Fractional Sobolev Spaces and Functions of Bounded Variation

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Abstract

In this paper, we investigate the 1D Riemann-Liouville fractional derivative to make the connection between the so called fractional Sobolev spaces and the classical spaces of functions of bounded variation whose derivatives are not functions but measures, BV and SBV.

1 Introduction

The aim of this work is to investigate the *fractional derivative* concepts and make the connection between the related (so called fractional) Sobolev spaces and the spaces of functions of bounded variation whose derivatives are not functions but measures. There are two main definitions of fractional differentiation whose connections are not clear (to our knowledge). The fractional derivative in the sense of Gagliardo is not explicitely defined (almost everywhere for example) but through the setting of fractional Sobolev Spaces and the underlying norm. It is, in some sense, a global definition which can be easily handled via the Fourier transform. The second approach is based on the Riemann-Liouville fractional derivative (in short RL) and may be pointwise defined. We choose to focus on the RL derivative : there are many variants of the fractional derivatives/integrals definition as the Grunwald-Letnikov, Caputo, Weyl ones [8, 9, 17] but the RL derivative can be considered as a generic one. For a complete study of these derivatives one can refer to the book by Samko and al. [20]. Moreover, in [23], the connection is made with metric and measure spaces, in particular the Hausdorff measure. We decided to use this derivative concept because it seems more adapted to applications than the Gagliardo one. The RL derivative is widely used by physicists [25, 26], in automatics, control theory and image processing as well, especially to deal with image enhancement and texture analysis (see [18] for example); in [16] calculus of variations problem where the cost functional involves fractional derivative are investigated. Nevertheless, the context is often a discrete one and there is not much analysis (to our knowledge) in the infinite dimensional setting. In particular, the link between the classical spaces of bounded variation functions and the fractional Sobolev spaces is not clear. The paper by Schmitt and Winkler[21] seems to be a pioneer work to compare the BV space and the fractional Sobolev spaces (in the Gagliardo sense but not in the RL sense). Indeed, the concept of fractional Sobolev spaces

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is not much developed for the RL derivative. One can refer to [6, 14, 15] however. Here we consider the 1D case and the paper is organized as follows: next section is devoted to the presentation of the two main approaches with a special focus on the Riemann-Liouville fractional derivative. We recall the main usual tools in this section. In section 3, we define RL-fractional Sobolev spaces $W^{s,1}(a,b)$ and give basic properties. In the last section, we perform a comparison between these fractional Sobolev spaces and the classical spaces BV and SBV. In particular we prove that $SBV \subset \bigcap_{s\in(0,1)} W^{s,1}$ and $\bigcap_{s\in(0,1)} W^{s,1} \setminus BV \neq \emptyset$.

2 Fractional Calculus and Fractional Sobolev Spaces

In this section we present the two main (different) definitions of fractional Sobolev spaces that we can find in the literature. We are in particular interested to the case where the differentiation order is $s \in [0,1)$ in order to study the fractional spaces between L^1 and $W^{1,1}$ and their relationship with BV. We recall that

$$W^{1,1}(a,b) := \left\{ u \in L^1(a,b) \mid u' \in L^1(a,b) \right\}.$$

In the sequel, we consider the 1D framework; therefore

$$W^{1,1}(a,b) \subset \mathcal{C}(a,b),$$

the space of continuous functions on $[a,b]$ (see [2, 5] for example) and

$$\forall u \in W^{1,1}(a,b), \forall y \in [a,b] \qquad \|u\|_{L^\infty} \leq |u(y)| + \|u'\|_{L^1} .$$

2.1 Gagliardo's fractional Sobolev Spaces

This section is devoted to recalling the classical definition of fractional Sobolev spaces:

Definition 2.1 (Gagliardo's spaces) *Let $s \in (0,1)$. For any $p \in [1,+\infty)$ we define the following space:*

$$W_G^{s,p}(a,b) = \left\{ u \in L^p(a,b) \ : \ \frac{|u(x)-u(y)|}{|x-y|^{\frac{1}{p}+s}} \in L^p([a,b]\times[a,b]) \right\}. \tag{2.1}$$

This is a Banach space endowed with the norm

$$\|u\|_{W_G^{s,p}(a,b)} = \left[\int_{[a,b]} |u(x)|^p dx \ + \ \int_{[a,b]} \int_{[a,b]} \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}} dx\, dy \right]^{\frac{1}{p}}.$$

$W_G^{s,p}(a,b)$ is an intermediary space between $L^p(a,b)$ and $\mathrm{W}^{1,p}(a,b)$ and the term

$$[u]_{W_G^{s,p}(a,b)} = \left[\int_{[a,b]} \int_{[a,b]} \frac{|u(x)-u(y)|^p}{|x-y|^{1+sp}} dx\, dy \right]^{\frac{1}{p}} \tag{2.2}$$

is the so-called Gagliardo semi-norm of u. We have in particular

$$W_G^{s,p}(a,b) \subseteq \mathrm{W}_G^{s',p}(a,b) \qquad \forall \quad 0 < s' < s < 1 .$$

If $s = m + k > 1$ with $m \in \mathbb{N}$, $k \in [0, 1[$, such a definition can be generalized to higher orders by setting

$$W_G^{s,p}(a,b) = \{u \in \mathrm{W}^{m,p}(a,b) \ : \ D^m u \in \mathrm{W}^{k,p}(a,b)\}.$$

This point of view is quite related to interpolation theory (see [2, 4, 10, 13, 22, 24] for example). Note that we are not interested in Hilbert framework so that a Fourier definition is useless here. There is a huge literature concerning these fractional differentiation methods that we cannot mention here.

2.2 Fractional integration and differentiation theory

Another point of view to deal with fractional derivatives is the one we describe in the sequel: the generic definition is the Riemann-Liouville one, though there are many variants that we do not consider. The point of view is different from the Gagliardo one. More precisely, we get a pointwise definition using fractional integrals while the Gagliardo's fractional Sobolev Spaces are defined by interpolation and global approach. As we already mentioned it, we decide to focus on this second type which seems more suitable with respect to applications.

2.2.1 Fractional integrals

In what follows $[a, b]$ is a non empty interval of $\mathbb{R}$. We start by defining the fractional integral for L^1-functions:

Definition 2.2 *Let $f \in L^1([a,b])$. For every $s \in (0,1)$ we define the left-side and right-side Riemann-Liouville fractional integrals by setting*

$$I_{a+}^s(f) : x \geq a \mapsto \frac{1}{\Gamma(s)} \int_a^x \frac{f(t)}{(x-t)^{1-s}} dt,$$

$$I_{b-}^s(f) : x \leq b \mapsto \frac{1}{\Gamma(s)} \int_x^b \frac{f(t)}{(t-x)^{1-s}} dt.$$

Here Γ stands for the classical Gamma function[1]. The properties of left-side and right-side integrals are similar and we prove in the following the main results in the case of the left-side integral $I_{a+}^s(u)$. The fractional integration theory has been extensively studied in [20]. Next proposition recall the main properties of the fractional integral (see Theorem 2.5 and 2.6 in [20]):

Proposition 2.1 *For any $s \in (0,1)$, the followings hold*

(i) the fractional integral is a continuous operator on the Lebesgue spaces:

$$\begin{aligned} &I_{a+}^s : L^p(a,b) \to L^p(a,b) \quad p \geq 1 \\ &\|I_{a+}^s u\|_{L^p(a,b)} \leq C(a,b,s) \|u\|_{L^p(a,b)} . \end{aligned} \tag{2.3}$$

