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On the role of affect for sense making in learning mathematics – aesthetic experiences in problem solving processes

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Sense making is one important goal of learning processes in school mathematics. Empirical studies on genuine sense constructions of mathematics students show the importance of sense construction categories with a subjective and an inner-mathematical orientation at the same time. From a theoretical point of view, we investigate two ways of fostering such sense constructions within school mathematics, by means of aesthetic experiences on the one hand and through reflection of problem solving processes on the other. In both cases, affective aspects play an important role. We also discuss the merits of intertwining aesthetic experiences and problem solving processes regarding sense construction. We finish with some remarks on possible ways of employing our theoretical results practically.

Keywords: Sense making, mathematical beauty, problem solving, beliefs, affect.

INTRODUCTION – THE ROLE OF SENSE MAKING IN LEARNING MATHEMATICS

Sense making is a general, and at least from the viewpoint of constructivism of course quite natural goal for the learning of mathematics. Nevertheless, the meaning of “sense” within the educational debate is underdetermined, sometimes vague. A great variety of different aspects of sense have been brought up in this debate, sometimes explicitly articulated, sometimes more implicitly. Such aspects are, e.g., purpose, benefit, intention, or merit. From an analytic point of view, most of these aspects have at least two dimensions: an objective-subjective and an inner-extra-mathematical dimension. With regard to these dimensions, a large part of the discussion in mathematics education focusses on rather objective, inner-

and extra-mathematical issues of “sense making” (e.g., what is or what should be the “sense” of negative numbers, which lifeworld contexts are adequate to teach them). Subjective issues of sense making oriented to the learning subject, on the other hand, are usually discussed with regard to rather extra-mathematical themes, like incentives, personal relation to teacher, beliefs about the subject’s own abilities in learning mathematics etc. However, there are also popular positions emphasizing subjective inner-mathematical aspects of sense making as of great importance for the learning of mathematics, as the work of Skovsmose (e.g., Skovsmose, 2005), or Ruf and Gallin’s work on “core ideas” (e.g., Ruf & Gallin, 1998).

The guiding question of the following considerations is: How can subjective, inner-mathematical aspects of sense making in learning mathematics be addressed, and triggered within learning processes, and what role does affect play in this regard? To our mind, this question is less a question about methods of teaching. It is rather meant as a question about contents of teaching mathematics, on two different levels: In second order, it addresses the object level of mathematics itself (negative numbers, triangles, functions and graphs, variables etc.). In first order, it shall deal with “meta-units” of doing mathematics (problem solving, proving, discovering new connections, reconstructing solution processes and so on) which can be a basis for subjective inner-mathematical sense making in a learning process.

Our approach is to take inert sense making categories of learning subjects as a starting point. Vollstedt (2011) investigates such inert categories of sense making, “sense constructions” in her terminology, for the case of students of the lower secondary level (15 till 16

years old) in a qualitative, comparative empirical interview study (34 participants, 17 from Hongkong and 17 from Germany).[1] In a first step, she classifies sense constructions into some 3 times 3 matrix (Table 1).

The category “high individual orientation–high inner-mathematical orientation” (cell 3) corresponds to our “subjective inner-mathematical” dimension:

High intensity of inner-mathematical orientation: The sense construction is in immediate relation to mathematical contents [and not to other aspects of learning processes in school, as, e.g., social interactions].

High intensity of individual orientation: The individual itself is the focus of the sense construction, intra-individual relations dominate [and not social institutions or demands]. (ibid, pp. 130f., German in original, added brackets give short descriptions of value “low”)

Vollstedt found that this category is empirically contentful, that is, sense constructions could be reconstructed from the interview data that fall into that category. In the following, we will concentrate on two of these sense constructions: “experience of autonomy” and “mathematical purism” (ibid, pp. 142–148). Vollstedt defines these two categories with reference to a “criterion of personal relevance” (ibid, p. 129, German in the original):

Mathematical purism: Doing or learning mathematics is personally relevant to the individual if it appreciates the purism of mathematics stemming from its formality and logical composition and benefits from this appreciation regarding its understanding of mathematics.

