Some hyperbolic three-manifolds that bound geometrically

Alexander Kolpakov, Bruno Martelli, Steven Tschantz

SOME HYPERBOLIC THREE-MANIFOLDS THAT BOUND GEOMETRICALLY

ALEXANDER KOLPAKOV, BRUNO MARTELLI, AND STEVEN TSCHANTZ

ABSTRACT. A closed connected hyperbolic n-manifold *bounds geometrically* if it is isometric to the geodesic boundary of a compact hyperbolic $(n+1)$-manifold. A. Reid and D. Long have shown by arithmetic methods the existence of infinitely many manifolds that bound geometrically in every dimension.

We construct here infinitely many explicit examples in dimension $n = 3$ using right-angled dodecahedra and 120-cells and a simple colouring technique introduced by M. Davis and T. Januszkiewicz. Namely, for every $k \geqslant 1$, we build an orientable compact closed 3-manifold tessellated by $16k$ right-angled dodecahedra that bounds a 4-manifold tessellated by $32k$ right-angled 120-cells.

A notable feature of this family is that the ratio between the volumes of the 4-manifolds and their boundary components is constant and, in particular, bounded.

1. INTRODUCTION

The study of hyperbolic manifolds that bound geometrically dates back to the works of D. Long, A. Reid [11, 12] and B. Niemershiem [17], motivated by a preceding work of M. Gromov [7, 8] and a question by F. Farrell and S. Zdravkovska [4]. This question is also related to hyperbolic instantons, as described by J. Ratcliffe and S. Tschantz [19, 20]. In particular, the following problems are of particular interest:

Question 1.1. Which compact orientable hyperbolic n-manifold $\mathscr{N}$ can represent the totally geodesic boundary of a compact orientable hyperbolic $(n+1)$-manifold $\mathscr{M}$?

Question 1.2. Which compact orientable flat n-manifold $\mathscr{N}$ can represent the cusp section of a single-cusped orientable hyperbolic $(n+1)$-manifold $\mathscr{M}$?

Once there exist such manifolds $\mathscr{N}$ and $\mathscr{M}$, we say that $\mathscr{N}$ *bounds* $\mathscr{M}$ *geometrically*. In this note, we shall concentrate on Question 1.1, devoted to compact geometric boundaries. The recent progress on Question 1.2, that involves cusp sections, is indicated by [10, 12, 14, 15]. However, this is still

The first named author is supported by the SNSF researcher scholarship P300P2-151316. The second named author was supported by the Italian FIRB project "Geometry and topology of low-dimensional manifolds", RBFR10GHHH.

an open problem in dimensions ≥ 5. On the other hand, by a result of M. Stover [21], an arithmetic orbifold in dimension ≥ 30 cannot have a single cusp.

In [11], D. Long and A. Reid have shown that many closed hyperbolic 3-manifolds do not bound geometrically: a necessary condition is that the eta invariant of the 3-manifold must be an integer. The first known closed hyperbolic 3-manifold that bounds geometrically was constructed by J. Ratcliffe and S. Tschantz in [19] and has volume of order 200.

Then, D. Long and A. Reid produced in [13], by arithmetic techniques, infinitely many orientable hyperbolic n-manifolds $\mathscr{N}$ that bound geometrically an $(n+1)$-manifold $\mathscr{M}$, in every dimension $n \geqslant 2$. Every such manifold $\mathscr{N}$ is obtained as a cover of some n-orbifold $O_{\mathscr{N}}$ geodesically immersed in a suitable $(n+1)$-orbifold $O_{\mathscr{M}}$.

In this paper, we construct an explicit infinite family in dimension $n = 3$, via a similar covering technique where the roles of $O_{\mathscr{N}}$ and $O_{\mathscr{M}}$ are played by the right-angled dodecahedron $\mathscr{D}$ and 120-cell $\mathscr{Z}$. These two compact Coxeter right-angled regular polytopes exist in $\mathbb{H}^3$ and $\mathbb{H}^4$ respectively, and the first is a facet of the second. The existence of suitable finite covers is guaranteed here by assigning appropriate *colourings* to their facets, following A. Vesnin [23, 24], M. Davis and T. Januszkiewicz [3], I. Izmestiev [9].

