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# HIGHER ORDER APPROXIMATION OF ANALYTIC SETS BY TOPOLOGICALLY EQUIVALENT ALGEBRAIC SETS

MARCIN BILSKI, KRZYSZTOF KURDYKA, ADAM PARUSIŃSKI,  
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ABSTRACT. It is known that every germ of an analytic set is homeomorphic to the germ of an algebraic set. In this paper we show that the homeomorphism can be chosen in such a way that the analytic and algebraic germs are tangent with any prescribed order of tangency. Moreover, the space of arcs contained in the algebraic germ approximates the space of arcs contained in the analytic one, in the sense that they are identical up to a prescribed truncation order.

## 1. INTRODUCTION

The problem of approximation of analytic sets (or functions) by algebraic ones is one of the most fundamental problems in singularity theory and an old subject of investigation (see e.g. [2], [4], [7], [15], [20], [24], [29]). In general this problem has been considered by two different approaches.

Firstly one may seek to approximate the germs of analytic sets by the germs of algebraic ones so that both objects are homeomorphic. For instance, it is well-known that a germ of analytic set with an isolated singularity is analytically equivalent to a germ algebraic set (see [13] for a general account of this case). But in general an analytic set is not even locally diffeomorphically equivalent to an algebraic one (cf. [29]). Nevertheless by a result of T. Mostowski [15] every analytic set is locally topologically equivalent to an algebraic one.

**Theorem 1.1** ([15], [4]). *Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . Let  $(X, 0) \subset (\mathbb{K}^n, 0)$  be an analytic germ. Then there is a homeomorphism  $h : (\mathbb{K}^n, 0) \rightarrow (\mathbb{K}^n, 0)$  such that  $h(X)$  is the germ of an algebraic subset of  $\mathbb{K}^n$ .*

Secondly one may seek to approximate analytic germs by the algebraic ones with the higher order tangency, cf. e.g. [5], [6], [9], [10], [11]. But the classical methods of such approximation do not provide the objects which are homeomorphic. In this paper we show how to construct approximations that satisfy both requirements that is approximate with homeomorphic objects and with a given order of tangency.

**Theorem 1.2.** *Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . Let  $(X, 0) \subset (\mathbb{K}^n, 0)$  be an analytic germ. Then there are  $c > 0$  and an open neighborhood  $U$  of 0 in  $\mathbb{K}^n$  such that for every  $m \in \mathbf{N}$  there are a subanalytic and arc-analytic homeomorphism  $\varphi_m : U \rightarrow \varphi_m(U) \subset \mathbb{K}^n$*

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and an algebraic subset  $V_m$  of  $\mathbb{K}^n$  with the following properties:

- (a)  $\varphi_m(X \cap U) = V_m \cap \varphi_m(U)$ ,
- (b)  $\|\varphi_m(a) - a\| \leq c^m \|a\|^m$  for every  $a \in U$ .

The proof of Theorem 1.2 is based on Mostowski's approach, its recent refinement [4], and a new result on Zariski equisingularity given in [16]. This proof will be divided into two parts. Firstly, using Popescu Approximation Theorem and a strong version of Varchenko's Theorem given in [16], we show that there is an arc-analytic homeomorphism  $\psi$  such that  $\psi(X)$  is a Nash set tangent to  $X$  at 0 with any prescribed order of tangency (cf. Proposition 3.4, Section 3). Next, using Artin-Mazur's Theorem of [3] (cf. also [7]), we prove that there is a Nash  $\mathbb{K}$ -analytic diffeomorphism  $\theta$  such that  $\theta(\psi(X))$  is an algebraic set tangent to  $\psi(X)$  (cf. Proposition 4.1, Section 4).

In Section 5 we give an application of Theorem 1.2 to the space of analytic arcs on a given germ of analytic set. Namely, we show that for the space of truncated arcs there is an algebraic germ, homeomorphic to the original one, with the identical space of truncated arcs. We shall use the following notation. For any analytic germ  $(Z, 0) \subset (\mathbb{K}^n, 0)$ , let  $\mathcal{A}_m^{\mathbb{R}}(Z)$  denote the space of all truncations up to order  $m$  of real analytic arcs contained in  $(Z, 0)$ . For a complex analytic germ  $(Z, 0)$ ,  $\mathcal{A}_m^{\mathbb{C}}(Z)$  denotes the space of all truncations up to order  $m$  of complex analytic arcs contained in  $(Z, 0)$ .

**Theorem 1.3.** *Let  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . Let  $(X, 0) \subset (\mathbb{K}^n, 0)$  be a germ of real (resp. complex) analytic set. Then for every  $m \in \mathbb{N}$ , there is an algebraic germ  $(V_m, 0) \subset (\mathbb{K}^n, 0)$  with the same embedded topological type as  $(X, 0)$  such that  $\mathcal{A}_m^{\mathbb{R}}(X) = \mathcal{A}_m^{\mathbb{R}}(V_m)$  if  $\mathbb{K} = \mathbb{R}$ , and  $\mathcal{A}_m^{\mathbb{C}}(X) = \mathcal{A}_m^{\mathbb{C}}(V_m)$ ,  $\mathcal{A}_m^{\mathbb{R}}(X) = \mathcal{A}_m^{\mathbb{R}}(V_m)$  if  $\mathbb{K} = \mathbb{C}$ .*

We also present an example showing that it is not enough in general to choose for  $V_m$  a germ of algebraic set which has a high tangency with  $X$ .