(ii) For every $u \in L^p(a,b)$, with $p \geq 1$, we have

$$\lim_{s \to 0} \|I_{a+}^s u - u\|_{L^p(a,b)} = 0 . \tag{2.4}$$

Next theorem concern the mapping properties of fractional integral on Lebesgue and Hölder spaces:

Proposition 2.2 *For any $s \in (0,1)$, we get*

(i) *I^s_{a+} is a continuous operator from $L^p(a,b)$ into $L^r(a,b)$ for every $p \in [1, 1/s)$ and $r \in [1, p/(1-sp))$;*

(ii) *For every $p > 1/s$ the fractional integral I^s_{a+} is a continuous operator from $L^p(a,b)$ into $\mathcal{C}^{0,s-\frac{1}{p}}(a,b)$;*

(iii) *For $p = 1/s$ the fractional integral I^s_{a+} is a continuous operator from $L^p(a,b)$ into $L^r(a,b)$ with $r \in [1, \infty)$;*

(iv) *the fractional integral I^s_{a+} is a continuous operator from $L^\infty(a,b)$ into $\mathcal{C}^{0,s}(a,b)$.*

Here $\mathcal{C}^{0,s}(a,b)$ denotes the space of Hölder (continuous) functions of order s. For the proofs we refer to [20] ((i): proof of Theorem 3.5 p.66, (ii):Theorem 3.6 p.67, (iii): paragraph 3.3 p.91, (iv): Corollary 2 p.56). The previous proposition shows that the fractional integration improves the function regularity.

Remark 2.1 (Fractional integral of BV-functions) *We point out that to ensure the Hölder-regularity of the fractional integral we need to work with L^p-functions with $p > 1$. The case $p = 1$ is not covered from the previous proposition. However, the point (iii) guarantees such a regularity for bounded functions, which helps to study an important subset of $L^1(a,b)$, namely $BV([a,b])$ (see section 4). Indeed, for dimension one, every function of bounded variation is bounded, so that we get*

$$I^s_{a+}(BV([a,b])) \subset \mathcal{C}^{0,s}(a,b) \qquad \forall\, s \in (0,1)\,. \tag{2.5}$$

Next result states a stronger result for Hölder continuous functions. For the proof we refer to [20] again (Theorem 3.4 p.65, Lemma 13.2 p. 240, and Theorem 13.13 p. 238).

Theorem 2.1 *Let $s \in (0,1)$ and $\alpha > 0$ be such that $s + \alpha < 1$. Then the fractional integral I^s_{a+} is an isomorphism between $\mathcal{C}^{0,\alpha}(a,b)$ and $\mathcal{C}^{0,s+\alpha}(a,b)$. Moreover, for every $f \in \mathcal{C}^{0,s+\alpha}(a,b)$, there exists $u \in \mathcal{C}^{0,s}(a,b)$ such that $f = I^s_{a+}(u)$ and*

$$C\|f\|_{\mathcal{C}^{0,s+\alpha}(a,b)} \leq \|u\|_{\mathcal{C}^{0,s}(a,b)} \leq D\|f\|_{\mathcal{C}^{0,s+\alpha}(a,b)}\,.$$

where C, D are two positive constants.

2.2.2 Fractional derivatives and representability

There are several different definitions of fractional derivatives. We recall next the definition of Riemann-Liouville and Marchaud derivatives and refer to [20, 19] for a deeper analysis of the fractional differentiation theory.

Definition 2.3 (Riemann-Liouville fractional derivative) *Let $u \in L^1(a,b)$ and $n - 1 \leq s < n$ (n integer). The left Riemann-Liouville derivative of u at $x \in [a,b]$ is defined by*

$$D^s_{a+}u(x) = \frac{d^n}{dx^n} I^{n-s}_{a+}u(x) = \frac{1}{\Gamma(n-s)} \frac{d^n}{dx^n} \int_a^x \frac{u(t)}{(x-t)^{s-n+1}} dt \tag{2.6}$$

if the previous term exists. If such a derivative exists then, for $s = 0$, it coincides with the function u at every point.

Similarly, we may define the right Riemann-Liouville derivative of u at $x \in [a, b]$ as

$$D^s_{b-}u(x) = \frac{d^n}{dx^n} I^{n-s}_{b-}u(x) = \frac{1}{\Gamma(n-s)} \frac{d^n}{dx^n} \int_x^b \frac{u(t)}{(t-x)^{s-n+1}} dt \tag{2.7}$$

if the last term exists

Remark 2.2 *We point out that D^s_{b-} is the adjoint operator of D^s_{a+} in the sense to be precised by Theorem 3.7.*

Example 2.1 (Power function) *We consider the function $u(x) = x^k$ $(k \geq 0)$ on $[0, 1]$ and $a = 0$. Then for every $s \in [n-1, n)$ and any $x \in (0, 1)$ the fractional derivative at x is defined as*

$$\begin{aligned} D^s_{0+}u(x) &= \frac{1}{\Gamma(n-s)} \frac{d^n}{dx^n} \int_0^x t^k (x-t)^{n-s-1} dt \\ &= \frac{1}{\Gamma(n-s)} \frac{d^n}{dx^n} \left[x^{n+k-s} \int_0^1 (1-v)^{n-s-1} v^k dv \right] \\ &= \frac{1}{\Gamma(n-s)} \frac{\Gamma(k+1)\Gamma(n-s)}{\Gamma(k+1+n-s)} \frac{d^n}{dx^n} x^{n+k-s} = \frac{\Gamma(k+1)}{\Gamma(k-s+1)} x^{k-s} \end{aligned}$$

where we used the fact that

$$\frac{d^n}{dx^n} x^k = \frac{\Gamma(k+1)}{\Gamma(k-n+1)} x^{k-n} \quad \forall\, k \geq 0\,.$$

If s is an positive integer number, then the fractional derivative coincides with the classical one. Moreover, if $k = 0$ then the fractional derivative is null only if $s \in \mathbb{N}$ (recall that $\Gamma(1-n) = \infty$). Remark also that

$$D^s_{0+} x^{s-k} = 0 \qquad \forall\, s > 0\,, k = 1, ..., 1 + [s]$$

where $[s]$ denotes the integer part of the real number s.

Now we focus on the case $p = 1$.

Definition 2.4 (Representability) *A function $f \in L^1(a, b)$ is said to be represented by a fractional integral if $f \in I^s_{a+}(L^1(a, b))$ for some $s \in (0, 1)$.*

Theorem 2.1 in [20] yields the following important result:

Theorem 2.2 **[L^1-representability]** *Let $f \in L^1(a, b)$. Then $f \in I^s_{a+}(L^1(a, b))$ for some $s \in (0, 1)$ if and only if*

$$I^{1-s}_{a+} f \in W^{1,1}(a, b) \quad \text{and} \quad I^{1-s}_{a+} f(a) = 0\,.$$

Moreover, there exists $u \in L^1(a, b)$ such that $f = I^s_{a+}u$, and it holds

$$u = D^s_{a+} f\,. \tag{2.8}$$

As an immediate consequence of Theorem 2.2, we have the following result:

Corollary 2.1 *Assume $s \in (0,1)$;*

$$\forall u \in L^1(a,b) \qquad D^s_{a+} I^s_{a+} u = u, \tag{2.9}$$

and

$$\forall u \in I^s_{a+}(L^1(a,b)) \qquad I^s_{a+} D^s_a u = u. \tag{2.10}$$

In the previous corollary, (2.9) proves that fractional differentiation can be seen as the inverse operator of the fractional integration.