A colouring determines a manifold covering, and the main factual observation is that a colouring of the dodecahedron $\mathscr{D}$ can be enhanced in a suitable way to a colouring of the right-angled hyperbolic Coxeter 120-cell $\mathscr{Z}$. We produce in this way a degree-32 orientable cover of $\mathscr{Z}$ that contains four copies of a non-orientable degree-8 cover of $\mathscr{D}$. By cutting along one such non-orientable geodesic submanifold we get a hyperbolic four-manifold $\mathscr{N}_1$ with connected geodesic boundary $\mathscr{M}_1 = \partial\mathscr{N}_1$.

The colouring technique applied to a single polytope can produce only finitely many manifolds. To get infinitely many examples we assemble n copies of $\mathscr{D}$ and $\mathscr{Z}$ to get more complicated right-angled polytopes, to which the above construction easily extends. We finally obtain the following. Let $V_{\mathscr{D}} \approx 4.3062...$ and $V_{\mathscr{Z}} = \frac{34}{3}\pi^2$ be the volumes of $\mathscr{D}$ and $\mathscr{Z}$, respectively.

Theorem 1.3. *For every $n \geqslant 1$ there exists an orientable compact hyperbolic three-manifold $\mathscr{N}_n$ of volume $16nV_{\mathscr{D}}$ which bounds geometrically an orientable compact hyperbolic four-manifold $\mathscr{M}_n$ of volume $32nV_{\mathscr{Z}}$.*

The manifolds $\mathscr{N}_n$ and $\mathscr{M}_n$ are tessellated respectively by $16n$ right-angled dodecahedra and $32n$ right-angled 120-cells.

An interesting feature of this construction is that it provides manifolds $\mathscr{N}_n$ and $\mathscr{M}_n$ of controlled volume. In particular, we deduce the following.

Corollary 1.4. *There are infinitely many hyperbolic three-manifolds $\mathscr{N}$ that bound geometrically some hyperbolic $\mathscr{M}$ with constant ratio:*

$$\frac{\mathrm{Vol}(\mathscr{M})}{\mathrm{Vol}(\mathscr{N})} = \frac{2V_{\mathscr{Z}}}{V_{\mathscr{D}}} < 53.$$

The manifold $\mathscr{M}_1$ has volume $16V_{\mathscr{D}} \approx 68.8992$ and is to our knowledge the smallest closed hyperbolic 3-manifold known to bound geometrically.

Structure of the paper. In Section 2 we introduce right-angled polytopes as orbifolds, and a simple colouring technique from [3] that produces manifold coverings of small degree. Then we show how this colouring technique passes easily from dimension n to $n+1$ and conversely, and may be used to produce n-manifolds that bound geometrically when $n = 3$. In Section 3 we assemble dodecahedra and 120-cells to produce the manifolds $\mathscr{N}_k$ and $\mathscr{M}_k$ of Theorem 1.3.

2. Colorings and covers of Coxeter orbifolds

A right-angled hyperbolic polytope $\mathscr{P} \subset \mathbb{H}^n$ may be interpreted as an orbifold with mirror boundary, where the mirrors correspond to its facets. As such an orbifold, it has a plenty of manifold coverings. A few of them may be constructed by colouring appropriately the facets of $\mathscr{P}$ as shown in [3, 6].

2.1. **Colourings and manifold covers.** Let $\mathscr{P} \subset \mathbb{H}^n$ be a convex compact right-angled polytope. Such objects exist only if $2 \leqslant n \leqslant 4$, see [18]; the two important basic examples we consider here are the right-angled dodecahedron $\mathscr{D} \subset \mathbb{H}^3$ and the right-angled 120-cell $\mathscr{Z} \subset \mathbb{H}^4$.

We consider $\mathscr{P}$ as an orbifold $\mathbb{H}^n/\Gamma$. The group Γ is a right-angled Coxeter group that may be presented as

$$\Gamma = \langle\ r_F \mid r_F^2, [r_F, r_{F'}]\ \rangle$$

where F varies over all the facets of $\mathscr{P}$ and the pair F, F' varies over all the pairs of adjacent facets. The isometry $r_F \in \mathrm{Isom}(\mathbb{H}^n)$ is a reflection in the hyperplane containing F.

