elements which show what historians actually do. Of course, our approach is descriptive and not normative.

REM and HM have at least this in common: contrary to what research mathematicians do, the object of their investigation is not mathematics, and this object is not studied primarily mathematically. Rather, they study how agents engage with mathematics, in a context which can be described; mathematics is necessary to make sense of this engagement and this context, but cannot possibly be the only background tool.

Beyond this common agent-based approach, dissimilarities become striking: MER study learners, HM tend to focus on experts (of course, both parts of this statement call for qualification). MER has direct access to the living agents it studies, which means empirical data can be gathered, hypotheses can be put to the test in finely-tuned conditions, and cognitive processes can be investigated; HM have indirect access to the agents they study, and it is part of the trade to attempt to assess what biases it entails (critique of sources, careful methodological reflection on corpus delineation etc.). HM have to deal with events which happened once, but can be understood, compared and, to some extent, fit into narratives; MER has an experimental side to it, and can aim for invariants and reproducibility.

The fact that historians depend heavily on the availability of sources and do not explicitly rely on theoretical frameworks does not imply that their work is purely descriptive and erudite. To use Kuhn's phrase, a historians solve puzzles, just as any researcher does, whatever the field. We would like to illustrate this agent-based, puzzle-solving approach from three different angles.

A SHARED INTEREST IN MATHEMATICAL AGENCY

First, let us mention the kind of questions that historians aim at tackling. A very general and context-free list of questions can be found, for instance, in Catherine Goldstein’s methodological paper (Goldstein, 1999, p. 187, trans. RC):

At a given period in time, what were the networks, the social groups, the institutions, the organizations where people practiced mathematics or engaged with mathematics? Who were mathematicians? In what conditions did they live; in what conditions did they carry out mathematical work? How were they educated and trained? What did they learn?

Why did they work in mathematics, in what preferred domain? What did this domain mean to them? (...) Where did mathematicians find problems to be solved? What were the form and origins of these problems? Why was some result considered as very important, or of lesser importance? According to which criteria? What was considered to be a solution to a problem? What had to be proven, and what did not require a proof (tacitly or explicitly)? Who decided so? When was a proof accepted or rejected? When was an explicit construction deemed indispensable, optional or altogether irrelevant?

When, where and how mathematics were written? Who wrote, and for whom? For instance, were new results taught, were they printed, were they applied? What got transmitted? To whom was it transmitted, in which material and intellectual conditions?

What changed and what remained fixed (and according to what scale, to which criteria)?

The variety of structural differences between history and didactics does not imply that no questions may be shared, in particular when one focuses on agency.
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For instance, Goldstein’s list strikingly echoes the list of questions which Guy Brousseau considered to be meaningful for RME when attempting to derive didactically-relevant insight from a study of mathematics from the past. Discussing Georges Glaeser’s paper on the epistemological obstacles relative to negative numbers (Glaeser, 1981), he critically summed up Glaeser’s “obstacle” approach, and then pointed to what he would consider to be the more relevant questions (as cited in Artigue, 1990, p. 252, trans. RC):

This formulation shows what failed Diophantus or Stevin, seen from our time and our current system. We thus spot some knowledge or possibility which failed 16th century authors and prevented them from giving the “right” solution or the proper formulation. But this formulation hides the necessity to understand by what means people tackled the problems which would have required the handling of isolated negative quantities. Were such problems investigated? How were they solved? (…) What we now see as a difficulty, how was it considered at the time? Why did this “state of knowledge” seem adequate; relative to what set of questions was it reasonably efficient? What were the advantages of this “refusal” to handle isolated negative quantities, or what drawbacks did it help avoid? Was this state stable? Why were the attempts at changing it doomed to fail, at that time? Maybe until some new conditions emerge and, some “side” work be done, but which one? These questions are necessary for an in-depth understanding of the construction of knowledge [pour entrer dans l’intimité de la construction de la connaissance] (…).