The converse is not true in general: a counterexample is given by the power function x^{s-k} $(k = 1, ..., 1 + [s])$ whose s-fractional derivative is null (Example 2.1). This is similar to the classical integration and differentiation theories where the integral of u' differs from u for a constant. However, according to (2.10), for every function that can be represented as a fractional integral, the fractional integration acts as the reciprocal operator of the fractional differentiation.

Using Theorem 2.2, it is easy to verify that the power function x^{s-1} is not represented by a fractional integral. In fact $I^{1-s}_{a+} x^{s-k} \equiv \Gamma(s-1)$ so that it belongs to $W^{1,1}(a,b)$ but it does not verify $I^{1-s}_{a+} x^{s-k}(0) = 0$.

Next proposition follows from Theorems 2.1 and 2.2:

Proposition 2.3 *Let $\alpha \geq 1$ and $u \in \mathcal{C}^{0,\alpha}(a,b)$. Then $D^s_{a+} u$ exists for every $s \in [0,\alpha)$ and $D^s_{a+} u \in \mathcal{C}^{0,\alpha-s}(a,b)$.*

Let us give a comment about the relationship between D^s_{a+} and I^s_{a+}. There are two kind of results on the fractional integral that can be very useful to study the properties of the fractional derivative :

- The first-one is a representability result (for instance Theorem 2.2) that gives conditions for a function f to be represented as the fractional integral of an other function u. This is quite important because it allows to easily prove that f admits a fractional derivative with (2.8).

  However, the representability of a function u (i.e., $u = I^s_{0+}\varphi$) is only a sufficient condition to get the existence of the derivative. The power function and the Heaviside function give two examples of functions that are not representable and whose fractional derivative exists (see previous discussion and Example 4.1).

- The second kind of result are embedding results, as (2.5), that give some informations on the regularity of the fractional integral to get the Riemann-Liouville fractional derivative existence.

2.2.3 Marchaud derivative and representability for $p \in (1,\infty)$

The representability result given by Theorem 2.2 can be improved by characterizing the set of the functions $u \in L^p(a,b)$ $(p > 1)$ represented by an other L^p-function

$$u = I^s_{a+}(f) \quad f \in L^p(a,b) \;\; s \in (0,1)\,.$$

In order to address this question we need to introduce a slightly different definition/notion of fractional derivative.

We remark that, for $\mathcal{C}^1$-functions and every $s \in (0,1)$, the use of integration by parts (and writing $u'(t) = (u(t) - u(x))'$), gives

$$D^s_{a+}u(x) = \frac{u(x)}{\Gamma(1-s)(x-a)^s} + \frac{s}{\Gamma(1-s)} \int_a^x \frac{u(x)-u(t)}{(x-t)^{1+s}}\, dt \quad \forall\, x \in (a,b]\,. \tag{2.11}$$

The *Marchaud fractional derivative* is defined as the right term of (2.11). To extend this setting to non-smooth functions we need to define the integral by a limit, which leads to the following definition:

Definition 2.5 (Marchaud fractional derivative) *Let $u \in L^1(a,b)$ and $s \in (0,1)$. The left-side Marchaud derivative of u at $x \in (a,b]$ is defined by*

$$\mathbf{D}^s_{a+}u(x) = \lim_{\varepsilon \to 0} \mathbf{D}^s_{a+,\varepsilon}u(x)$$

with

$$\mathbf{D}^s_{a+,\varepsilon}u(x) = \frac{u(x)}{\Gamma(1-s)(x-a)^s} + \frac{s}{\Gamma(1-s)}\, \psi_\varepsilon(x)$$

and

$$\psi_\varepsilon(x) = \begin{cases} \displaystyle\int_a^{x-\varepsilon} \frac{u(x)-u(t)}{(x-t)^{1+s}}\, dt & \text{if } x \geq a+\varepsilon\,, \\ \displaystyle\int_a^{x-\varepsilon} \frac{u(x)}{(x-t)^{1+s}}\, dt & \text{if } a \leq x \leq a+\varepsilon\,. \end{cases} \tag{2.12}$$

Note that the definition of ψ_ε for $a \leq x \leq a+\varepsilon$ is obtained by continuing the function u by zero beyond the interval $[a,b]$. The passage to the limit depends on the functional space we are working with. In our case we will consider L^p-functions and the limit will be defined in the sense of the L^p-strong topology.

We remark that such a derivative is not defined at $x = a$ and that a necessary condition for the derivative to exist is $u(a) = 0$

The right-side derivative can be defined similarly by using the integral between x and b. In the following we state the main results about Marchaud differentiation for the left-side derivative, but we similar results can be obtained for the right-side one.

As expected,that, the Marchaud and Riemann-Liouville derivative coincide for every $s \in (0,1)$, if u is a $\mathcal{C}^1$-function, and their expression is given by (2.11):

$$\forall\, u \in \mathcal{C}^1([a,b]) \qquad D^s_{a+}u(x) = \mathbf{D}^s_{a+}u(x) \quad \forall\, x \in (a,b].;$$

Next results generalize Theorem 2.2 and Corollary 2.1 :

Theorem 2.3 *Let $s \in (0,1)$ and $p \in (1,\infty)$. For any $f \in L^p(a,b)$, we get $f = I^s_{a+}(u)$ with $u \in L^p(a,b)$ if and only if the limit of the family $\{\psi_\varepsilon\}$ as $\varepsilon \to 0$, where ψ_ε is defined in (2.12) exists (for the L^p norm topology).*

Theorem 2.4 *Let be $f = I^s_{a+}u$ where $u \in L^p(a,b)$ with $p \geq 1$ and $s \in (0,1)$. Then*

$$\mathbf{D}^s_{a+}f = u\,.$$

Proofs can be found in [20] (Theorem 13.2 p. 229 and Theorem 13.1 p. 227, respectively.

Remark 2.3 (Marchaud vs Riemann-Liouville derivative) *We point out that, for every $s \in (0,1)$, we have*

$$\forall\, u \in I^s_{a+}(L^1(a,b)),\ \forall\, x \in (a,b] \quad D^s_{a+}u(x) = \mathbf{D}^s_{a+}u(x),$$

because of Theorems 2.2 and 2.4. With Theorem 2.1, this implies in particular that

$$\forall\, u \in \mathcal{C}^{0,s+\alpha}(a,b),\ \forall x \in (a,b] \qquad D^s_{a+}u(x) = \mathbf{D}^s_{a+}u(x),$$

if $s+\alpha < 1$. This is a useful result in order to study the fractional derivative because the Marchaud derivative it is easier to handle.

3 Riemann-Liouville Fractional Sobolev space ($p = 1$)

In this section, we define the Sobolev spaces associated to the Riemann-Liouville fractional derivative for $p = 1$. The case $p = 1$ is of particular interest since we aim to study the relationship between these spaces and the spaces of functions of bounded variation.

A first possible definition could be given by the following set for $s \in (0,1)$:

$$\{u \in L^1(a,b) \mid D^s_{a+}u \in L^1(a,b)\}\ ,$$

which contains all the L^1-functions such that the -Riemann-Liouville fractional derivative or order s for a given $s \in (0,1)$ belongs to L^1.

Now, we noticed that if the Riemann-Liouville fractional derivative of u exists (for some s), then $I^s_{a+}(u)$ is differentiable almost everywhere. However, we have no information on the differential properties of the fractional integral. These differential properties are not completely described by the pointwise derivative though it exists a.e. (it could be for example a SBV-function). This shows that the previous definition is not suitable to obtain a generalized integration by parts formula.

Therefore, to develop a satisfactory theory of fractional Sobolev spaces we use a more suitable definition in the next section.