## 2. PRELIMINARIES

**2.1. Nash functions and sets.** Let  $\mathbb{K} = \mathbb{C}$  or  $\mathbb{R}$ . Let  $\Omega$  be an open subset of  $\mathbb{K}^q$  and let  $f$  be a  $\mathbb{K}$ -analytic function on  $\Omega$ . We say that  $f$  is a *Nash function* at  $\zeta \in \Omega$  if there exist an open neighbourhood  $U$  of  $\zeta$  in  $\Omega$  and a non zero polynomial  $P \in \mathbb{K}[Z_1, \dots, Z_q, W]$  such that  $P(z, f(z)) = 0$  for  $z \in U$ . A  $\mathbb{K}$ -analytic function on  $\Omega$  is a Nash function if it is a Nash function at every point of  $\Omega$ . A  $\mathbb{K}$ -analytic mapping  $\varphi : \Omega \rightarrow \mathbb{K}^N$  is a *Nash mapping* if each of its components is a Nash function on  $\Omega$ .

A subset  $X$  of  $\Omega$  is called a *Nash subset* of  $\Omega$  if for every  $\zeta \in \Omega$  there exist an open neighbourhood  $U$  of  $\zeta$  in  $\Omega$  and Nash functions  $f_1, \dots, f_s$  on  $U$ , such that  $X \cap U = \{z \in U ; f_1(z) = \dots = f_s(z) = 0\}$ . A germ  $X_\zeta$  of a set  $X$  at  $\zeta \in \Omega$  is a *Nash germ* if there exists an open neighbourhood  $U$  of  $\zeta$  in  $\Omega$  such that  $X \cap U$  is a Nash subset of  $U$ . Note that  $X_\zeta$  is a Nash germ if its defining ideal can be generated by convergent power series algebraic over the ring  $\mathbb{K}[Z_1, \dots, Z_q]$ . For more details on real and complex Nash functions and sets see [8], [25].

**2.2. Nested Artin-Płoski-Popescu Approximation Theorem.** We set  $x = (x_1, \dots, x_n)$  and  $y = (y_1, \dots, y_m)$ . The ring of convergent power series in  $x_1, \dots, x_n$  is denoted by  $\mathbb{K}\{x\}$ . If  $A$  is a commutative ring then the ring of algebraic power series with coefficients in  $A$  is denoted by  $A\langle x \rangle$ .

The following result of [4] is a generalization of Płoski's result [17]. It is a corollary of Theorem 11.4 [22] which itself is a corollary of the Néron-Popescu Desingularization (see [18], [22] or [23] for the proof of this desingularization theorem in whole generality or [19] for a proof in characteristic zero).

**Theorem 2.1.** [4] *Let  $f(x, y) \in \mathbb{K}\langle x \rangle[y]^p$  and let us consider a solution  $y(x) \in \mathbb{K}\{x\}^m$  of*

$$f(x, y(x)) = 0.$$

*Let us assume that  $y_i(x)$  depends only on  $(x_1, \dots, x_{\sigma(i)})$  where  $i \mapsto \sigma(i)$  is an increasing function. Then there exist a new set of variables  $z = (z_1, \dots, z_s)$ , an increasing function  $\tau$ , convergent power series  $z_i(x) \in \mathbb{K}\{x\}$  vanishing at 0 such that  $z_1(x), \dots, z_{\tau(i)}(x)$  depend only on  $(x_1, \dots, x_{\sigma(i)})$ , and an algebraic power series vector solution  $y(x, z) \in \mathbb{K}\langle x, z \rangle^m$  of*

$$f(x, y(x, z)) = 0,$$

*such that for every  $i$ ,*

$$y_i(x, z) \in \mathbb{K}\langle x_1, \dots, x_{\sigma(i)}, z_1, \dots, z_{\tau(i)} \rangle \text{ and } y(x) = y(x, z(x)).$$

**2.3. Arc-analytic and arc-wise analytic maps.** Let  $Z, Y$  be  $\mathbb{K}$ -analytic spaces. A map  $f : Z \rightarrow Y$  is called *arc-analytic* if  $f \circ \delta$  is analytic for every real analytic arc  $\delta : I \rightarrow Z$ , where  $I = (-1, 1) \subset \mathbb{R}$  (cf. [14]). By an *arc-analytic homeomorphism* we mean a homeomorphism  $\varphi$  such that both  $\varphi$  and  $\varphi^{-1}$  are arc-analytic.

Let  $T$  be a nonsingular  $\mathbb{K}$ -analytic space. We say that a map  $f(t, z) : T \times Z \rightarrow Y$  is *arc-wise analytic in  $t$*  if it is  $\mathbb{K}$ -analytic in  $t$  and arc-analytic in  $z$ . This means that for every real analytic arc  $z(s) : I \rightarrow Z$ , the map  $f(t, z(s))$  is real analytic and moreover, if  $\mathbb{K} = \mathbb{C}$ , complex analytic with respect to  $t$  (cf. [16]).

**2.4. Algebraic Equisingularity of Zariski.** Notation: Let  $x = (x_1, \dots, x_n) \in \mathbb{K}^n$ . Then we set  $x^i = (x_1, \dots, x_i) \in \mathbb{K}^i$ .

**2.4.1. Assumptions.** Let  $V$  be an analytic hypersurface in a neighborhood of the origin in  $\mathbb{K}^l \times \mathbb{K}^n$  and let  $T = V \cap (\mathbb{K}^l \times \{0\})$ . Suppose there are given pseudopolynomials<sup>1</sup>

$$F_i(t, x^i) = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(t, x^{i-1})x_i^{p_i-j}, \quad i = 0, \dots, n,$$

$t \in \mathbb{C}^l, x^i \in \mathbb{C}^i$ , with analytic coefficients  $a_{i-1,j}$ , that satisfy

- (1)  $V = F_n^{-1}(0)$ ,
- (2) for all  $i, j, a_{i,j}(t, 0) \equiv 0$ ,
- (3)  $F_{i-1}$  vanishes on the set of those  $(t, x^{i-1})$  for which  $F_i(t, x^{i-1}, x_i) = 0$ , considered as an equation on  $x_i$ , has fewer complex roots than for generic  $(t, x^{i-1})$ ,
- (4) Either  $F_i(t, 0) \equiv 0$  or  $F_i \equiv 1$  (and in the latter case  $F_k \equiv 1$  for all  $k \leq i$  by convention),

---

<sup>1</sup>A pseudopolynomial is a polynomial in  $x_i$  with coefficients that are analytic in the other variables. The pseudopolynomials  $F_i$  that we consider are moreover distinguished polynomials in  $x$ , i.e. are of the form  $x_i^p + \sum_{j=1}^p a_j(x^{i-1})x_i^{p-j}$  with  $a_j(0) = 0$  for all  $j$ . They may depend analytically on  $t$  that is considered as a parameter.