Experience of autonomy: Doing or learning mathematics is personally relevant to the individual if it experiences self-reliance in doing or learning mathematics, e.g., in terms of learning autonomy

and original development of solutions to mathematical problems.

Answers coded as “mathematical purism” deal with a special fascination of pure mathematics, and describe a positive affective relationship to a special mathematical discipline (e.g., geometry or algebra) or express personal relevance of clear and logical mathematical structure (ibid, p. 146). Answers belonging to the category “experience of autonomy” describe positive experiences by, e.g., choosing exercises autonomously or finding individual ways of solving problems (ibid, pp. 142–144).

In the following sections, we discuss two kinds of “meta-units” of doing mathematics in relation to these two sense constructions. Both have a certain affective potential, and therefore specific merits regarding the initiation of sense making in learning mathematics. First, we argue that *aesthetic mathematical experiences* provide good opportunities for the learning subject to appreciate mathematical purism. For the case of autonomy, aesthetic judgments are distinctly different from right-or-wrong-judgments. Aesthetic mathematical experiences are often closely related to so called AHA! experiences, a highly affective type of individual understanding processes. Second, we look at *mathematical problem solving processes*, and, in particular, the affective components of those, in relation to the sense constructions “mathematical purism” and “experience of autonomy”, and then turn to discussing the value of intertwining both perspectives to arrive at a particularly fruitful basis for sense construction within learning processes at school.

AESTHETIC EXPERIENCES AND PROBLEM SOLVING PROCESSES AS SOURCES OF SENSE CONSTRUCTION IN LEARNING MATHEMATICS

Sense construction and mathematical beauty
Relying on Spies (2013), we adopt the premise that though mathematical beauty isn’t definable entirely, a concept of beautiful pieces of mathematics, cover-

Intensity of inner-mathematical orientation of sense construction	Intensity of individual orientation of sense construction		
	Low	medium	high
High	1	2	3
Medium	4	5	6
Low	7	8	9

Table 1: “Typology of sense construction” (ibid, p. 133, German in original)

ing for instance theorems, proofs, argumentation schemes, and heuristic strategies, can be explicated. Informed by mathematics as a scientific discipline it is also fruitfully employable within discussions of mathematics and beauty in mathematics education. The concept of mathematical beauty developed in (Spies, 2013) is explicated by four relevant attributes: *range*, *economy*, *epistemic transparency*, and *emotional effectiveness*. At least three of these attributes, range, epistemic transparency, and economy, are directly related to the sense constructions “mathematical purism” and “experience of autonomy” investigated by Vollstedt. Regarding the sense construction “mathematical purism” in detail, the following relations hold:

Range: There are two aspects of range often referred to when it comes to aesthetic value judgments in mathematics. On the one hand, a beautiful piece of mathematics connects different parts or branches of mathematics, for example the use of algebraic tools to solve a geometrical problem.

A beautiful proof often makes unexpected connections between seemingly disparate parts of mathematics. A proof which suggests further development in the subject will be more pleasing than one which closes off the subject. (Stout, 1999, p. 10)

On the other hand, an argumentation owns a certain kind of beauty if the idea at the core of the argument is applicable to a variety of other cases, if the chosen heuristic is paradigmatic in some sense. Becoming aware of the range of an argument or a result may deliver *insight* in the system of mathematics itself. The awareness of broad connectivity of an argumentation or of the paradigmatic character of a heuristic may help to establish an appreciation of mathematics also as a self-contained system besides its applicability in extra-mathematical contexts.

Epistemic transparency: This attribute of beautiful mathematics underlines the subjective character of aesthetic experiences within mathematics. It explicitly stresses the importance of a subjective *understanding* of mathematical structures in connection with an aesthetic mathematical experience: A beautiful proof offers a special kind of deep understanding of *why* the result is true. Often, this is described as an illumination, as a spontaneous grasping of the whole

argument from one moment to the other, as an AHA! experience, together with strong positive emotions.

The mathematician’s “aesthetic buzz” comes not only from simply contemplating a beautiful piece of mathematics, but, additionally, from achieving insight. (Borwein, 2006, p. 25)

Accordingly, an aesthetic mathematical experience is not only a product of positive feelings, but also linked to a special kind of deep individual understanding why.