In both list, we can see that a focus on agency does not mean that the object of study is a freely creative cognitive agent. Quite the contrary. Agents are born in a world which pre-exists, and constrains their actions. When it comes to mathematical activity, constraints come from a great variety of sources, ranging from the material environment (a Chinese abacus is not an electronic calculator) to epistemic values (rigor, generality, simplicity, accuracy, applicability, etc.) and epistemic categories (definition, justification, proof, example, algorithm, analysis/synthesis, principle, etc.). The historical contingency of these constraints does not imply that they have no a-historical components, be they mathematical properties (a rule such as “minus times minus equals minus” is not compatible with distributivity of × over +) or semiotic properties (an algebraic shorthand with no parentheses – such as Cardano’s – has different properties from Bombelli’s). Making historical sense of how actors engage with mathematics involves understanding how they act within a given set of constraint, what meaning they give to their actions, and in what respect these actions alter the system of constraints.

FOCUSING ON AGENCY: USES OF A DIAGRAM

Let us now flesh out this notion of agent-based approach – this focus on mathematical agency – from another angle. We will use the example of this diagram to illustrate several methodological points.

The very same diagram (Figure 1) appears in two of the most influential works in the history of mathematics: Euclid’s Elements and Descartes’ La Géométrie. One could argue that not only the diagram is the same, but the mathematical content is the same; however, the parts these diagrams play in both works are strikingly different.

In Euclid’s Elements (Heath, 1908), this diagram comes with proposition 14 of book II; a proposition which solves the following construction problem: to construct a square equal (in area) to a given rectangle. If the sides of the rectangle are equal (in length) to FG and GH, then the perpendicular IG is the side of the sought-for square, which Euclid proves using proposition 47 of book I (which we call Pythagoras’ theorem). At the end of book I, a series of proposition established that, for any given polygon, a rectangle with the same area could be constructed (with ruler and compass only), hence prop. 14 provides the final positive solution to the problem of quadrature of polygons (i.e. to transform any polygonal area into a square). In turn, this fact implies that – at least for polygons – area is a well behaved magnitude: areas can be compared (since square areas can), and added (since the Pythagorean construction provides a means
to add square areas). On this basis, a modern reader would conclude that a theory of measure in possible for polygonal areas; the modern reader also knows that this requires the set of real numbers. Euclid was very well aware of the fact that the theory of well-behaved geometrical magnitude (even line-segments, for which comparison and addition are straightforward) requires more than natural numbers and their ratios. The solution he presented in book V is a number-free solution, based on the notion of ratio of magnitudes and not of measure. The positive result of II.14 also points to open questions in the theory of magnitudes, in particular the extension of the theory beyond the case of polygons (the case of the circle being of prime importance).

The same diagram (Figure 1) appears on the second page of Descartes’ *La Geométrie* (1637). Along with another diagram (Figure 2), he aims at defining operations on segments; operations for which he would use the same names as for arithmetical operations. In Figure 2, if AB denotes a unit segment, then BE will be called the “product” of segments BC and BD. In Figure 1, if FG is the unit segment, then IG will be called the “square root” of GH. Descartes then adds that he would not only use the same names as those of arithmetical operations, but that he would also resort to the same signs as in algebra: letters for segments (known or unknown), symbols such as √ and × for the above mentioned constructions. The project is to use the means of algebra (rewriting rules, elimination in simultaneous equations, identification in polynomial equalities, method of indeterminate coefficients) to capture and analyze geometrical relations between segments; among such relations, those expressed by one equation in two unknowns capture plane curves. This specific Cartesian project is quite different both from Euclid’s, and from what we call either algebra or coordinate geometry. In the *Elements*, prop. II.14 solved an area problem; in terms of magnitudes, considering two line-segments could lead either to a new segment (by concatenation, which can be seen as a form of addition), or to an area (that of a rectangle, which can be seen as a form of multiplication), or to a ratio (which is not a geometrical entity, but not a number either). On the contrary, Descartes uses elementary construction (with an *ad hoc* unit segment) to define operations such as “time”, “divide” or “root” as internal operations within the domain of segments; this enables him to make free use of algebraic symbolism while warranting geometrical interpretability. This system, however, involves no global coordinate system; it does not even involve coordinates, since no numbers play any explicit part in the system.