(5)  $F_0 \equiv 1$ .

For condition (3) it suffices that the discriminant of  $(F_i)_{red}$ , that is of  $F_i$  reduced, divides  $F_{i-1}$ . Equivalently, the first not identically equal to zero generalized discriminant of  $F_i$  divides  $F_{i-1}$ , see subsection 3.0.2.

We call a system  $\{F_i(t, x^i)\}$  satisfying the above conditions *Zariski equisingular*. We also say that the family of germs  $V_t := V \cap (\{t\} \times \mathbb{K}^n)$  is *Zariski equisingular* if such a system with  $V = \{F_n = 0\}$  exists, maybe after a local analytic change of coordinates  $x$ . Answering a question posed by O. Zariski, A. Varchenko [26, 27, 28] showed that a Zariski equisingular family  $V_t$  is locally topologically trivial. The following stronger version of Varchenko's Theorem follows from Theorems 3.3 and 4.3 of [16].

**Theorem 2.2.** ([26, 27, 28], [16]) *If a system  $\{F_i(t, x^i)\}$  is Zariski equisingular then the family  $V_t$  is locally topologically trivial along  $T$ , that is there exist  $\varepsilon, \delta > 0$  and a homeomorphism*

$$\Phi : B_\varepsilon \times \Omega_0 \longrightarrow \Omega,$$

where  $B_\varepsilon = \{t \in \mathbb{K}^l ; \|t\| < \varepsilon\}$ ,  $\Omega_0 = \{x \in \mathbb{K}^n ; \|x\| < \delta\}$  and  $\Omega$  is a neighborhood of the origin in  $\mathbb{K}^{l+n}$ , that preserves the family  $V_t : \Phi^{-1}(V) = B_\varepsilon \times V_0$ . Moreover,  $\Phi$  has a triangular form

$$\Phi(t, x) = (t, \Psi(t, x)) = (t, \Psi_1(t, x^1), \Psi_2(t, x^2), \dots, \Psi_n(t, x^n))$$

and satisfies the following additional properties:

- (1) if  $F_n$  is a product of pseudopolynomials then  $\Phi$  preserves the zero set of each factor,
- (2)  $\Phi$  is subanalytic (semi-algebraic if all  $F_i$  are polynomials or Nash functions),
- (3)  $\Phi$  is arc-wise analytic in  $t$  and  $\Phi^{-1}$  is arc-analytic.

Moreover, if the multiplicity of  $F_i(0, x^i)$  at  $0 \in \mathbb{K}^i$  is equal to  $p_i$  for every  $2 \leq i \leq n$  then there exists a constant  $C$  such that for all  $(t, x) \in B_\varepsilon \times \Omega_0$

$$\left\| \frac{\partial \Psi}{\partial t}(t, x) \right\| \leq C \|\Psi(t, x)\|.$$

*Remark 2.3.* The last condition can be written equivalently, maybe for different  $\varepsilon > 0$  and  $\delta > 0$ , as follows. There exists a constant  $C_1 > 0$  such that for all  $(t, x) \in B_\varepsilon \times \Omega_0$

$$C_1^{-1} \|x\| \leq \|\Psi(t, x)\| \leq C_1 \|x\|,$$

see Proposition 1.6 of [16]. That means geometrically that the trivialization  $\Phi$  preserves the size of the distance to the origin.

*Remark 2.4.* The condition (1) of the conclusion can be given much stronger form, cf. Proposition 1.9 of [16]. For any analytic function  $G$  dividing  $F_n$ , there is a constant  $C > 0$  such that

$$C_1^{-1} \|G(0, x)\| \leq \|G(\Phi(t, x))\| \leq C_1 \|G(0, x)\|.$$

### 3. NASH APPROXIMATION OF ANALYTIC SETS

3.0.2. *Generalized discriminants.* (see e.g. [30] Appendix IV) Let  $f(T) = T^p + \sum_{j=1}^p a_j T^{p-j} = \prod_{j=1}^p (T - T_j)$ . Then the expressions

$$\sum_{r_1, \dots, r_{j-1}} \prod_{k < l, k, l \neq r_1, \dots, r_{j-1}} (T_k - T_l)^2$$

are symmetric in  $T_1, \dots, T_p$  and hence polynomials in  $a = (a_1, \dots, a_p)$ . We denote these polynomials by  $\Delta_j(a)$ . Thus  $\Delta_1$  is the standard discriminant and  $f$  has exactly  $p - j$  distinct roots if and only if  $\Delta_1 = \dots = \Delta_j = 0$  and  $\Delta_{j+1} \neq 0$ .

3.0.3. *Construction of a normal system of equations.* We recall here the main construction of [15], [4].