3.1 Riemann-Liouville Fractional Sobolev spaces

The Riemann-Liouville Fractional Sobolev spaces are defined as follows:

Definition 3.1 *Let $s \in [0,1)$. We denote by*

$$W^{s,1}_{RL,a+} := \{u \in L^1(a,b) \mid I^{1-s}_{a+}u \in W^{1,1}(a,b)\ \}.$$

A similar space can be defined by using the right-side fractional integral.

Note that this definition does not mean that u is representable but its fractional integral $f = I^{1-s}_{a+}u$ is representable. We shall describe the representable functions of $W^{s,1}_{RL,a+}$ next (it is related to their trace).
We can make the connection with what would be the *natural* definition

Proposition 3.1 *Let $s \in [0,1)$. Then*

$$W^{s,1}_{RL,a+} \subset \{u \in L^1(a,b) \mid D^s_{a+}u \in L^1(a,b)\ \}.$$

Proof. Let be $u \in W^{s,1}_{RL,a+}(\subset L^1(a,b))$. With deefinition 2.3 (with $n = 1$) we get $D^s_{a+}u = \dfrac{dI^{1-s}_{a+}u}{dx}$. In addition, Theorem 1 p. 235 in [12], yields that if $I^{1-s}_{a+}(u) \in W^{1,1}(a,b)$ its Frechet derivative exists a.e. and coincides with its weak derivative.
This means that if $u \in W^{s,1}_{RL,a+}(a,b)$ then the fractional derivative $D^s_{a+}u$ exists a.e. and belongs to $L^1(a,b)$. □

Remark 3.1 *If $I^s_{a+}(u)$ is differentiable at every point and its derivative is a L^1-function, then $u \in W^{s,1}_{RL,a+}(a,b)$. This is due to the fact that if a function is differentiable at every point and its derivative is L^1, then it belongs to $W^{1,1}(a,b)$.*

In the sequel, when there is no ambiguity, we omit the index "RL,a+" in the notation.
Analogously to the usual theory we introduce the following space:

Definition 3.2 *Let $s \in [0,1)$ and $p \in [1,\infty)$. We denote by*

$$W^{s,1}_0(a,b) := \{u \in L^1(a,b) \mid I^{1-s}_{a+}u \in W^{1,1}_{0,a}(a,b)\}.$$

where $W^{1,1}_{0,a}(a,b) = \{v \in W^{1,1}(a,b) \mid v(a) = 0\}$.

Note that $u \in W^{s,1}_0(a,b)$ does not mean that $u(a) = 0$ (it may even not be defined).
The following result is a direct consequence of Theorem 2.2:

Theorem 3.1 *Let $s \in [0,1)$. Then $u \in W^{s,1}_0(a,b)$ if and only if u is L^1-representable.*

Remark 3.2 **Case** $s = 0$**.** *For every $u \in L^1(a,b)$, if $s = 1$ then*

$$I^{1-s}_{a+}u(x) = \int_a^x u(t)dt.$$

Then, as $u \in L^1(a,b)$, we get $I^{1-s}_{a+}u \in W^{1,1}_{0,a}(a,b)$ and

$$D^s_{a+}u(x) = u(x)\,.$$

This proves that

$$L^1(a,b) = W^{s,1}(a,b) = W^{s,1}_0(a,b) \quad \text{if} \quad s = 0\,.$$

Case $s = 1$**.** *This case deserves more attention. Indeed, the integral operator is not well defined for $s = 0$ and $I^{1-s}_{a+}u$ cannot be calculated explicitely if $s = 1$.*

3.2 Main properties

Before performing comparisons between these fractional Sobolev spaces and the spaces of bounded variation function spaces, we investigate their basic properties.

Theorem 3.2 *For any $s \in (0,1)$, the Riemann-Liouville fractional Sobolev space $W^{s,1}(a,b)$ is a Banach space endowed with the norm*

$$\|u\|_{W^{s,1}(a,b)} := \|u\|_{L^1(a,b)} + \|I^{1-s}_a u\|_{W^{1,1}(a,b)}.$$

Proof. It is easy to verify that $\|\cdot\|_{W^{s,1}(a,b)}$ is a norm so that we have just to prove the completeness. Let $(u_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{s,1}(a,b)$ which implies that $(u_n)_{n\in\mathbb{N}}$ and $(I_{a+}^{1-s}u_n)_{n\in\mathbb{N}}$ are Cauchy sequences in $L^1(a,b)$ and $W^{1,1}(a,b)$, respectively. Then there exists $u\in L^1(a,b)$ and $v\in W^{1,1}(a,b)$ such that

$$u_n \overset{L^1}{\to} u\ , \qquad I_{a+}^{1-s}u_n \overset{W^{1,1}}{\to} v$$

Because of the definition we have $I_{a+}^{1-s}(u_n)\in W^{1,1}(a,b)$, and, because of (2.3), as $u_n\to u$ strongly in L^1, we have $I_{a+}^{1-s}(u_n)\to I_{a+}^{1-s}(u)$ strongly in L^1 as well. This proves that $I_{a+}^{1-s}(u)=v$.

Now, we have to prove the (strong) L^1 convergence of the first distributional derivative of $(I_{a+}^{1-s}(u_n))'$ toward the derivative of $I_{a+}^{1-s}(u)$ that is : $v'=(I_{a+}^{1-s}(u))'$, where v' denotes the weak derivative of v.
For every $\varphi\in C_c^\infty(a,b)$ we get

$$\int_a^b \varphi\,(I_{a+}^{1-s}(u_n))' = -\int_a^b \varphi' I_{a+}^{1-s}(u_n)$$

and, by taking the limit, we obtain

$$\int_a^b \varphi\, v' = -\int_a^b \varphi' I_{a+}^{1-s}(u).$$

This proves that $I_{a+}^{1-s}(u)$ is a $W^{1,1}$-function and its weak derivative is v'. This implies that $I_{a+}^{1-s}u_n \overset{W^{1,1}}{\to} I_{a+}^{1-s}u$ and this gives the result. □
An immediate consequence of this result is the following theorem

Theorem 3.3 *$W_0^{s,1}(a,b)$ equipped with the norm*

$$\|u\|_{W_0^{s,1}(a,b)} := \|u\|_{L^1(a,b)} + \|D_{a+}^s u\|_{L^1(a,b)}$$

is a Banach space.

Proof. The space $W_0^{s,1}(a,b)$ is complete with respect to the norm $\|\cdot\|_{W^{s,1}(a,b)}$. This is done as in the previous proof since $W_{0,a}^{1,1}(a,b)=\{u\in W^{1,1}(a,b)\mid u(a)=0\}$ is a Banach space.