A right-angled polyhedron $\mathscr{P}$ is *simple* [25, Theorem 1.8], which means that it looks combinatorially at every vertex v like the origin of an orthant in $\mathbb{R}^n$. In particular, v is the intersection of exactly n facets.

Let V be a finite-dimensional vector space over $\mathbb{F}_2$, thus isomorphic to $\mathbb{F}_2^s$ for some s. A *V-colouring* (or simply, a colouring) λ is the assignment of a vector $\lambda_F \in V$ to each facet F of $\mathscr{P}$ (called its *colour*) such that the following holds: at every vertex v, the n colours assigned to the n adjacent facets around v are linearly independent vectors in V.

A colouring induces a group homomorphism $\lambda\colon \Gamma \to V$, defined by sending r_F to λ_F for every facet F. Its kernel $\Gamma_\lambda = \ker \lambda$ is a subgroup of Γ which determines an orbifold $M_\lambda = \mathbb{H}^n/\Gamma_\lambda$ covering $\mathscr{P}$.

Proposition 2.1. *The orbifold M_λ is a manifold.*

Proof. We follow [23, Lemma 1]. A torsion element in Γ fixes some face F of the tessellation of $\mathbb{H}^n$ obtained by reflecting $\mathscr{P}$ in its own facets. Up to conjugacy, we can suppose that $F \subset \mathscr{P}$. The stabilizer of F is generated by the reflections in the facets containing F (see [2, Theorem 12.3.4]) and

is hence mapped injectively into V by λ. Thus, Γ_λ is torsion-free and M_λ is a manifold. □

At each vertex v the colours λ_F of the n incident facets F are independent: therefore the image of λ has dimension at least n and the covering $M_\lambda \to \mathscr{P}$ has degree $|\Gamma : \Gamma_\lambda| \geqslant 2^n$; if the equality holds the manifold M_λ is called a *small cover* of $\mathscr{P}$. The manifold coverings of $\mathscr{P}$ of smallest degree are precisely its small covers, see [6, Proposition 2.1].

We say that the colouring *spans* V if the vectors λ_F span V as F varies, which is equivalent to the map $\lambda\colon \Gamma \to V$ being surjective. In that case the covering $M_\lambda \to \mathscr{P}$ has degree $|\Gamma : \Gamma_\lambda| = 2^{\dim V}$.

2.2. **k-colourings.** Here, we give an example: recall that a *k-colouring* of a polytope is the assignment of a colour from the set $\{1, \dots, k\}$ to each facet so that two adjacent facets have distinct colours. A k-colouring for $\mathscr{P}$ produces an $\mathbb{F}_2^k$-colouring that spans $\mathbb{F}_2^k$: simply replace each colour $i \in \{1, \dots, k\}$ with the element e_i of the canonical basis for $\mathbb{F}_2^k$.

Example 2.2. The dodecahedron has precisely one four-colouring, up to symmetries. This induces an $\mathbb{F}_2^3$-colouring (of lower dimension 3 rather than 4) on the hyperbolic right-angled dodecahedron $\mathscr{D}$ described in [23] and hence a manifold covering having degree $2^3 = 8$.

Example 2.3. The 120-cell has a five-colouring (in fact, ten five-colourings up to symmetries [5]). Each produces a manifold covering of the hyperbolic right-angled 120-cell $\mathscr{Z}$ of degree $2^5 = 32$.

2.3. **Orientable coverings.** The following lemma gives an orientability criterion analogous to that for small covers in [16] or Löbell manifolds in [23, Lemma 2]. Let λ be a V-colouring of a right-angled polytope $\mathscr{P}$.

Lemma 2.4. *Suppose λ spans V. The manifold M_λ is orientable if and only if, for some isomorphism $V \cong \mathbb{F}_2^s$, each colour λ_F has an odd number of 1's.*

Proof. Let $\Gamma^+ \lhd \Gamma$ be the index two subgroup consisting of orientation-preserving isometries. Then Γ^+ is the kernel of the homomorphism $\phi\colon \Gamma \to \mathbb{F}_2$ that sends r_F to 1 for every facet F.