Let  $g_1, \dots, g_l \in \mathbb{K}\{x\}$  be a finite set of pseudopolynomials:

$$g_s(x) = x_n^{r_s} + \sum_{j=1}^{r_s} a_{n-1,s,j}(x^{n-1})x_n^{r_s-j}.$$

We arrange the coefficients  $a_{n-1,s,j}$  in a row vector  $a_{n-1} \in \mathbb{K}\{x^{n-1}\}^{p_n}$ ,  $p_n = \sum_s r_s$ . Let  $f_n$  be the product of the  $g_s$ 's. The generalized discriminants  $\Delta_{n,i}$  of  $f_n$  are polynomials in  $a_{n-1}$ . Let  $j_n$  be a positive integer such that

$$(3.1) \quad \Delta_{n,j_n}(a_{n-1}) \neq 0 \quad \text{and} \quad \Delta_{n,i}(a_{n-1}) \equiv 0 \text{ for } i < j_n.$$

Then, after a linear change of coordinates  $x^{n-1}$ , we may write

$$\Delta_{n,j_n}(a_{n-1}) = u_{n-1}(x^{n-1}) \left( x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2})x_{n-1}^{p_{n-1}-j} \right)$$

where  $u_{n-1}(0) \neq 0$  and for all  $j$ ,  $a_{n-2,j}(0) = 0$ , and

$$f_{n-1} = x_{n-1}^{p_{n-1}} + \sum_{j=1}^{p_{n-1}} a_{n-2,j}(x^{n-2})x_{n-1}^{p_{n-1}-j}$$

is the Weierstrass polynomial associated to  $\Delta_{n,j_n}$ . We denote by  $a_{n-2} \in \mathbb{K}\{x^{n-2}\}^{p_{n-1}}$  the vector of its coefficients  $a_{n-2,j}$ .

Similarly we define recursively a sequence of pseudopolynomials  $f_i(x^i)$ ,  $i = 1, \dots, n-1$ , such that  $f_i = x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j}$  is the Weierstrass polynomial associated to the first non identically equal to zero generalized discriminant  $\Delta_{i+1,j_{i+1}}(a_i)$  of  $f_{i+1}$ , where we denote in general  $a_i = (a_{i,1}, \dots, a_{i,p_{i+1}})$  and

$$(3.2) \quad \Delta_{i+1,j_{i+1}}(a_i) = u_i(x^i) \left( x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1,j}(x^{i-1})x_i^{p_i-j} \right), \quad i = 0, \dots, n-1.$$

Thus the vector of functions  $a_i$  satisfies

$$(3.3) \quad \Delta_{i+1,k}(a_i) \equiv 0 \quad k < j_{i+1}, \quad i = 0, \dots, n-1.$$

This means in particular that

$$\Delta_{1,k}(a_0) \equiv 0 \quad \text{for } k < j_1 \text{ and } \Delta_{1,j_1}(a_0) \equiv u_0,$$

where  $u_0$  is a non-zero constant.

*Remark 3.1.* At each step of this construction we may use a linear change of coordinates in order to assume that the multiplicity of  $f_i$  at the origin of  $\mathbb{K}^i$  is equal to  $p_i$ . We will assume this in the following.

3.0.4. *Approximation by Nash functions.* If we consider (3.2) and (3.3) as a system of polynomial equations on  $a_i(x^i)$ ,  $u_i(x^i)$  then, by construction, this system admits convergent solutions. Therefore, by Theorem 2.1, there exist a new set of variables  $z = (z_1, \dots, z_s)$ , an increasing function  $\tau$ , convergent power series  $z_i(x) \in \mathbb{K}\langle x \rangle$  vanishing at 0, algebraic power series  $u_i(x^i, z) \in \mathbb{K}\langle x^i, z_1, \dots, z_{\tau(i)} \rangle$  and vectors of algebraic power series  $a_i(x^i, z) \in \mathbb{K}\langle x^i, z_1, \dots, z_{\tau(i)} \rangle^{p_i}$  such that the following holds:

$$\begin{aligned} z_1(x), \dots, z_{\tau(i)}(x) &\text{ depend only on } (x_1, \dots, x_i), \\ a_i(x^i, z), u_i(x^i, z) &\text{ are solutions of (3.2), (3.3)} \\ \text{and } a_i(x^i) = a_i(x^i, z(x^i)), \quad u_i(x^i) &= u_i(x^i, z(x^i)). \end{aligned}$$

Then we define

$$\begin{aligned} F_n(z, x) &= \prod_s G_s(z, x), \quad G_s(z, x) = x_n^{r_s} + \sum_{j=1}^{r_s} a_{n-1, s, j} (x^{n-1}, z(x^{n-1}) - z) x_n^{r_s - j}, \\ F_i(z, x) &= x_i^{p_i} + \sum_{j=1}^{p_i} a_{i-1, j} (x^{i-1}, z^{\tau(i-1)}(x^{i-1}) - z^{\tau(i-1)}) x_i^{p_i - j}, \quad i = 0, \dots, n-1. \end{aligned}$$

Finally we set  $F_0 \equiv 1$ . Because  $u_i(0, 0) = u_i(0, z(0)) \neq 0$  and by Remark 3.1, the system  $\{F_i(z, x)\}$  is Zariski equisingular deformation of  $F_n(0, x) = \prod_s g_s(x)$ . Hence Theorem 2.2 implies the following.

**Lemma 3.2.** *There are  $\varepsilon, \delta > 0$  and a subanalytic, arc-wise analytic in  $z$ , arc-analytic homeomorphism*

$$(3.4) \quad \Phi(z, x) = (z, \Psi(z, x)) : B_\varepsilon \times \Omega_0 \rightarrow \Omega,$$

where  $B_\varepsilon = \{z \in \mathbb{K}^l ; \|z\| < \varepsilon\}$ ,  $\Omega_0 = \{x \in \mathbb{K}^n ; \|x\| < \delta\}$ , and  $\Omega$  is a neighborhood of the origin in  $\mathbb{K}^{l+n}$ , preserving the sets  $\{G_s = 0\}$  for  $1 \leq s \leq l$ , i.e.  $\Phi^{-1}(\{G_s = 0\}) = B_\varepsilon \times (\{G_s = 0, z = 0\})$ .