Moreover, with Poincaré's inequality for spaces $W_{0,a}^{1,1}(a,b)$, the norms $\|\cdot\|_{W^{s,1}(a,b)}$ and $\|\cdot\|_{W_0^{s,1}(a,b)}$ are equivalent. □

Theorem 3.4 *Assume $0<s<s'<1$ and consider $u\in I_{a+}^{s'}(L^1(a,b))$. Then $u\in W_0^{s,1}(a,b)$ and*

$$\|u\|_{W^{s,1}(a,b)} \le C_{s,s'}\|u\|_{W^{s',1}(a,b)}\,.$$

Proof. Let be $u\in I_{a+}^{s'}(L^1(a,b))$. Then $u=I_{a+}^{s'}f$ with $f\in L^1(a,b)$. So with Theorem 2.5 p.46 in [20] we get $u=I_{a+}^{s'}f=I_{a+}^{s}(I_{a+}^{s'-s}f)$. Therefore $u\in I_{a+}^{s}(L^1(a,b))$ for every $s\in(0,s']$.
As u is represented by a fractional integral of a L^1-function, then with Theorem 3.1, $u\in W_0^{s,1}(a,b)$ for every $s\in(0,s']$. As $f\in L^1(a,b)$, Corollary 2.1 gives

$$\|D_{a+}^{s'}u\|_{L^1} = \|D_{a+}^{s'}I_a^{s'}f\|_{L^1} = \|f\|_{L^1}\ .$$

Similarly, u is represented by $g = I_{a+}^{s'-s} f$ and

$$\|D_{a+}^s u\|_{L^1} = \|g\|_{L^1} = \|I_{a+}^{s'-s} f\|_{L^1} \ .$$

Moreover, with the continuity of the fractional integral operator we get

$$\|I_{a+}^{s'-s} f\|_{L^1} \leq C_{s,s'} \|f\|_{L^1},$$

where $C_{s,s'}$ is a generic constant depending on s et s'. Finally

$$\|D_{a+}^s u\|_{L^1} \leq C_{s,s'} \|D_{a+}^{s'} u\|_{L^1} \ .$$

This proves the result. □
Next theorem gives a relationship between Riemann-Liouville Sobolev spaces and Gagliardo Sobolev spaces:

Theorem 3.5 *Let be $s, s' \in (0,1)$ such that $s' > s$. Then*

$$W_G^{s',1}(a,b) \cap I_{a+}^s(L^1(a,b)) \subset W_{RL,a+}^{s,1}(a,b)$$

with continuous injection. More precisely,

$$\forall u \in W_G^{s',1}(a,b) \cap I_{a+}^s(L^1(a,b)) \qquad \|u\|_{W_{RL,a+}^{s,1}(a,b)} \leq C \|u\|_{W_G^{s',1}(a,b)} .$$

Proof. Let us choose $s \in (0,1)$ and $s' > s$ (in $(0,1)$). Let be $u \in I_{a+}^s(L^1(a,b))$. It is represented by a fractional integral of a L^1-function, so its Riemann-Liouville and Marchaud derivative coincide. The norm of the fractional derivative can be evaluated by the Marchaud derivative. Recall that if $a \leq x \leq a + \varepsilon$ we get

$$\mathbf{D}_{a+,\varepsilon}^s u(x) = \frac{u(x)}{\varepsilon^s \Gamma(1-s)},$$

and if $x \geq a + \varepsilon$

$$\mathbf{D}_{a+,\varepsilon}^s u(x) = \frac{1}{\Gamma(1-s)} \frac{u(x)}{(x-a)^s} + \frac{s}{\Gamma(1-s)} \int_a^{x-\varepsilon} \frac{u(x)}{(x-t)^{1+s}} \, dt.$$

$$\begin{aligned} \|\mathbf{D}_{a+,\varepsilon}^s u\|_{L^1(a+\varepsilon,b)} \leq & \frac{1}{\Gamma(1-s)} \int_{a+\varepsilon}^b \frac{|u(x)|}{(x-a)^s} ds + \frac{s}{\Gamma(1-s)} \int_{a+\varepsilon}^b \int_a^{x-\varepsilon} \frac{|u(x)-u(t)|}{|x-t|^{1+s}} \, dt \, dx \\ \leq & \frac{1}{\Gamma(1-s)} \int_{a+\varepsilon}^b \frac{|u(x)|}{(x-a)^s} ds + \frac{s}{\Gamma(1-s)} [u]_{W_G^{s,1}(a,b)}. \end{aligned}$$

where the Gagliardo semi-norm $[u]_{W_G^{s,1}(a,b)}$ is given by (2.2). Moreover

$$\|\mathbf{D}_{a+,\varepsilon}^s u\|_{L^1(a,a+\varepsilon)} \leq \frac{1}{\varepsilon^s \Gamma(1-s)} \int_a^{a+\varepsilon} |u(x)| ds;$$

finally

$$\|\mathbf{D}_{a+,\varepsilon}^s u\|_{L^1(a,b)} \leq \frac{1}{\Gamma(1-s)} \left(\int_{a+\varepsilon}^b \frac{|u(x)|}{(x-a)^s} ds + \varepsilon^{-s} \int_a^{a+\varepsilon} |u(x)| ds + s[u]_{W_G^{s,1}(a,b)} \right). \quad (3.1)$$

Now, we know ([11] - section 6. for example) that

$$W_G^{s',1}(a,b) \subset L^p(a,b)$$

with continuous injection and $p \in [1, \frac{1}{1-s'})$. As $u \in W_G^{s',1}(a,b)$, then $u \in L^p(a,b)$ where p can be chosen such as $\frac{1}{1-s} < p < \frac{1}{1-s'}$ since that $s' > s$. Let us call p^* the conjugate exponent and apply Hölder inequalities to relation 3.1. Note that p^* satisfies $1 - sp^* > 0$. We get

$$\begin{aligned}
\|\mathbf{D}^s_{a+,\varepsilon} u\|_{L^1(a,b)} \leq & \frac{\|u\|_{L^p(a+\varepsilon,b)}}{\Gamma(1-s)} \left[\frac{(b-a)^{1-sp^*} - \varepsilon^{1-sp^*}}{1-sp^*} \right]^{1/p^*} \\
& + \frac{\|u\|_{L^p(a,a+\varepsilon)}}{\Gamma(1-s)} \varepsilon^{(1-sp^*)/p^*} \\
& + \frac{s}{\Gamma(1-s)} [u]_{W_G^{s,1}(a,b)} \\
\leq & \frac{\|u\|_{L^p}}{\Gamma(1-s)} \left(\left[\frac{(b-a)^{1-sp^*} - \varepsilon^{1-sp^*}}{1-sp^*} \right]^{1/p^*} + \varepsilon^{(1-sp^*)/p^*} \right) \\
& + \frac{s}{\Gamma(1-s)} [u]_{W_G^{s,1}(a,b)} \, .
\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ gives

$$\|\mathbf{D}^s_{a+} u\|_{L^1(a,b)} \leq \frac{\|u\|_{L^p}}{\Gamma(1-s)} \left(\frac{(b-a)^{(1-sp^*)}}{1-sp^*} \right)^{1/p^*} + \frac{s}{\Gamma(1-s)} [u]_{W_G^{s,1}(a,b)} \, .$$

As $\|u\|_{L^p} \leq C \|u\|_{W_G^{s',1}(a,b)}$ and

$$[u]_{W_G^{s,1}(a,b)} \leq [u]_{W_G^{s',1}(a,b)} \leq \|u\|_{W_G^{s',1}(a,b)}$$

we finally get

$$\|\mathbf{D}^s_{a+} u\|_{L^1(a,b)} \leq C(s,s',a,b) \|u\|_{W_G^{s',1}(a,b)} \, .$$

This ends the proof. □

Note that $s = s'$ is the critical case in the above theorem. We cannot handle this case with the same arguments.

3.3 Integration by parts and relationship with $W^{1,1}(a,b)$

We have the following integration by parts formula ([6] Theorem 2).