The manifold M_λ is orientable if and only if Γ_λ is contained in Γ^+, and this in turn holds if and only if there is a homomorphism $\chi : V \to \mathbb{F}_2$ such that $\phi = \chi \circ \lambda$. The latter is equivalent to the existence of an isomorphism $V \cong \mathbb{F}_2^s$ that transforms λ_F into a vector with an odd number of 1's for each F. Indeed, if such an isomorphism exists, then χ can be taken to be the sum of the coordinates of a vector.

Conversely, suppose such an isomorphism does not exist. Since the vectors λ_F span V we may take some of them as a basis for V and write them as $e_1 = (1, 0, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0)$, $\dots$, $e_s = (0, \dots, 0, 0, 1)$. By hypothesis, there exists a facet F such that λ_F has an even number of 1's.

Up to reordering, we may write $\lambda_F = \sum_{i=1}^{2k} e_i$ for some k. Now we can see that the homomorphism χ does not exist, since its existence would imply $1 = \phi(r_F) = \sum_{i=1}^{2k} \phi(e_i) = \sum_{i=1}^{2k} 1 = 0$. □

Corollary 2.5. *Let there be facets F, F' and F'' of $\mathscr{P}$ such that $\lambda_F + \lambda_{F'} + \lambda_{F''} = \mathbf{0}$. Then λ is a non-orientable colouring.*

Proof. For a vector $v = (v_1, v_2, \ldots, v_s) \in \mathbb{F}_2^s$ let $\epsilon(v) := \sum_{i=1}^{s} v_i$. Suppose that λ is orientable, so there exists an isomorphism $V \cong \mathbb{F}_2^s$, such that each λ_F has an odd number of 1's. Then we arrive at a contradiction, since $0 = \epsilon(\mathbf{0}) = \epsilon(\lambda_F + \lambda_{F'} + \lambda_{F''}) = \epsilon(\lambda_F) + \epsilon(\lambda_{F'}) + \epsilon(\lambda_{F''}) = 1$. □

Example 2.6. The manifolds in Examples 2.2 and 2.3 are orientable.

Example 2.7. Consider the 25 small covers of the hyperbolic right-angled dodecahedron $\mathscr{D}$ found by A. Garrison and R. Scott in [6]. The list is complete, up to isometries between the corresponding manifolds. Using the orientability criterion one sees immediately from [6, Table 1] that 24 of them are non-orientable and exactly 1 is orientable and corresponds to Example 2.2. Another example carried out in [6] is a small cover of the hyperbolic right-angled 120-cell $\mathscr{Z}$. This cover is again non-orientable. There is no classification of small covers of $\mathscr{Z}$ known at present.

2.4. **Induced colouring.** A facet F of a n-dimensional right-angled polytope $\mathscr{P} \subset \mathbb{H}^n$ is a $(n-1)$-dimensional right-angled polytope. A V-colouring λ of $\mathscr{P}$ induces a W-colouring μ of F with $W = V/\langle \lambda_F \rangle$: simply assign to every face of F the colour of the facet of $\mathscr{P}$ that is incident to it. The following lemma generalises [6, Proposition 2.3].

Lemma 2.8. *The manifold M_μ is contained in M_λ as a totally geodesic sub-manifold, so that the cover $M_\lambda \to \mathscr{P}$ restricts to the cover $M_\mu \to F$.*

Proof. Let Γ be the Coxeter group of $\mathscr{P}$ and $\mathbb{H}^{n-1} \subset \mathbb{H}^n$ be the hyperplane containing F. We regard F as the orbifold $\mathbb{H}^{n-1}/\Gamma_F$ where Γ_F is the Coxeter group of F. The following natural diagram commutes:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \langle r_F \rangle & \longrightarrow & \Gamma \cap \mathrm{Stab}(\mathbb{H}^{n-1}) & \xrightarrow{f} & \Gamma_F \longrightarrow 0 \\ & & & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \mu} \\ & & & & V & \xrightarrow[\pi]{} & V/\langle \lambda_F \rangle \end{array}$$

The first line is an exact sequence. We deduce easily that f restricts to an isomorphism

$$f\colon \ker \lambda \cap \mathrm{Stab}(\mathbb{H}^{n-1}) \longrightarrow \ker \mu.$$

Hence $M_\mu = \mathbb{H}^{n-1}/\Gamma_\lambda \cap \mathrm{Stab}(\mathbb{H}^{n-1})$ is naturally contained in $M_\lambda = \mathbb{H}^n/\Gamma_\lambda$. □

The pre-image of F in M_λ with respect to the regular covering $M_\lambda \to \mathscr{P}$ consists of possibly several copies of M_μ.