Economy: Under the sense construction “mathematical purism”, Vollstedt subsumes statements concerning the shortness of mathematical arguments, or the number of formulas you have to remember to solve a certain range of mathematical problems (Vollstedt, 2011, p. 147). The properties expressed by these statements correspond to economy as an attribute of beautiful mathematics, and link the attributes of range and epistemic transparency. As G.H. Hardy points out in his famous *Apology*:

In both theorems [mentioned as examples for great mathematical beauty; author’s remark] there is a very high degree of unexpectedness, combined with inevitability and economy. The arguments take so odd and surprising a form; the weapons used seem so childishly simple when compared with the far-reaching results; but there is no escape from the conclusions. There are no complications of detail—one line of attack is enough in each case. (Hardy, 1940, p. 113)

Hence, it is not shortness of an argument as an extrinsic property that releases an aesthetic value judgment. It is the impression of economy, that is, shortness of argumentation in relation to its range and with regard to epistemic transparency for the judging subject. There is a necessary connection between individually oriented and inner-mathematically oriented aspects to trigger aesthetic judgments, and thus, possible sense constructions.

Regarding the sense construction “experiencing autonomy”, we take into account that aesthetic experiences of beauty undergone by an individual, by becoming aware of them, lead to conscious judgements of beauty, stated or not. These aesthetic value judgements are in a specific way opposed to the

prominent excluded middle character of mathematics, and the focus on formal mathematical correctness (Müller-Hill & Spies, 2011). While the latter are rather related to experiences of special conformity and authority of mathematics, aesthetic value judgements transcend formal correctness and are, at least in part, subjectively justified. Nevertheless, they can usually be explained on the basis of inter-subjectively graspable criteria by the judging individual. In this sense, aesthetic mathematical experiences give room for autonomous, but negotiable judgements of beauty, and as a consequence, for responsible, self-relying decisions and actions based on these judgements.

The fourth attribute of mathematical beauty according to (Spies, 2013), *emotional effectiveness*, explicates the affective character of aesthetic mathematical experiences. When mathematicians talk about the aesthetic pieces of mathematics, they use a highly emotional and affective language. Leone Burton reports from an interview study with practicing mathematicians:

The mathematicians discussed aesthetics [...] in terms that were emotive, full of expressed feelings. (Burton, 2004, p. 63)

Often, the expressed emotions are used to qualify one of the other attributes of mathematical beauty described above.[2] Terms of special relevance according to aesthetic value judgement seem to be *unexpectedness* and *surprise*. A beautiful argument may evoke the feeling of surprise about its (economical) form, of a “surprising twist” in the argumentation, or of unexpectedness regarding the heuristics employed in the argumentation. The feeling of *inevitability* of an argumentation seems to be another aspect of the emotional effectiveness of beautiful proofs (see also the above quote from Hardy, 1940, p. 113), as the mathematician Gregory Chaitin states vividly:

After the initial surprise it [a beautiful proof] has to seem inevitable. You have to say, of course, how come I didn't see this! (Chaitin, 2002, p. 61)

Sense construction and central types of problem solving processes

Of course, problem solving is a widely discussed issue in mathematics education, and in turn there are a number of different conceptions of problem solving and ways or conceptual tools to describe and investigate problem solving processes. In the following, we

discuss problem solving processes as a basis for sense construction in learning mathematics, and ask for the role of affect regarding this relation.[3] To this end, we lean on the synopsis given in (Schoenfeld, 1992), which explicitly embraces and explicates “beliefs and affects” as an important (cognitively effective) aspect of problem solving processes (see *ibid*, p. 348). The aspect “beliefs and affects” contains students beliefs, teachers beliefs, and also “general societal beliefs” about the nature of mathematics and doing mathematics. We will focus on the “belief system” (Schoenfeld, 1985) of the problem solver, encompassing beliefs, attitudes and opinions about mathematics itself (about „formal mathematics“, his „sense of the discipline“; Schoenfeld, 1992, p. 359), about mathematics as a subject matter, and about doing and learning mathematics. Schoenfeld emphasizes the extraordinary powerful impact of the belief system on mathematical problem solving.