Theorem 3.6 *If $0 \leq \frac{1}{p} < s < 1$ and $0 \leq \frac{1}{r} < s < 1$, then for every $u \in W^{s,1}_{RL,a+}(a,b)$ and $v \in W^{s,1}_{RL,b-}(a,b)$ we get*

$$\int_a^b (D^s_{a+} u)(t) v(t) \, dt = \int_a^b (D^s_{b-} v)(t) u(t) \, dt + u(b)(I^{1-s}_{b-} v)(b) - (I^{1-s}_{a+} u)(a) v(a) \, . \qquad (3.2)$$

Here

$$W^{s,1}_{RL,b-} := \{u \in L^1(a,b) \mid I^{1-s}_{b-}u \in W^{1,1}(a,b)\}.$$

This is a generalization of the following proposition proving that D^s_{a+} and D^s_{b-} are adjoints operators on the set of functions represented by fractional integral (see Corollary 2 p. 46 in [20])

Theorem 3.7 *Let $u \in I^s_{a+}(L^p(a,b))$ and $v \in I^s_{b-}(L^q(a,b))$ with $p,q \in [1,+\infty[$, $1/p+1/q \leq 1+s$ and $p,q \neq 1$ if $1/p+1/q = 1+s$. Then*

$$\int_a^b (D^s_{a+}u)(t)v(t)\,dt = \int_a^b (D^s_{b-}v)(t)u(t)\,dt\,.$$

Example 3.1 (Smooth functions) *Set $[a,b] = [0,1]$. For every $u \in C^\infty([0,1],\mathbb{R})$ and for every $s \in [0,1)$ we get for any $x \neq 0$:*

$$\begin{aligned}
D^s_{0+}u(x) &= \frac{1}{\Gamma(1-s)}\frac{d}{dx}\int_0^x u(t)(x-t)^{-s}dt \\
&= \frac{1}{(1-s)\Gamma(1-s)}\frac{d}{dx}\left[u(0)x^{1-s} + \int_0^x u'(t)(x-t)^{1-s}dt\right] \\
&= \frac{1}{\Gamma(1-s)}\left[u(0)x^{-s} + \int_0^x u'(t)(x-t)^{-s}dt\right] \\
&= \frac{1}{\Gamma(1-s)}u(0)x^{-s} + \frac{1}{\Gamma(2-s)}u'(0)x^{1-s} + \frac{1}{\Gamma(2-s)}\int_0^x u''(t)(x-t)^{1-s}dt
\end{aligned}$$

since $\Gamma(z+1) = z\Gamma(z)$ for every $z > 0$. Note that, in order to make such a derivative well defined at $x = 0$, we have to suppose $u(0) = 0$.

As $x \mapsto x^{-s}$ belongs to $L^1(0,1)$ and u'' is bounded and $s \in [0,1)$, we get that $D^s_{0+}u$ belongs to $L^1(0,1)$. Moreover, we have

$$\begin{aligned}
\|u' - D^s_{0+}u\|_{L^1([0,1],\mathbb{R})} &= \int_0^1 \left|u'(x) - \frac{u(0)x^{-s}}{\Gamma(1-s)} - \frac{u'(0)x^{1-s}}{\Gamma(2-s)} - \int_0^x \frac{u''(t)(x-t)^{1-s}}{\Gamma(2-s)}dt\right| dx \\
&\leq \frac{|u(0)|}{(1-s)\Gamma(1-s)} + \int_0^1 \left|u'(x) - \frac{u'(0)x^{1-s}}{\Gamma(2-s)} - \int_0^x \frac{u''(t)(x-t)^{1-s}}{\Gamma(2-s)}dt\right| dx \\
&\leq \frac{|u(0)|}{\Gamma(2-s)} + \int_0^1 \left|u'(x) - \frac{u'(0)x^{1-s}}{\Gamma(2-s)} - \int_0^x \frac{u''(t)(x-t)^{1-s}}{\Gamma(2-s)}dt\right| dx
\end{aligned}$$

Since $\Gamma(1) = 1$ by the Lebesgue convergence theorem, we get

$$\|u' - D^s_{0+}u\|_{L^1([0,1],\mathbb{R})} \overset{s\to 1}{\longrightarrow} |u(0)| + \int_0^1 \left|u'(x) - u'(0) - \int_0^x u''(t)dt\right| dx = |u(0)|\,.$$

Moreover, if $u(0) = 0$

$$\forall x \in [0,1] \qquad \lim_{s\to 1} D^s_{0+}(u)(x) = u'(x),$$

and not only almost everywhere.

Now, we compare the usual Sobolev space $W^{1,1}(a,b)$ and the Riemann-Liouville fractional Sobolev spaces.

Theorem 3.8 *The space $W^{1,1}(a,b)$ is continuously embedded in $W^{s,1}_{RL,a+}(a,b)$ for any $s \in [0,1)$ and*

$$D^s_a u \rightharpoonup u'\mathcal{L}^1 + u(a)\delta_a \quad as \quad s \to 1\,,$$

weakly in $\mathcal{M}(a,b)$ (the space of Radon measures on (a,b)). Here δ_a is the Dirac measure at a and $\mathcal{L}^1$ denotes the 1D Lebesgue measure. Moreover, if $u(a) = 0$ we have

$$D^s_{a+} u \overset{L^1(a,b)}{\longrightarrow} u' \quad as \quad s \to 1.$$

Proof. According to Lemma 2.1 p. 32 in [20] and its corollary, every $u \in W^{1,1}(a,b)$ admits a fractional derivative for every $s \in (0,1)$. The fractional integral and derivatives are given by

$$I^{1-s}_{a+} u(x) = \frac{1}{\Gamma(2-s)} \left[u(a)\,(x-a)^{1-s} + \int_a^x u'(t)(x-t)^{1-s} dt \right] \tag{3.3}$$

and

$$\begin{aligned} \forall\, x \in [a,b] \quad D^s_{a+} u(x) &= \frac{1}{\Gamma(1-s)} \left[u(a)\,(x-a)^{-s} + \int_a^x u'(t)(x-t)^{-s} dt \right] \\ &= \frac{u(a)(x-a)^{-s}}{\Gamma(1-s)} + I^{1-s}_a u'(x)\,. \end{aligned} \tag{3.4}$$

As $s \in (0,1)$ with (2.3) we get that the fractional derivative is defined at every point of $(a,b]$, belongs to $L^1(a,b)$, and

$$\begin{aligned} \|D^s_{a+} u\|_{L^1(a,b)} &\leq \frac{|u(a)|(b-a)^{1-s}}{\Gamma(2-s)} + \|I^{1-s}_{a+} u'\|_{L^1}(a,b) \\ &\leq C\left(|u(a)| + \|u'\|_{L^1}(a,b) \right). \end{aligned}$$

As $W^{1,1}(a,b)$ is continuously embedded in $\mathcal{C}([a,b])$(see [2, 7]), we get $|u(a)| \leq \|u\|_\infty \leq C\|u\|_{W^{1,1}}$ and

$$\|D^s_{a+} u\|_{L^1(a,b)} \leq C\|u\|_{W^{1,1}}$$

which proves the continuous embedding. (Here C is a generic constant that only depends on a, b and s).
Now, if $u(a) = 0$, then

$$\forall\, x \in [a,b] \quad D^s_{a+} u(x) = I^{1-s}_{a+} u'(x)\,. \tag{3.5}$$

Then the strong convergence of the derivative follows from (2.4).
Otherwise, because of (2.4), $I^{1-s}_{a+} u'$ weakly star converges to u' in $L^1(a,b)$. Moreover, we verify that

$$\frac{u(a)\,(x-a)^{-s}}{\Gamma(1-s)} \overset{*}{\rightharpoonup} u(a)\delta_a$$