2.5. **Extended colouring.** Conversely, we can also extend a colouring from a facet to the whole polytope. We say that two colourings λ and λ' on $\mathscr{P}$ are *equivalent* if they have isomorphic kernels $\Gamma_\lambda \cong \Gamma_{\lambda'}$ (cf. the definition before [6, Proposition 2.4]).

Proposition 2.9. *Let F be a facet of a compact right-angled polytope $\mathscr{P} \subset \mathbb{H}^n$. Every colouring of F is equivalent to one induced by an orientable colouring of $\mathscr{P}$.*

Proof. Let λ be a V-colouring of the facet F. Fix an isomorphism $V \cong \mathbb{F}_2^s$. Define $W = \mathbb{F}_2 \oplus \mathbb{F}_2^s \oplus \mathbb{F}_2^f$ where f is the number of facets of $\mathscr{P}$ that are not adjacent to F. We define a W-colouring μ of $\mathscr{P}$ as follows:

- set $\mu_F = (1, \mathbf{0}, \mathbf{0})$;
- set $\mu_G = (\epsilon(\lambda_{G\cap F}) + 1, \lambda_{G\cap F}, \mathbf{0})$ where $\epsilon(v) = \sum_{i=1}^s v_i$, for every facet G adjacent to F;
- set $\mu_{G_i} = (\mathbf{0}, \mathbf{0}, e_i)$ for the remaining facets $G_1, \dots G_f$.

Indeed, the map μ is a colouring: the linear independence condition is satisfied at each vertex. Moreover, each vector μ_F, μ_G, μ_{G_i} has an odd number of 1's. Finally, by construction $\mu|_F$ is equivalent to λ. □

We call the colouring μ an *extension* of λ.

2.6. **A more efficient extension.** Proposition 2.9 shows how to extend a V-colouring of a facet F to an orientable W-colouring of the polytope $\mathscr{P}$. The proof shows that the dimension of W can grow considerably during the process, since $\dim W = \dim V + 1 + f$ where f is the number of facets of $\mathscr{P}$ not adjacent to F.

We may use Proposition 2.9 with $\mathscr{P}$ being the right-angled 120-cell and F its dodecahedral facet. However, in this case we could find examples where both V and W have smaller dimension via computer. The following was proved by using "Mathematica".

Proposition 2.10. *Each of the* 24 *non-orientable $\mathbb{F}_2^3$-colourings of $\mathscr{D}$ from* [6, Table 1] *is equivalent to one induced by an orientable $\mathbb{F}_2^5$-colouring of $\mathscr{Z}$.*

Proof. The Mathematica program code given in [22] takes a non-orientable $\mathbb{F}_2^3$-colouring of $\mathscr{D}$ and produces an orientable $\mathbb{F}_2^5$-colouring of $\mathscr{Z}$, as required.

Each vector $v = (v_1, v_2, \dots, v_s) \in \mathbb{F}_2^s$ is encoded by the binary number $n_v = v_1 \cdot 2^0 + v_2 \cdot 2^1 + \dots + v_s \cdot 2^{s-1}$. Let $P_0 := \mathscr{D}$, be the right-angled dodecahedron. We enumerate its faces exactly as shown in [6, Figure 3] and the corresponding 12-tuple of numbers encodes its colouring. Let P_i, $i = 1, \dots, 12$ be the dodecahedral facets incident to P_0 at the respective faces F_i, $i = 1, \dots, 12$. We start extending the colouring of $P_0 := \mathscr{D}$ as follows:

- set $\lambda_{P_0} = (0, 0, 0, 0, 1)$;
- if $\mu_{F_i} = v = (v_1, v_2, v_3)$, then $\lambda_{P_i} := (v_1, v_2, v_3, 0, \epsilon(v) + 1)$.