In the following, we distinguish two ways in which problem solving processes can generally be connected to sense construction in the above sense, that is, in which they can be meaningful and personally relevant to the problem solver. The first way refers to the *decisions* that are made throughout the process by the problem solver, and the motivations behind them. The second way considers *changes in the belief system* as responsible for subjective sense making. Such changes may be initiated by actual performance of, or by later reflection on, problem solving processes.

Decisions guiding the course of action throughout a problem solving process are potentially meaningful components of these processes. At least to a certain degree, they allow to infer something about what is personally relevant to the problem solver in terms of guiding motives and reasons for these decisions. Potentially subjectively meaningful decisions, and actions in turn, have to be autonomous at least to a certain degree, intentional, and goal oriented. Usually, decisions are also affectively driven, which strengthens their subjective relevance and, in the case of several alternatives, can even be the last instance to decide. Undergoing a process of problem solving also induces an interaction with the individual belief system of the problem solver. The belief system, being the network of conceptions, opinions, attitudes and beliefs related to mathematics, is per definition the basis of all subjective sense making in learning mathematics. Sense constructions on the basis of problem solving processes

will therefore necessarily incorporate changes in the belief system, in the form of expansion, overwriting, or readjustment.

Regarding both decisions and changes in the belief system, it is not sufficient for fostering sense making just to get learning subjects involved in concrete problem solving processes. Sense construction in the way described above will usually not take place automatically. We suggest that *explicit reflection* of problem solving processes including decisions, action guiding motives, and affective aspects like emotions felt during the process or conscious changes of certain beliefs is necessary.

Reflecting on decisions as part of a problem solving process obviously encourages sense constructions of the type “experience of autonomy”. But problem solving processes also have a specific sense making potential with regard to „mathematical purism“, by focusing on recognizing *structural types* of mathematical problem solving processes. Reflections on characteristic elements of problem solving processes of a certain type can both lead to an appreciation of the formal structure of mathematics (regarding corresponding changes in the belief system), and lead to a better subjective understanding, as it may promote the ability to transfer problem solving approaches and strategies (regarding the orientation on a certain type of problem solving processes as a guiding motive for decisions in a concrete process). The latter also provides the opportunity for experiences of autonomy. The inner-mathematical focus can be increased by highlighting *central structural types* of mathematical problem solving processes. Mathematical problem solving processes of a central type employ certain, structural elements that are characteristic for working with central concepts of (branches of) mathematics like “function”, “gauge”, “number”, or “area”. [4]

Now in turn, we will argue that there are at least two ways in which a combination of problem solving processes and experiences of mathematical beauty can be particularly fruitful with regard to sense construction in learning mathematics, exploiting their affective elements in a specific way.

AN INTERTWINED PERSPECTIVE

As stated above, the decisions made during a problem solving process can be meaningful and relevant to the

problem solver in terms of ideas, reasons or motives guiding them and the corresponding actions. These guiding ideas, reasons or motives are usually not merely rational, but decisions and actions are also guided by affect. Goldin (2000), e.g., investigates the relation between affective states (including aesthetic experiences) and chosen heuristics in problem solving processes. Though criticizing Goldin’s subsumption of aesthetics under what he conceives as affective states, Sinclair in her (2008) also stresses the importance of aesthetics as coupled both with affective and cognitive aspects of mathematical problem solving. However, she emphasizes that

the aesthetic and the affective domains each *function* differently in the problem-solving process: the aesthetic draws the attention of the perceiver to a phenomenon, while the affective can bring these perceptions to the conscious attention (ibid, p. 55, our emphasis).

In this sense, we consider decisions and actions driven by experiences of mathematical beauty as important, potentially sense-constitutive affective elements of mathematical problem solving processes. This is additionally underpinned and emphasized by a number of famous practicing mathematicians. For example, in his famous essay *The psychology of invention in the mathematical field*, Jacques Hadamard reaches the following “double conclusion” after reviewing psychological and philosophical literature on general and mathematical invention:

That invention is choice. That this choice is imperatively governed by the sense of scientific beauty. (Hadamard, 1954, p. 31)

This observation from scientific mathematics can at least partly be adapted for problem solving processes in school mathematics: Choice is a meaningful element of problem solving processes. Choices are i.a. governed by subjective motives like perceptions of mathematical beauty.[5] To foster sense construction sustainably, the conscious attention of such aesthetic perceptions should be supplemented by explicit reflection on their role for the course of concrete problem solving processes.