$L^1(a,b)$ which implies that $D^s_{a+} u$ weakly star converges to $u' + u(a)\delta_a$ in $\mathcal{M}(a,b)$. Indeed, for every $\varphi \in \mathcal{C}^1(a,b)$

$$\frac{u(a)}{\Gamma(1-s)} \int_a^b (x-a)^{-s} \varphi(x) dx = \frac{u(a)}{\Gamma(2-s)} \left[\varphi(b)(b-a)^{1-s} - \int_a^b (x-a)^{1-s} \varphi'(x) dx \right] \overset{s\to 1}{\to} u(a)\varphi(a).$$

We conclude with the density of the $\mathcal{C}^1$ functions in the space of continuous functions. □

Remark 3.3 *We give an example of a smooth function u with $u(a) \neq 0$ and such that the fractional derivative does not converge (as $s \to 1$) to u' strongly in L^1.*
It suffices to consider $u(x) = 1$ on $[a, b]$. Then

$$u' \equiv 0 \,, \qquad D^s_{a+}u(x) = \frac{(x-a)^{-s}}{\Gamma(1-s)}$$

and

$$\|D^s_{a+}u\|_{L^1(a,b)} \;=\; \frac{(b-a)^{1-s}}{\Gamma(2-s)} \;\stackrel{s\to 1}{\longrightarrow}\; 1 \;\neq\; \|u'\|_{L^1(a,b)} \,.$$

4 Comparison with BV and SBV

Let us recall the definition and the main properties of the space of functions of bounded variation (see [3, 5] for example), defined by

$$BV(a,b) = \{u \in L^1(\Omega) \mid TV(u) < +\infty\},$$

where Ω is a bounded, open subset of $\mathbb{R}^d$ and

$$TV(u) := \sup \left\{ \int_\Omega u(x) \operatorname{div} \xi(x)\, dx \mid \xi \in \mathcal{C}^1_c(\Omega),\ \|\xi\|_\infty \leq 1 \right\}. \tag{4.1}$$

The space $BV(a,b)$, endowed with the norm $\|u\|_{BV(a,b)} = \|u\|_{L^1} + TV(u)$, is a Banach space. The derivative in the sense of distributions of every $u \in BV(a,b)$ is a bounded Radon measure, denoted Du, and $TV(u) = \int_\Omega |Du|$ is the total variation of u. We next recall standard properties of functions of bounded variation in the case $d = 1$.

Proposition 4.1 *Let (a,b) be an open subset of $\mathbb{R}$.*

1. *For every $u \in BV(a,b)$, the Radon measure Du can be decomposed into $Du = \nabla u\, dx + D^s u$, where $\nabla u\, dx$ is the absolutely continuous part of Du with respect of the Lebesgue measure and $D^s u$ is the singular part.*
2. *The mapping $u \mapsto TV(u)$ is lower semi-continuous from $BV(a,b)$ to $\mathbb{R}^+$ for the $L^1(a,b)$ topology.*
3. *$BV(a,b) \subset L^\sigma(a,b)$ with continuous embedding, for $\sigma \in [1, \infty]$.*
4. *$BV(a,b) \subset L^\sigma(a,b)$ with compact embedding, for $\sigma \in [1, \infty)$.*

The singular part $D^s u$ of the derivative has a jump part and a Cantor component. The $SBV(\Omega)$ space (see [3] for example) is the space of functions in $BV(\Omega)$ whose derivative has no singular Cantor component. The functions of $SBV(a,b)$ have two components: one is regular and belongs to $W^{1,1}(a,b)$ and the other one is a countable summation of characteristic functions. More precisely, any increasing function in $SBV(a,b)$ can be written as

$$u(x) = u(a) + \int_a^x u'(t)dt + \sum_{x_k \in J_u} p_k \chi_{[x_k,1]}(x) \qquad x \in [a,b]$$

where J_u denotes the (at most countable) set of jump points of u and $p_k = u^+(x_k) - u^-(x_k)$ denotes the positive jump of u at x_k. This describes all the functions of $SBV(a,b)$ since any BV-function can be written as the difference of two increasing functions.

Next example shows that there exists a SBV-function that belongs to $W^{s,1}_{RL,a+}(a,b)$ for any $s \in (0,1)$. This confirms the regularizing behavior of the fractional integral operator and represents a preliminary result in order to prove the relationship between SBV function and fractional Sobolev space.

Example 4.1 (Heaviside function) *Let $u : [0,1] \to \mathbb{R}$ $(a = 0)$, $u(x) = \chi_{[\alpha,1]}(x)$ with $\alpha \in (0,1)$. We consider $s \in [0,1)$. For every $x \in [\alpha,1]$ we get*

$$I^{1-s}_{0+}u(x) = \begin{cases} 0 & \text{if} \quad x \in [0,\alpha) \\ \dfrac{(x-\alpha)^{1-s}}{\Gamma(2-s)} & \text{if} \quad x \in [\alpha,1] \end{cases} \tag{4.2}$$

which proves that $I^{1-s}_{0+}u \in W^{1,1}([0,1])$ so that $u \in W^{s,1}_{RL,0+}([0,1])$.

The fractional derivative is given by

$$D^{s}_{0+}u(x) = \begin{cases} 0 & \text{if} \quad x \in [0,\alpha] \\ \dfrac{(x-\alpha)^{-s}}{\Gamma(1-s)} & \text{if} \quad x \in (\alpha,1] \end{cases} \tag{4.3}$$

and, for every $s \in (0,1)$, we have that

$$\int_\alpha^1 \frac{(x-\alpha)^{-s}}{\Gamma(1-s)} = \left[\frac{(x-\alpha)^{1-s}}{\Gamma(2-s)}\right]_\alpha^1 = \frac{(1-\alpha)^{1-s}}{\Gamma(2-s)}$$

which implies that

$$\|D^{s}_{0+}u\|_{L^1([0,1])} \overset{s\to 1}{\longrightarrow} 1 = |Du|([0,1]) \tag{4.4}$$

where $|Du|([0,1])$ denotes the total variation of u on $[0,1]$.

Next result is useful to prove Theorem 4.1:

Lemma 4.1 *Let $\{f_k\} \subset W^{1,1}(a,b)$ a sequence of non-negative functions with $f_k(a) = 0$ and with non-negative derivative. We suppose also that*

$$\sum_k f_k\,,\ \sum_k f_k' \in L^1(a,b)\,. \tag{4.5}$$

Then

$$\left(\sum_k f_k(x)\right)' = \sum_k f_k'(x) \qquad \text{a.e. } x \in [a,b]\,. \tag{4.6}$$

Proof. The result follows from the monotone convergence theorem and the hypothesis on f_k. In fact, for every $x \in [a,b]$ we get

$$\sum_k f_k(x) = \sum_k \int_a^x f_k'(t)\,dt = \int_a^x \sum_k f_k'(t)\,dt\,,$$

which proves that $\sum_k f_k \in W^{1,1}(a,b)$ and (4.6) follows. □

Theorem 4.1 *For every $s \in (0,1)$, it holds that $SBV(a,b) \subset W^{s,1}_{RL,a+}(a,b)$ and*

$$D^s_{a+}u \overset{\mathcal{M}(a,b)}{\rightharpoonup} u'\mathcal{L}^1 + u(a^+)\delta_a + \sum_{x_k \in J_u} p_k \delta_{x_k} \quad \text{as} \quad s \to 1\,.$$

Moreover, if $u(a^+) = 0$ then

$$\|D^s_{a+}u\|_{L^1(a,b)} \to \|u\|_{SBV(a,b)} \quad \text{as} \quad s \to 1\,,$$

where $u(a^+)$ denotes the right limit of u at zero and $J_u \subset (a,b)$ is the jump set of u.