We obtain a 13-tuple, which is the initial segment of the colouring of $\mathscr{Z}$. Then the Mathematica code [22] attempts to produce an orientable 120-tuple, which encodes the entire colouring. □

We were not able to find any orientable $\mathbb{F}_2^4$-colouring of $\mathscr{Z}$ extending a non-orientable $\mathbb{F}_2^3$-colouring of $\mathscr{D}$: our examples are not small covers.

Let now M_λ be the manifold obtained by one $\mathbb{F}_2^5$-colouring λ of $\mathscr{Z}$: it covers $\mathscr{Z}$ with degree $2^5 = 32$. The colouring λ restricts to a non-orientable colouring μ of the facet $\mathscr{D}$, which gives rise, by Lemma 2.8, to a non-orientable codimension-1 geodesic submanifold $M_\mu \subset M_\lambda$, covering $\mathscr{D}$ with degree $2^3 = 8$.

In complete analogy to [13, Lemma 3.2], by cutting M_λ along M_μ we get a compact orientable hyperbolic manifold tessellated by 32 right-angled 120-cells having a geodesic boundary isometric to a connected manifold $\widetilde{M_\mu}$ that double-covers M_μ and is hence tessellated by $2 \cdot 2^3 = 16$ right-angled dodecahedra.

To prove Theorem 1.3 it only remains to extend this argument from one 120-cell to an appropriate assembling of $n \geq 2$ distinct 120-cells.

3. Assembling right-angled dodecahedra and 120-cells

Here, we assemble right-angled dodecahedra and 120-cells in order to construct more complicated convex compact right-angled convex compact polytopes.

3.1. **Connected sum of polytopes.** Let $\mathscr{P}_1$ and $\mathscr{P}_2$ be two right-angled polytopes in $\mathbb{H}^n$. If there is an isometry between two facets of $\mathscr{P}_1$ and $\mathscr{P}_2$, we may use it to glue them: the result is a new right-angled polytope in $\mathbb{H}^n$ which we call a *connected sum* of $\mathscr{P}_1$ and $\mathscr{P}_2$ along these facets.

3.2. **Assembling.** An *assembling* of right-angled dodecahedra (or 120-cells) is a right-angled polytope constructed from a finite sequence

$$\mathscr{P}_1 \# \mathscr{P}_2 \# \mathscr{P}_3 \# \ldots \# \mathscr{P}_k$$

of connected sums performed from the left to the right, where each $\mathscr{P}_i$ is a right-angled dodecahedron (or 120-cell).

Lemma 3.1. *An assembling of k right-angled dodecahedra is a facet of an assembling of k right-angled 120-cells.*

Proof. Consider $\mathbb{H}^3$ inside $\mathbb{H}^4$ as a geodesic hyperplane. Consider $\mathscr{D} \subset \mathbb{H}^3$ as a facet of $\mathscr{Z} \subset \mathbb{H}^4$. Every time we attach a new copy $\mathscr{D}_i$ of $\mathscr{D}$ in $\mathbb{H}^3$, we correspondingly attach a new copy $\mathscr{Z}_i$ of $\mathscr{Z}$ having $\mathscr{D}_i$ as a facet. □

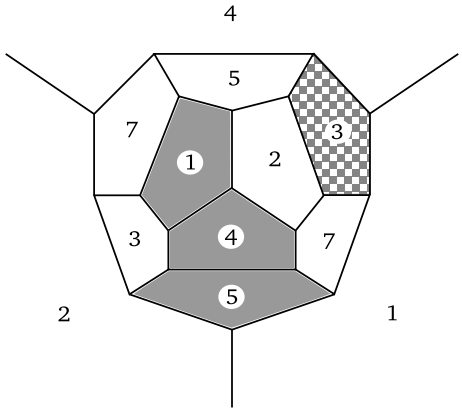

FIGURE 1. A non-orientable colouring of $\mathscr{D}$ from [6, Table 1]. Face colours are encoded by binary numbers

3.3. **Proof of Theorem 1.3.** We have described all the ingredients necessary to prove Theorem 1.3.

By Proposition 2.10, pick an orientable $\mathbb{F}_2^5$-colouring of the right-angled 120-cell $\mathscr{Z}$ that induces a non-orientable $\mathbb{F}_2^3$-colouring of one dodecahedral facet $\mathscr{D}$.