Moreover, there exist constants  $C, C_1 > 0$  such that for all  $(z, x) \in B_\varepsilon \times \Omega_0$

$$(3.5) \quad \left\| \frac{\partial \Psi}{\partial z}(z, x) \right\| \leq C \|\Psi(z, x)\| \leq C_1 \|x\|.$$

Let us consider for  $m \in \mathbb{N}$

$$z(x) = z_m(x) + \tilde{z}_m(x),$$

where  $z_m(x)$  is the  $m$ -th Taylor polynomial of  $z(x)$ . Set

$$\tilde{f}_n(t, x) = \prod_s \tilde{g}_s(t, x), \quad \tilde{g}_s(t, x) = G_s(t\tilde{z}_m(x), x)$$

$$\tilde{f}_i(t, x) = F_i(t\tilde{z}_m(x), x), \quad i = 0, \dots, n-1.$$

Let us fix the trivialization  $\Phi$  given by Lemma 3.2. Then we have the following result.

**Lemma 3.3.** *There are  $R > 1$  and  $0 < \delta' \leq \delta$  such that for all  $m \in \mathbb{N}$  the system  $\{\tilde{f}_i\}$  is Zariski equisingular for  $|t| < R$  and such that the zero sets of  $\tilde{g}_s(t, x)$ ,  $s = 1, \dots, l$ , can be trivialized by the homeomorphism*

$$\varphi(t, x) = \Phi(z_m(x) + t\tilde{z}_m(x), x),$$

defined on  $\{(t, x) ; |t| < R, \|x\| < \delta'\}$ .

*Proof.* The system is Zariski equisingular as it follows by construction. The only point to check is that

$$(3.6) \quad \begin{aligned} a_{i-1,j}(t, x^{i-1}) &:= a_{i-1,j}(x^{i-1}, z^{\tau(i-1)}(x^{i-1}) - t\tilde{z}_m^{\tau(i-1)}(x^{i-1})) \\ &= a_{i-1,j}(x^{i-1}, z_m^{\tau(i-1)}(x^{i-1}) + (1-t)\tilde{z}_m^{\tau(i-1)}(x^{i-1})) \end{aligned}$$

depends only on  $x^{i-1}$ . Fix  $R > 1$ . Then to show the second part it is enough to note that we may choose  $\delta'$  so that

$$\|t\tilde{z}_m(x)\| < \varepsilon$$

for  $\|x\| < \delta'$ ,  $|t| < R$ , and all  $m \in \mathbb{N}$ .  $\square$

Write  $\varphi(t, x) = (t, \psi(t, x))$ , i.e.  $\psi(t, x) := \Psi(t\tilde{z}_m(x), x)$ . The inequalities (3.5) then imply

$$(3.7) \quad \left\| \frac{\partial \psi}{\partial t}(t, x) \right\| \leq C \|\Psi(t\tilde{z}_m(x), x)\| \|\tilde{z}_m(x)\| \leq C(m)C_1 \|x\|^{m+2},$$

if we bound  $\|\tilde{z}_m(x)\| \leq C(m)\|x\|^{m+1}$ . By classical formulae we may bound  $C(m) \leq C \sup_{\|x\| < \delta'} \|z(x)\| \delta'^{-(m+1)}$  for a universal  $C$ . Then by integration we obtain

$$(3.8) \quad \|\psi(1, x) - x\| \leq CC_1 \left( \sup_{\|x\| < \delta'} \|z(x)\| \delta'^{-(m+1)} \right) \|x\|^{m+2}.$$

This allows us to give the following result:

**Proposition 3.4.** *Let  $X$  be an analytic subset of an open neighborhood  $U_0$  of  $0 \in \mathbb{K}^n$  with  $0 \in X$ . Then there are a constant  $c > 0$  and an open neighborhood  $U \subset U_0$  of  $0$  in  $\mathbb{K}^n$  such that for every  $m \in \mathbb{N}$  there are a subanalytic arc-analytic homeomorphism  $\varphi_m : U \rightarrow \varphi_m(U) \subset \mathbb{K}^n$  and a Nash subset  $V_m$  of  $\varphi_m(U)$  with the following properties:*

- (a)  $\varphi_m(X \cap U) = V_m$ ,
- (b)  $\|\varphi_m(a) - a\| \leq c^m \|a\|^m$  for every  $a \in U$ .

*Proof.* We may assume that  $X$  is defined by the pseudopolynomials  $g_1, \dots, g_l \in \mathbb{K}\{x\}$ . We set  $\hat{g}_i(x) = G_i(1, x)$  and we denote by  $V_m$  the zero locus of the  $\hat{g}_i$ . Then by Lemma 3.3 the set  $X \cap U$  is homeomorphic to  $V_m \cap U$  where  $U = \{x \in \mathbb{K}^n ; \|x\| < \delta'\}$ . The homeomorphism is given by  $\varphi_m : x \mapsto \psi(1, x)$ . Condition (b) follows from (3.8).  $\square$

#### 4. ALGEBRAIC APPROXIMATION OF NASH SETS

The main result of this section is the following

**Proposition 4.1.** *Let  $X$  be a Nash subset of an open neighborhood  $U_0$  of  $0 \in \mathbb{K}^n$  with  $0 \in X$ . Then there are a constant  $c > 0$  and an open neighborhood  $U \subset U_0$  of  $0$  in  $\mathbb{K}^n$  such that for every  $m \in \mathbb{N}$  there are a Nash  $\mathbb{K}$ -analytic diffeomorphism  $\varphi_m : U \rightarrow \varphi_m(U) \subset \mathbb{K}^n$  and an algebraic subset  $V_m$  of  $\mathbb{K}^n$  with the following properties:*

- (a)  $\varphi_m(X \cap U) = V_m \cap \varphi_m(U)$ ,
- (b)  $\|\varphi_m(a) - a\| \leq c^m \|a\|^m$  for every  $a \in U$ .