The fact that Descartes’ system is an algebra of segments has other far-reaching consequences. Let us mention one of general epistemological importance. At first, when we read in *La Géometrie* that the solution of equation \( z^2 = az + b^2 \) (\( z \) being unknown, \( a \) and \( b \) known) can be expressed by formula:

\[
  z = \frac{1}{2} a \pm \sqrt{\frac{1}{4} a^2 + b^2},
\]

we feel we are on pretty familiar ground. However, we need to recall that this formula is not a symbolic summary for a list of arithmetical operation on numbers, but is a symbolic summary for a geometrical construction program; a ruler-and-compass construction program which involves two concatenations, two multiplications (Figure 2), the construction of a “square-root” segment (Figure 1), and three midpoints. This, in turn, means that the algebraic manipulation of formulae and equations deals with the transformation and comparison of geometric construction programs. Here, the comparison with Al-Khwarizmi (ca 820 CE) is striking:

Roots plus numbers equal squares; for instance, when you say: three roots and four in numbers equal one square.

Procedure: halve the number of roots, you get one and one half; multiply it by itself, you get two and one quarter; add four, you get six and one quarter; take its root, which is two and one half, add it to half the number of roots, that is one and one half, you get four, which is the root of the square, and the square is sixteen. (Rashed, 2007, p. 106. Trans. RC)
With its purely rhetorical algebra and its use of generic examples, this excerpt from Al-Khwarizmi’s *Algebra* may look less familiar than Descartes’ formula. However, it presents a *bona fide* list of operations which enables one to solve an equation, in a numerical context. In the rhetorical context, algorithms are easy to express, but not so easy to compare, transform and calculate upon. One of the properties of Descartes’ system is that its symbolic algebra allows for calculation to operate on algorithms; the fact that the basic steps of the algorithms involved are ruler and compass constructions and not numerical operations is irrelevant, and testifies to the meta-level function of symbolic algebra.

This interplay between the familiar and the not-so-familiar (yet understandable) may feel disorienting at first, but this disorienting effect is a positive effect, as Artigue stressed. It has a *critical* function, helping the researcher to distance himself from his own mathematical culture ; and a *heuristic* function, suggesting new viewpoints on seemingly familiar notions, for instance on the role of symbolism in algebra, or the role of real numbers in geometry (as measures and as coordinates). At least two other functions can be mentioned. First, it helps identify *problems* to which there are no straightforward answers. For instance, what should we consider to be the geometrical analogue of multiplication, at least for one-dimensional objects? In particular, should the analogue of the product be one-dimensional or two-dimensional? A long series of different – yet mathematically sound – constructions provide different answers to this question, including dimension changing solutions (going down with the dot product, or up with the exterior product). Secondly, it helps question the notion of *identity*. It could be argued that, from a purely mathematical point of view, Euclid and Descartes rely on the same content associated to Figure 1; this probably makes sense, but it is probably not very helpful, either to the historian of to the RME. Indeed, researchers in both fields aim at analyzing how content depends on – for instance – semiotic resources, or intended use.

To finish with this Euclid-Descartes example, we would like to make plain what it took to come up with such an example. On the one-hand it is relatively small scale: we did not need to include it in any large-scale narrative on the “stages” in the history of geometry for this sketchy comparison to serve the four above listed functions of epistemological inquiry; on an even smaller scale, the comparison with a short passage of Al-Khwarizmi could play a relevant part even with no background “big picture” on the history of algebra, or even on the Kitāb al-Jabr. On the other hand, to compare the uses of the same diagram required that its role in the whole structure of the works (the *Elements*, and *La Géométrie*) be analyzed. It requires some knowledge of history to make sense of highly sophisticated but largely forgotten theoretical constructs such as the classical theory of ratios, or the 17th century research program of construction of equations. This knowledge cannot derive from a quick look at short extracts from the original sources, and probably not even from a long look at the whole books; here we depend on professional historians such as (Netz, 1999) for Euclid, and (Bos, 2001) for Descartes.

**SOLVING PUZZLES, DESIGNING RESEARCH-QUESTIONS**

Finally, we would like to illustrate the fact that erudite analysis is not all there is to historical research. Finding answers and grounding answers through erudite analysis of documents come only in a second phase; in a first phase, historians strive to identify challenges, and craft non-trivial (and possibly innovative) questions. Let us give five pretty different examples.