Then assemble n copies of $\mathscr{D}$ as

$$\mathscr{P} = \mathscr{D}_1 \# \mathscr{D}_2 \# \ldots \# \mathscr{D}_n$$

and consider, as in Lemma 3.1, the resulting right-angled polyhedron $\mathscr{P}$ as a facet of a a right-angled polytope $\mathscr{Q}$ made of n copies of the right-angled 120-cell, each having a $\mathscr{D}_i$ as a facet.

Every time we assemble a new copy of $\mathscr{D}_i$, we give $\mathscr{D}_i$ the colouring of the adjacent dodecahedron, mirrored along the glued pentagonal face, and we do the same with each corresponding new 120-cell $\mathscr{Z}_i$. The resulting polytope $\mathscr{Q}$ inherits an orientable $\mathbb{F}_2^5$-colouring λ that induces a non-orientable $\mathbb{F}_2^3$-colouring μ of $\mathscr{P}$, if $\mathscr{P}$ is assembled appropriately. Indeed, each of the 24 non-orientable colourings of $\mathscr{D}$ has three faces F, F' and F'' satisfying the conditions of Corollary 2.5. Moreover, we can find a fourth face F^* which is disjoint from each of them. These properties are easily verifiable by using [6, Table 1]. Then, we start assembling $\mathscr{D}_i$'s by forming a connected sum along F^*. Then the colouring of the resulting polytope $\mathscr{P}$ again satisfies Corollary 2.5 and hence is non-orientable, as required.

By cutting M_λ along M_μ we get a compact orientable hyperbolic manifold tessellated by $32n$ right-angled 120-cells having a geodesic boundary that is isometric to a connected manifold $\widetilde{M_\mu}$ that double-covers M_μ and is hence tessellated by $2 \cdot 2^3 n = 16n$ right-angled dodecahedra. □

We conclude the paper by providing an example of the construction carried in the proof above.

Example 3.2. Let us choose a non-orientable colouring μ of the dodecahedron $\mathscr{D}$ from [6, Table 1], say the one having the maximal symmetry group $(\mathbb{Z}_2 \times \mathbb{Z}_2) \ltimes \mathbb{Z}_6$. This is an $\mathbb{F}_2^3$-colouring depicted in Fig. 1, with face colours encoded by binary numbers in the decimal range $1, \dots, 7$. Here, the grey shaded faces of $\mathscr{D}$ are exactly the faces F, F' and F'', satisfying the conditions of Corollary 2.5. The face F^*, along which we take a connected sum in the proof of Theorem 1.3, has a chequerboard shading. Now, we take a connected sum of $\mathscr{D}_1 := \mathscr{D}$ along F^* with its isometric copy $\mathscr{D}_2$, having the same colouring. Then we can choose a face of $\mathscr{D}_1 \# \mathscr{D}_2$, distinct from any of F, F' or F'' and continue assembling until we use all n given copies of $\mathscr{D}$, which produce a polyhedron $\mathscr{P} = \mathscr{D}_1 \# \dots \# \mathscr{D}_n$. The faces F, F' and F'' of $\mathscr{D}_1$ still remain among those of $\mathscr{P}$. Thus, the colouring of $\mathscr{P}$ is again non-orientable by Corollary 2.5.

Finally, by applying Proposition 2.9 and Lemma 3.1, we obtain a four-dimensional polytope $\widetilde{\mathscr{P}}$ with an orientable $\mathbb{F}_2^5$-colouring, that induces a non-orientable colouring on one of its facets, which is isometric to $\mathscr{P}$. Indeed, the non-orientable $\mathbb{F}_2^3$-colouring of each $\mathscr{D}_i$ can be extended by Proposition 2.9 to an orientable $\mathbb{F}_2^5$-colouring of a 120-cell $\mathscr{Z}_i$. Then, by Lemma 3.1, $\mathscr{P} = \mathscr{D}_1 \# \dots \# \mathscr{D}_n$ is a facet of a polytope $\widetilde{\mathscr{P}} = \mathscr{Z}_1 \# \dots \# \mathscr{Z}_n$. Since the colourings of $\mathscr{D}_i$'s match under taking connected sums, the colourings of $\mathscr{Z}_i$'s also match. The polytope $\widetilde{\mathscr{P}}$ gives rise to a covering manifold with totally geodesic boundary, as described in the proof of Theorem 1.3.