*Proof.* First we discuss the case  $\mathbb{K} = \mathbb{C}$ . We may assume that  $\dim(X) = k < n$  because otherwise there is nothing to prove. By Artin-Mazur's construction [3] (cf. also [7]), there are  $s \in \mathbb{N}$ , an algebraic set  $M \subset \mathbb{C}^n \times \mathbb{C}^s$  with  $0 \in M$  and a complex Nash submanifold  $N$  of a polydisc  $E \times F \subset \mathbb{C}^n \times \mathbb{C}^s$  centered at zero such that:

- (i)  $\pi|_M : M \rightarrow \mathbb{C}^n$  is a proper map, where  $\pi : \mathbb{C}^n \times \mathbb{C}^s \rightarrow \mathbb{C}^n$  denotes the natural projection, and  $M \cap (E \times F) \subset N$ ,
- (ii)  $\pi|_N : N \rightarrow E$  is a biholomorphism and  $\pi(M \cap (E \times F)) = X \cap E$ .

We may assume that  $s = 1$  (i.e.  $F$  is a disc in  $\mathbb{C}$ ). Indeed, we may replace  $M, N$  by  $v_L(M), v_L(N) \subset \mathbb{C}^n \times \mathbb{C}$ , respectively. Here  $v_L$  is defined by the formula  $v_L(x, z) = (x, L(z))$ , where  $L : \mathbb{C}^s \rightarrow \mathbb{C}$  is a linear form such that  $v_L$  restricted to  $(\{0\}^n \times \mathbb{C}^s) \cap M$  is injective. (Note that it may be necessary to replace  $E \times F$  by a smaller polydisc.)

We may also assume (applying a change of variables) that  $\pi(M) \subset \mathbb{C}^n = \mathbb{C}^{n-1} \times \mathbb{C}$  has proper projection onto  $\mathbb{C}^{n-1}$  and  $X \cap (E' \times E'')$  has proper projection onto  $E'$ , where  $E = E' \times E'' \subset \mathbb{C}^{n-1} \times \mathbb{C}$ .

Denote  $\pi = (\pi_1, \dots, \pi_{n-1}, \pi_n) = (\pi', \pi_n)$ . Then after replacing  $\pi_n$  by any polynomial  $w$  we obtain a polynomial map  $\rho = (\pi', w)$  such that  $\rho|_M$  is proper (so  $\rho(M)$  is an algebraic subset of  $\mathbb{C}^n$ ). Now, by (ii), there is a holomorphic function  $\tau : E' \times E'' \rightarrow F$  such that  $N$  is the graph of  $\tau$ . We will choose  $w = w_m$  such that (perhaps after shrinking  $E', E'', F$ ) for  $\rho_m = (\pi', w_m)$  the map  $\varphi_m(x) = \rho_m((x, \tau(x)))$  satisfies (a) and (b) with  $V_m = \rho_m(M)$ . The size of  $E' \times E'' \times F$  will be independent of  $m$ . The idea to obtain algebraic  $V_m$  (equivalent to or approximating some  $X$ ) as the image of an algebraic set by a certain polynomial map  $\rho_m$  appeared before (see e.g. [7], [1], [6]). But here  $\rho_m$  must be chosen in a special way to ensure that  $V_m$  is both analytically equivalent and higher order tangent to  $X$ .

Let us verify that after shrinking  $E', E'', F$  the following hold:

- (x)  $(\overline{E'} \times \mathbb{C} \times \partial F) \cap M = \emptyset$ ,
- (y)  $(\overline{E'} \times \partial E'' \times \overline{F}) \cap M = \emptyset$ ,
- (z)  $(\overline{E} \times \partial F) \cap \overline{N} = \emptyset$ .

First, by the fact that  $N \subset E \times F$  is a graph of a holomorphic function defined on  $E$ , we know that (z) holds after replacing  $E$  by any of its relatively compact subsets. Hence, we may assume to have (z). Observe also that if (z) is true, then it remains such after shrinking  $F$  slightly with fixed  $E$ .

Next, by the facts that  $\pi|_M$  is proper and  $\pi(M)$  has proper projection onto  $\mathbb{C}^{n-1}$ , we know that  $(\{0\}^{n-1} \times \mathbb{C} \times \mathbb{C}) \cap M$  is a finite set. Therefore after shrinking  $F$  slightly (in such a way that (z) remains true) we have  $(\{0\}^{n-1} \times \mathbb{C} \times \partial F) \cap M = \emptyset$ . Thus, in view of the properness of the projections, (x) holds for  $F$  chosen above with any sufficiently small neighborhood  $E'$  of 0. It follows that both (x) and (z) hold with any sufficiently small  $E = E' \times E''$ . We can pick  $E$  in such a way that  $(\overline{E'} \times \partial E'') \cap \pi(M) = \emptyset$ , which implies (y).

Let  $\delta$  denote the radius of  $F$ . Define  $w_m : E \times F \rightarrow \mathbb{C}$  by the formula  $w_m(x, z) = \pi_n(x, z) + (\frac{z}{\delta})^m$  with large  $m$ . Let us check that  $\varphi_m : U \rightarrow \mathbb{C}^n$  given by  $\varphi_m(x) = \rho_m((x, \tau(x)))$  has all the required properties, where  $U = E' \times E''$ . For  $z \in F$  we have  $|z| < \delta$  and the graph of  $\tau$  is contained in  $E' \times E'' \times F$ , hence in view of (z), for  $m$  large enough,  $\varphi_m$  is a biholomorphism onto its image. The injectivity of  $\varphi_m$

requires a brief explanation. First, by the definition of  $\varphi_m$ , we can explicitly write

$$(4.1) \quad \varphi_m(x) = \left( x_1, \dots, x_{n-1}, x_n + \left( \frac{\tau(x)}{\delta} \right)^m \right).$$

Thus  $\varphi_m$  is injective on  $U$  if the map  $x_n \mapsto x_n + \left( \frac{\tau(x)}{\delta} \right)^m$  is injective for every fixed  $(x_1, \dots, x_{n-1})$ . The latter assertion is true for large  $m$  if  $\sup_{x \in U} \left| \frac{\tau(x)}{\delta} \right| < 1$  (because then the modulus of the derivative of the map  $x_n \mapsto \left( \frac{\tau(x)}{\delta} \right)^m$  is small).