In (Proust, 2012), Christine Proust studies the algorithm displayed in paleo-Babylonian tablets when working out the reciprocals of large numbers in the sexagesimal system. The clay-tablets display instances of calculations, but no general descriptions of the method (much less any justifications), which is why historians endeavor to come up with reconstructions of the algorithm. Pioneer in the history of Babylonian mathematics Otto Neugebauer (1899–1990) reconstructed an algorithm on the basis few tablets; an algorithm which required that additions be used along the way. However, in the *floating point* sexagesimal number system, and in the purely numerical context of these tablets, addition is not possible (whereas products and reciprocals make perfect sense)! On the basis of a much larger sample of tablets, Proust reconstructed a different algorithm; one which is fully compatible with a floating point arithmetic.

In his now classic *The Shaping of Deduction in Greek Mathematics* (Netz, 1999), Reviel Netz attempted to re-historicize the endeavor of the Greek mathema-
The question of the circulation of mathematics between different cultural areas— and not only different periods— is also a central field of investigation. In (Chemla, 1996), Karine Chemla discussed the introduction of “western” mathematics in 17th century China by Jesuit missionaries. It was usually thought that, in this period, the indigenous Chinese tradition of mathematics was to a large extent forgotten in China, and that western mathematics had been adopted passively. Actually, studying the Chinese sources leads to a more nuanced picture. In particular, when Jesuit Matteo Ricci and Chinese scholar Li Zhizao collaborated to write a treatise of arithmetic based on Clavius, they ended up with much more than a translation: Li added many elements from the indigenous tradition, in particular the fangcheng algorithm to solve simultaneous linear equations (similar to Gaussian elimination). This work of synthesis did not stir interest in the West; in China however, the introduction of western mathematics revived scholarly interest in classical Chinese mathematics, and triggered comparative studies of both traditions.

It is well-known that for the founders of the calculus the prime goal was the study of curves defined by ordinary differential equations, in a geometrical context. Pen and paper, and formulaic solutions were not all there was to it, as is demonstrated by the deep and original work of Dominique Tournès. In (Tournès, 2003), he studied the intense work on graphical methods and graphing devices carried out from the very beginning (Leibniz, Newton, Jean Bernoulli, Euler), up until the advent of digital instruments in the second half of the 20th century. This work brings to light a great wealth of largely forgotten mathematical ideas and techniques; shows the continuity between the algebraic research program on the “construction of equations” (as in Descartes) and the late-17th and 18th century researches on ODEs; and documents the deep connections between the most theoretical considerations on the one hand, and the demand for effective approximation methods in the engineering communities on the other hand.

The distinction between “local” and “global” is now standard in the scholarly mathematical world, but it was not always the case. Studying the emergence of an explicit local-global articulation is tricky for a number of reasons: it concerns more or less all mathematics; the meta level terms “local” and “global” have definitions which differ in every specific mathematical context; actually they can be used with no definitions at all. Moreover, the question of the explicit is crucial. When, at the turn of the 20th century, some mathematicians began to explicitly express a such distinction, was the general context one in which it was actually clear to everyone that this mattered (though it went without saying), or one in which no clear distinction was made between local and global statements, resulting in a wealth of faulty proofs and ambiguously-worded theorems? These questions were addressed in (Chorlay, 2011), who provided answers based on a combination of quantitative and qualitative methods.

CONCLUSION

This short list of examples illustrates how historians endeavor to design non-trivial questions; what means they use to answer these questions; and what kind of answers they tend to consider relevant and innovative. Although we feel historians provide a great wealth of material of prime interest for mathematics education research, it is a fact that they do not usually provide this material in a form which directly meets the needs or wishes of the mathematics education research community.

To further this paper, we can identify at least three avenues for research: (1) to analyze papers in mathematics education research which depend heavily on historical analysis, such as (Sierpinska, 1985), (Katz, 2007) or (Dorier, 2000); (2) to discuss the relevance of standard conceptual tools such as “epistemological
obstacle”, “(mis)-conceptions”, “historical genesis”; (3) to review the main theoretical frameworks in mathematics education research in order to identify which (if any) role they assign to epistemological or historical investigation.

REFERENCES


ENDNOTE

1. An altered version of this paper makes up the first, introductory part of a chapter in a collective volume in the honour of M. Artigue (scientific editors: B. Hodgson, J.-B. Lagrange, A. Kuzniak).