With regard to the second form of sense making in problem solving processes, on the other hand, aesthetic experiences can help to initiate *changes* in the

inner-mathematical component of the belief system due to their specific affective character as described above. This might be necessary when students fail in completing a problem solving process on their own because of holding certain constraining beliefs. Even if they succeed to finish the problem solving process with some help, the constraining beliefs might be abstracted from years of classroom experience (Schoenfeld, 1985). Hence, rational reflections of such single problem solving processes will usually have only little effect on their belief system, and therefore hardly trigger new sense making. Aesthetic experiences might catalyze the reflective impact in this regard. In particular, aesthetic mathematical experiences are often conceived as AHA! experiences. Empirical studies as (Liljedahl, 2005) show that there is a strong relation between the reflection of AHA! experiences and (even drastic) changes of student's beliefs about their learning of mathematics because of the affective, especially the emotional potential of AHAs (ibid., 231). AHA! experiences of mathematical beauty work in a quite similar way regarding unexpectedness, surprise, sudden inevitability (compare ibid., 226), and emotional effectiveness, but they are directed to attributes and relations of mathematical argumentations, formulas, diagrams, theorems, etc. The four attributes of mathematical beauty described above, range, economy, epistemic transparency, and emotional effectiveness, can be experienced jointly or in different combinations as aesthetic aspects of a certain piece of mathematics dealt with in a concrete problem solving process. Therefore, it seems promising to assume a similar impact of reflecting on aesthetic AHA! experiences in problem solving processes on student's beliefs on mathematics itself.

SHORT OUTLOOK ON DESIGN ISSUES

When we now direct the focus to designing issues in a short outlook, we rather have in mind developmental questions and the design of supporting learning material, not measurement issues. Our contribution must obviously be conceived as rather theoretical with regard to the design of concrete learning environments or teaching material. One reason for this is that from our point of view, sense construction is strongly tied to individual parameters of learners and learning groups. This makes it difficult to argue in favor of concrete designs in detail. Nevertheless, we think that we have generally identified good candidates to direct concrete designing attempts to.

Problem solving is standard to school mathematics today. What might not be standard is guided reflection of concretely undergone processes with an explicit focus on action guiding motives, decisions, beliefs, or affects. Also still non-standard is the upgrading of aesthetic experiences as admissible and negotiable justifying reasons for choosing between alternative courses of action within problem solving processes. Future work will expand the view onto the scope of content didactics, aiming at a specification, e.g., of mathematical concepts, appropriate mathematical problems, and (types of) corresponding problem solving processes and beautiful pieces of mathematics as the subject matter of possible learning environments to foster subjective inner-mathematical sense construction. A point of practical interest in the design of learning environments will be the degree of guidance and instruction accompanying the problem solving activities of learners. Through a theoretical lense, this will link the perspective opened up here to questions about the role of creativity.

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3. See also (Presmeg, 2014) for another most recent, programmatic emphasis.
4. “Central types of problem solving processes” corresponds to “fundamental mathematical ideas” famously introduced by Bruner in the 1960s and discussed under a variety of readings. We will not enter this discussion here, because the explication of „central” is not necessary for the general arguments discussed. Further qualification of “central” will be necessary for future work, esp. from a content didactical point of view.
5. According to Hadamard, this assumption is quite obvious: “Between the work of the student who tries to solve a problem in geometry or algebra and a work of invention, one can say that there is only a difference of degree, a difference of level, both works being of a similar nature.” (Hadamard, 1954, p. 104)

ENDNOTES

1. Vollstedt uses grounded theory methodology, not operationally defined items or coding categories. Due to space, we refer to the original source for more details on Vollstedts shaping of the categories based on her data.
2. Similarly, “emotional relation to mathematics” is described as part of “purism” (Vollstedt, 2011, p. 146).