Let us check that  $\sup_{x \in U} \left| \frac{\tau(x)}{\delta} \right| < 1$ . Recall that  $N = \text{graph}(\tau)$ . Hence, by (z), we have

$$0 < \inf \{ \|a - b\| ; a \in \overline{\text{graph}(\tau)}, b \in \overline{E} \times \partial F \} \leq \inf \{ |\tau(x) - c| ; x \in \overline{E}, c \in \partial F \}.$$

Since  $\delta$  is the radius of  $F$ , and  $U = E$ , we obtain  $\sup_{x \in U} |\tau(x)| < \delta$ , as required.

Let us verify that for large  $m$ ,

$$\varphi_m(X \cap (E' \times E'')) = \rho_m(M) \cap \varphi_m(E' \times E'').$$

By (i), (ii) and the definition of  $\varphi_m$ , we have  $\varphi_m(X \cap (E' \times E'')) = \rho_m(\text{graph}(\tau) \cap M)$ . Moreover,  $\rho_m(\text{graph}(\tau)) = \varphi_m(E' \times E'')$ , hence it is sufficient to show that  $\rho_m(\text{graph}(\tau) \cap M) = \rho_m(\text{graph}(\tau)) \cap \rho_m(M)$ . In fact, the inclusion " $\subset$ " is trivial so we prove " $\supset$ ".

Fix  $a \in \rho_m(\text{graph}(\tau)) \cap \rho_m(M)$ . Then there are  $z \in \text{graph}(\tau)$  and  $v \in M$  such that  $\rho_m(z) = \rho_m(v) = a$ . Observe that  $z, v \in E' \times \mathbb{C} \times \mathbb{C}$ . Indeed, write  $z = (z_1, \dots, z_{n-1}, z_n, z_{n+1})$ ,  $v = (v_1, \dots, v_{n-1}, v_n, v_{n+1})$ . Since  $\text{graph}(\tau) \subset E' \times E'' \times F$  we have  $(z_1, \dots, z_{n-1}) \in E'$ . By the definition of  $\rho_m$  and by  $\rho_m(z) = \rho_m(v)$  we have  $(z_1, \dots, z_{n-1}) = (v_1, \dots, v_{n-1})$ .

Moreover,  $(E' \times \mathbb{C} \times \mathbb{C}) \cap M$  is bounded so  $v$  must belong to  $E' \times \mathbb{C} \times F$  because otherwise (for large  $m$ ) in view of the definition of  $w_m$  and (x),  $\rho_m(v) = a$  lies outside  $\rho_m(\text{graph}(\tau))$ , which is a contradiction. Now if  $v \notin E' \times E'' \times F$ , then by (y), (z), (x) (for large  $m$ )  $\rho_m(v) \notin \rho_m(\text{graph}(\tau))$ , again a contradiction. Consequently, in view of (i), we have  $v \in M \cap (E' \times E'' \times F) \subset \text{graph}(\tau)$ . Since  $\rho_m|_{\text{graph}(\tau)}$  is injective, we have  $v = z$ , hence  $a \in \rho_m(\text{graph}(\tau) \cap M)$ . This shows that

$$\varphi_m(X \cap (E' \times E'')) = \rho_m(M) \cap \varphi_m(E' \times E'')$$

and completes the proof of (a).

Finally, by 4.1 we have

$$\|\varphi_m(a) - a\| \leq \frac{|\tau(a)|^m}{\delta^m}$$

for every  $a \in E' \times E''$ . Since  $\tau(0) = 0$ , we immediately obtain (b), which completes the proof in the case  $\mathbb{K} = \mathbb{C}$ .

As for  $\mathbb{K} = \mathbb{R}$ , we cannot simply repeat the procedure above because the image of a real algebraic set by a proper polynomial map need not be algebraic. Instead we proceed as follows. First for the given Nash set  $X$  let  $X_{\mathbb{C}}$  denote its complexification. More precisely,  $X_{\mathbb{C}}$  is a representative of the smallest complex Nash germ in  $(\mathbb{C}^n, 0)$  containing the germ  $(X, 0)$ . Note that  $X_{\mathbb{C}}$  is defined by real equations. For  $X_{\mathbb{C}}$  one can repeat the construction described above obtaining  $M, N$  also defined by real equations. In particular, the Taylor expansion of  $\varphi_m$  around zero has real coefficients i.e. the restriction of  $\varphi_m$  to  $\mathbb{R}^n$  is a real Nash isomorphism. It is not

difficult to observe that  $(\varphi_m(X), 0)$  is the germ of a real algebraic set as required.  $\square$

## 5. PROOF OF THEOREM 1.3

Let  $(X, 0) \subset (\mathbb{K}^n, 0)$  be a germ of analytic set. In both real and complex cases  $(X, 0)$  can be considered as a real analytic set germ defined by equations

$$g_1 = \cdots = g_l = 0$$

where the  $g_i$  are convergent power series with real coefficients. A real analytic arc on  $(X, 0)$  is a germ of real analytic map  $(\mathbb{R}, 0) \rightarrow (X, 0)$ . Then the space of real analytic arcs of  $(X, 0)$  is in bijection with set of morphisms

$$\frac{\mathbb{R}\{x_1, \dots, x_n\}}{(g_1, \dots, g_l)} \rightarrow \mathbb{R}\{t\}$$

which are defined by the data of  $n$  convergent power series  $x_1(t), \dots, x_n(t) \in \mathbb{R}\{t\}$  such that