References

[1] E.M. Andreev, *On convex polyhedra in Lobačevskiĭ space*, Math. USSR Sb. **10** (1970), 413–440.

[2] M. Davis, *The geometry and topology of Coxeter groups*, Princeton Univ. Press, (2008), 1–594.

[3] M. Davis – T. Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), 417–451.

[4] F.T. Farrell – S. Zdravkovska, *Do almost flat manifolds bound?*, Michigan Math. J. **30** (1983), 199–208.

[5] S. Fisk, *Coloring the* 600*-cell*, `arXiv:0802.2533`

[6] A. Garrison – R. Scott, *Small covers of the dodecahedron and the* 120*-cell*, Proc. Amer. Math. Soc. **131** (2003), 963–971.

[7] M. Gromov, *Manifolds of negative curvature*, J. Differential Geom. **13** (1987), 223–230.

[8] ———, *Almost flat manifolds*, J. Differential Geom. **13** (1978), 231–241.

[9] I.V. Izmestiev, *Three-dimensional manifolds defined by coloring a simple polytope*, Math. Notes **69** (2001), 340–346.

[10] A. Kolpakov – B. Martelli, *Hyperbolic four-manifolds with one cusp*, Geom. Funct. Anal. (2013), on-line first.

[11] D.D. Long – A.W. Reid, *On the geometric boundaries of hyperbolic 4-manifolds*, Geom. Topol. **4** (2000) 171–178.

[12] ———, *All flat manifolds are cusps of hyperbolic orbifolds*, Alg. Geom. Topol. **2** (2002), 285–296.

[13] ———, *Constructing hyperbolic manifolds which bound geometrically*, Math. Research Lett. **8** (2001), 443–456.

[14] D.B. McReynolds, *Controlling manifold covers of orbifolds*, Math. Res. Lett. **16** (2009), 651–662.
[15] D.B. McReynolds – A.W. Reid – M. Stover, *Collisions at infinity in hyperbolic manifolds*, Math. Proc. Cambridge Philos. Soc. **155** (2013), 459–463.
[16] H. Nakayama – Y. Nishimura, *The orientability of small covers and coloring simple polytopes*, Osaka J. Math. **42** (2005), 243–256.
[17] B.E. Nimershiem, *All flat three-manifolds appear as cusps of hyperbolic four-manifolds*, Topology and Its Appl. **90** (1998), 109–133.
[18] L. Potyagailo – È. Vinberg, *On right-angled reflection groups in hyperbolic spaces*, Comment. Math. Helv. **80** (2005), 63–73.
[19] J.G. Ratcliffe – S.T. Tschantz, *Gravitational instantons of constant curvature*, Class. Quantum Grav. **15** (1998) 2613–2627.
[20] ———, *On the growth of the number of hyperbolic gravitational instantons with respect to volume*, Class. Quantum Grav. **17** (2000) 2999–3007.
[21] M. Stover, *On the number of ends of rank one locally symmetric spaces*, Geom. Topol. **17** (2013), 905–924.
[22] S.T. Tschantz, *Mathematica program code available on-line at the author's web-page* `http://www.math.vanderbilt.edu/∼tschantz/geosubmanincovers120cell.nb`
[23] A.Yu. Vesnin, *Three-dimensional hyperbolic manifolds of Löbell type*, Siberian Math. J. **28** (1987), 731–733.
[24] ———, *Three-dimensional hyperbolic manifolds with a common fundamental polyhedron*, Math. Notes **49** (1991), 575–577.
[25] È. Vinberg (Ed.), "Geometry II", Berlin: Springer-Verlag, 1993.

University of Toronto, Department of Mathematics, 40 St. George Street, Toronto ON, M5S 2E4, Canada
Email address: `kolpakov dot alexander at gmail dot com`

Università di Pisa, Dipartimento di Matematica "Tonelli", Largo Pontecorvo 5, 56127 Pisa, Italy
Email address: `martelli at dm dot unipi dot it`

Vanderbilt University, Department of Mathematics, 1326 Stevenson Center, Nashville, TN 37240, USA
Email address: `steven dot tschantz at vanderbilt dot edu`