Proof. With a simple change of variables, we can assume that $[a,b] = [0,1]$. This will make the proof easier to read. Every BV-function can be written as the difference of two increasing functions. Then, in the following we prove the result for a SBV-increasing function. Every SBV-function u can be written as

$$u(x) = u(0^+) + \int_0^x u'(t)dt + \sum_{x_k \in J_u} p_k \chi_{[x_k,1]}(x) \qquad x \in [0,1]$$

where $u(0^+)$ denotes the right limit of u at zero, J_u denotes the (at most countable) set of jump points of u and $p_k = u^+(x_k) - u^-(x_k)$ denotes the positive jump of u at x_k. In particular

$$\|u\|_{SBV([0,1])} = |u(0^+)| + \int_0^1 |u'(t)|dt + \sum_{x_k \in J_u} p_k\,.$$

In particular, u can be written as the sum of two functions

$$\begin{aligned} u(x) &= f(x) + g(x)\,,\\ f(x) &= u(0^+) + \int_0^x u'(t)dt\,,\\ g(x) &= \sum_{x_k \in J_u} p_k \chi_{[x_k,1]}(x)\,. \end{aligned}$$

Now f belongs to $W^{1,1}$ and using Theorem 3.8, its fractional derivative is given by

$$f_s(x) = \frac{u(0^+)x^{-s}}{\Gamma(1-s)} + I^{1-s}_{0+}u'(x)\,.$$

So we have $f_s \in L^1([0,1])$ and

$$\begin{array}{lllll} \text{if } u(0^+) = 0 & f_s \overset{L^1(0,1)}{\longrightarrow} u' & & \text{as} & s \to 1\,,\\ \text{otherwise} & f_s \overset{\mathcal{M}(0,1)}{\rightharpoonup} u'\mathcal{L}^1 + u(0^+)\delta_0 & & \text{as} & s \to 1\,. \end{array} \tag{4.7}$$

Concerning g, by the monotone convergence theorem, we have

$$I^{1-s}_{0+}g = \sum_{x_k \in J_u} I^{1-s}_{0+}(p_k \chi_{[x_k,1]})(x)\,.$$

Now, because of (4.2) and (4.3), the previous series verify the hypothesis of Lemma 4.1. Condition (4.5) can be verified by observing that

$$\sum_{x_k \in J_u} I^{1-s}_{0+}(p_k \chi_{[x_k,1]}) \le \sum_{x_k \in J_u} \frac{p_k}{\Gamma(2-s)}$$

and $\sum_{x_k \in J_u} D^s_{0+} p_k \chi_{[x_k,1]}$ is well defined for every $x \notin J_u$ and

$$\int_0^1 \sum_{x_k \in J_u} D^s_{0+} p_k \chi_{[x_k,1]}(t)\, dt = \sum_{x_k \in J_u} I^{1-s}_{0+}(p_k \chi_{[x_k,1]})(1) \leq \sum_{x_k \in J_u} \frac{p_k}{\Gamma(2-s)}\,.$$

With the previous proposition, the fractional derivative of g is given by

$$g_s(x) = (I^{1-s}_{0+} g)' = \sum_{x_k \in J_u} \frac{(x-x_k)^{-s}}{\Gamma(1-s)} \chi_{[x_k,1]}\,.$$

Moreover, by applying the monotone convergence theorem, we get

$$\|g_s\|_{L^1(a,b)} = \sum_{x_k \in J_u} p_k \int_{x_k}^1 \frac{(t-x_k)^{-s}}{\Gamma(1-s)} dt = \sum_{x_k \in J_u} p_k \frac{(1-x_k)^{1-s}}{\Gamma(2-s)}$$

and, similarly to (4.4) and using the fact that the series in the right-side is normally convergent, we have

$$\|g_s\|_{L^1(a,b)} \overset{s\to 1}{\longrightarrow} \sum_{x_k \in J_u} p_k\,. \tag{4.8}$$

The *-weak convergence of g_s towards $\sum_{x_k \in J_u} p_k \delta_{x_k}$ is obtained by the same arguments by writing every function test as the sum of its positive and negative part. Finally, as the fractional derivative is a linear operator, we get

$$D^s_{0+} u(x) = f_s(x) + g_s(x) \quad \forall\, x \in [0,1]\,.$$

Then, $D^s_{0+} u$ is defined at every point and belongs to $L^1(a,b)$, which implies that $u \in W^{s,1}_{RL,0+}([0,1])$ for every $s \in (0,1)$. The result ensues from (4.7) and (4.8). □

The next two remarks point out that the fractional Sobolev spaces are larger than SBV and give some relationship between BV and $W^{s,1}$.

Remark 4.1 (Cantor function) *The Cantor function is an example of increasing continuous function on* $[0,1]$ *whose standard derivative is defined and null at a.e. point. It is well known that such a function is of bounded variation but is not a* SBV*-function. More precisely such a function is Hölder-continuous with exponent* $\alpha = \ln 2/\ln 3$ *(i.e., the Hausdorff dimension of the Cantor set). Then, with Proposition 2.3 the Riemann-Liouville derivative is well defined at every point for every* $s \in (0,\alpha)$*. Moreover the Riemann-Liouville derivative of order* s *belongs to* $\mathcal{C}^{0,\alpha-s}([0,1])$*, and we get that the Cantor function belongs to* $W^{s,1}_{RL,0+}([0,1])$ *for every* $s \in (0,\alpha)$*. This proves in particular that in general*

$$(BV \setminus SBV) \cap W^{s,1}_{RL,0+} \neq \emptyset\,.$$

Remark 4.2 (Fractional vs a.e. differentiation) *In [19] the authors investigate the relationship between usual a.e. differentiation and the fractional Riemann-Liouville derivative definition. Several interesting examples are given. One of them is given by the Weierstrass function defined as*

$$W(x) = \sum_{n=0}^{\infty} q^{-n}(e^{iq^n x} - e^{iq^n a}) \qquad x \in [a,b]$$

where $q > 1$. It is proved that W has continuous and bounded fractional Riemann-Liouville derivatives of all orders $s < 1$. However, since W is nowhere differentiable it cannot be of bounded variation.

This implies that Riemann-Liouville fractional Sobolev spaces are larger than $BV([a,b])$. Then we can state

$$SBV \subset \bigcap_{s\in(0,1)} W^{s,1}\,, \qquad \bigcap_{s\in(0,1)} W^{s,1} \setminus BV \neq \emptyset\,.$$

5 Conclusion

In this paper we try to make connections between the two main definitions of fractional derivatives : the *local* (pointwise) one whose typical representation is the RL derivative and the *global* one which is typically the Gagliardo one. In view of a more precise description of the derivative of order $s \in (0,1)$ with respect to BV functions we have also proved preliminary results to compare $SBV(a,b)$ and $W^{s,1}(a,b)$. Open problems are numerous. In particular, it remains to strongly connect the Riemann-Liouville theory with the Gagliardo one. This would allow to perform comparison between $W^{s,1}(a,b)$ and Besov-spaces for example. In addition, we have to understand precisely how $W^{s,1}(a,b)$ behaves with respect to $BV(a,b)$ to get some information about the $BV\backslash W^{1,1}$ functions. In particular, we proved that $SBV(a,b) \subset W^{s,1}(a,b)$ and exhibit a function in $W^{s,1}(a,b)$, that does not belong to $BV(a,b)$; however we still don't know if $BV(a,b) \subsetneq W^{s,1}(a,b)$ (can we find a function in $BV(a,b)\backslash W^{s,1}(a,b)$?) In addition, it remains to prove density results and continuity/compactness results in view of variational models involving the RL derivative.

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