$$g_i(x_1(t), \dots, x_n(t)) = 0 \quad \forall i.$$

Let  $m$  be a positive integer and let  $\varphi_m$  be the homeomorphism given by Theorem 1.2. Let  $\delta$  be an arc on  $(X, 0)$ , i.e.  $\delta$  is a real analytic map  $(-\varepsilon, \varepsilon) \rightarrow X \cap U$  for some  $\varepsilon > 0$ . Then we define  $\delta' : (-1, 1) \rightarrow X \cap U$  by  $\delta'(t) = \delta(\varepsilon t)$  for all  $t \in (-1, 1)$ . For such a real analytic arc  $\delta'$ ,  $\varphi_m \circ \delta'$  is a real analytic arc on  $V_m \cap \varphi_m(U)$  since  $\varphi_m$  is an arc-analytic map (cf. Theorem 1.2). Thus  $\varphi_m \circ \delta$  is a real analytic arc on  $(V_m, 0)$ . On the other hand  $\varphi_m^{-1}$  is arc-analytic thus the same procedure applies. This shows that  $\varphi_m$  induces a bijection between the space of real analytic arcs on  $(X, 0)$  and the space of real analytic arcs on  $(V_m, 0)$ .

Now if  $\delta$  is a real analytic arc on  $(X, 0)$  then

$$\|\varphi_m(\delta(t)) - \delta(t)\| \leq c^m \|\delta(t)\|^m \leq C|t|^m$$

for all  $t$  small enough and some positive constant  $C$ . This shows that  $x_i \circ \varphi_m \circ \delta(t)$  has the same Taylor expansion as  $x_i \circ \delta$  up to order  $m - 1$ . Thus  $\varphi_m$  induces the identity map between the space of  $m$ -truncations of real analytic arcs on  $(X, 0)$  and the space of  $m$ -truncations of real analytic arcs on  $(V_m, 0)$ .

If  $(X, 0)$  is complex analytic then any real analytic arc germ  $\delta : (\mathbb{R}, 0) \rightarrow (X, 0)$  extends uniquely to a complex analytic arc germ  $\delta_{\mathbb{C}} : (\mathbb{C}, 0) \rightarrow (X, 0)$  and similarly for the arc-germs in  $(V_m, 0)$ . Moreover, any complex arc germ in  $(X, 0)$ , resp. in  $(V_m, 0)$ , is the complexification of a real arc germ. Thus  $\mathcal{A}^{\mathbb{C}}(X) = \mathcal{A}^{\mathbb{C}}(V_m)$  follows from  $\mathcal{A}^{\mathbb{R}}(X) = \mathcal{A}^{\mathbb{R}}(V_m)$ .

*Remark 5.1.* By a result of M. Greenberg (cf. [12] or [21] for the analytic case) for a given analytic germ  $(X, 0) \subset (\mathbb{K}^n, 0)$  there exists a constant  $a = a_X > 0$  such that for every integer  $m$  we have that

$$\mathcal{A}_m^{\mathbb{K}}(X) = \pi_m(\mathcal{B}_{am}^{\mathbb{K}}(X))$$

where  $\mathcal{B}_k^{\mathbb{K}}(X)$  denotes the space of  $k$ -jets on  $(X, 0)$ , i.e. the space of  $\mathbb{K}$ -analytic arcs on  $(\mathbb{K}^n, 0)$  whose contact order with  $(X, 0)$  is at least  $k + 1$ , and  $\pi_m$  is the truncation map at order  $m$ , i.e. the map sending an arc on  $(\mathbb{K}^n, 0)$  onto its truncation at order  $m$ . If we denote by  $(X', 0)$  an analytic germ defined by equations that coincide with the equations defining  $(X, 0)$  up to order  $am$ , then we obviously have that

$$\mathcal{B}_{am}^{\mathbb{K}}(X) = \mathcal{B}_{am}^{\mathbb{K}}(X').$$

But while  $\mathcal{A}_m^{\mathbb{K}}(X)$  is the truncation of  $\mathcal{B}_{am}^{\mathbb{K}}(X)$ ,  $\mathcal{A}_m^{\mathbb{K}}(X')$  has no reason in general to be equal to the truncation of  $\mathcal{B}_{am}^{\mathbb{K}}(X')$  since the constant  $a_{X'}$  of Greenberg's Theorem may be strictly greater than  $a = a_X$ . Thus we cannot prove, using Greenberg's Theorem, that  $\mathcal{A}_m^{\mathbb{K}}(X)$  is equal to  $\mathcal{A}_m^{\mathbb{K}}(X')$  when  $(X', 0)$  is an analytic germ whose equations coincide with those of  $(X, 0)$  up to a high order.

In fact a high order of tangency of two analytic germs does not guarantee that the spaces of truncated arcs associated with these germs are equal. We can explicitly show this on the following example. Let

$$f(x, y, z) = z^2 - xy^4, \quad f_k(x, y, z) = z^2 - x(y^4 + x^{2k})$$

and define  $X = \{(x, y, z) : f(x, y, z) = 0\}$ ,  $X_k = \{(x, y, z) : f_k(x, y, z) = 0\}$  for any positive integer  $k$  which is not divisible by 2. Then the arc  $\gamma$  given by  $\gamma(t) = (t, 0, 0)$  satisfies  $\gamma \in \mathcal{A}_1^{\mathbb{K}}(X)$  but in both cases real or complex  $\gamma \notin \mathcal{A}_1^{\mathbb{K}}(X_k)$ . Indeed, let  $\tilde{\gamma}$  be any arc whose truncation up to order 1 equals  $\gamma$ . Then after substituting  $\tilde{\gamma}$  to  $x(y^4 + x^{2k})$  and to  $z^2$ , we obtain power series with odd and even order of zero, respectively, so  $f_k \circ \tilde{\gamma} \neq 0$ . Thus  $\mathcal{A}_1^{\mathbb{K}}(X_k) \neq \mathcal{A}_1^{\mathbb{K}}(X)$ , although  $2k$ -truncations of  $f$  and  $f_k$  are equal and the multiplicities of  $X$  and  $X_k$  at 0 are also equal for  $k$  large enough.

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