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# Invented or discovered?

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*Does mathematics pre-exist and hence is discovered or is it invented and owes its being to humans? What do students believe and how does this interact with their beliefs about the production and the meaningfulness of mathematical knowledge? This paper presents results based on 18 Greek students' interviews about their relationship with mathematics through an epistemological lens. The findings diverge from what the literature suggests especially with respect to whether mathematics is perceived as a meaningful human activity and to what extent it produces certain and fixed conclusions. Ideally educators could foster beliefs which promote students' engagement and understanding of mathematics.*

**Keywords:** Mathematics ontology, epistemology, existence.

## INTRODUCTION

The ontology of mathematics is a hot debate in the philosophy of mathematics. The key question is whether mathematics pre-exists or comes into existence through human activity. Does mathematics transcend humans or is it simply yet another sector of human knowledge. The question is complicated with respect to mathematics because it is entangled with its epistemology. Although mathematical concepts may not appear to be materially substantiated – at least not in the same sense that a table is – mathematical conclusions have long been endowed with a certainty that would be strange to assume for any creation of the human mind (Hersh, 1999).

Moreover, it seems that mathematicians and mathematics educators do not share the same views on this issue. Most mathematicians tend to embrace the belief that mathematics is independent of the human mind. On the contrary, most educators advocate the belief that mathematics is constructed by humans (Sfard, 1998). Research has generally associated the belief that mathematics pre-exists with traditional

teaching practices. Teachers who view mathematics as an independent entity would present mathematical knowledge as fixed. Consequently, their role is to transmit it to the students while the latter's role is to passively absorb it. Educators, of course, opt for an active engagement of the students (Lerman, 2002). However, there have been great mathematicians (e.g., Hardy, Gödel) who have been actively engaged with mathematics and who have done wonders holding the belief that educators dread.

Consequently it is contestable what we would like students to believe about mathematics' ontology. Should they follow the steps of great mathematicians or will this render them passive learners? Nevertheless, before aiming at such a question, we need to know more about students' beliefs on this issue and how they affect the student's relationship with mathematics? Although there has been abundant research in students' beliefs about mathematics (e.g., Schoenfeld, 1992) the issue of ontology seems to have been neglected. This paper focuses on it, investigating the second of the above mentioned questions in the traditional teaching context of Greece.

## THEORETICAL FRAMEWORK

The distinction between finding something that already exists and something that is novel is captured by the verbs 'discover' and 'invent'. We discover something that already exists the same way that Columbus discovered America. To the contrary when we invent something it owes its existence to this very process of invention<sup>1</sup>.

The predominant opinion in the history of the philosophy of mathematics speaks of discovery. This tradition may be traced back to Plato and has been called Platonism<sup>2</sup> after the philosopher. Platonism is nicely captured in the words of the mathematician G. H. Hardy who maintains that

mathematical reality lies outside us, that our function is to discover or observe it and that the theorems which we prove... are simply our notes of our observations. (1967, pp. 123–124).

This is an ontological assertion related to the ‘mode of existence’ of mathematics. However, it has been this ontological assertion that underlain the predominant epistemological conviction about the certainty of mathematical knowledge. Mathematical truth is absolute and objective since the truth of any mathematical statement is judged against an extra-human mathematical reality.

Nevertheless, many modern philosophers reject Platonism as an absurd idea; we can see and touch the physical reality, but where is this purported mathematical reality (Hersh, 1999)? If Platonism is rejected, then mathematics can no longer be discovered. Mathematics is now claimed to be invented, and again an ontological conviction is coupled with an epistemological claim. Mathematics does not exist and mathematical knowledge becomes fallible. Lakatos (1976) argues that no proof guarantees the truth of the theorem it proves; there is always the possibility of a hitherto unknown counterexample which will refute the theorem’s generality. Moreover, Paul Ernest (1991) presents mathematics as a socially constructed field of knowledge; there is no longer a need to assume an external mathematical reality and no longer a need for this craving for certainty.

Paul Ernest also relates this to mathematics education. If mathematics is invented it acquires a human face. It is not a timeless, unerring entity which imposes itself on students. It is only a human creation and students can re-invent it through the process of learning. Consequently, mathematics could become meaningful for students as a product of a human activity. Nevertheless, mathematics seems to retain this potential even if it is discovered. According to Galileo ‘the book of nature is written in the language of mathematics’ and understanding the world around us has always been meaningful to humans.

In any case, philosophy of mathematics suggests that it is hard to disentangle ontological from epistemological beliefs about mathematics. Therefore, in the following I also discuss epistemological beliefs of the students, but only in relation to the main question of ontology.

## METHODS

This article reports some preliminary results of a study investigating epistemic beliefs of Greek students at the last grade of upper secondary school (17–18 years old). The study follows a qualitative interpretivist paradigm. Twenty eight students were interviewed twice. The interviews investigated their relationship with mathematics through an epistemological lens touching upon subjects such as truth, certainty, logic, rules and usefulness and comparing mathematics to other courses or to life in general.

Before the second interview was conducted, the first one was transcribed and used as a stimulus for a further and more detailed discussion. Effectively, generating questions for the second interview with a particular student was influenced both from that student’s first interview and earlier first interviews; while later first interviews were also affected by this process. The duration between the two interviews varied between 10 days to one month and on average each interview lasted 70 minutes.

All students come from the same middle-class school of Athens. Practical reasons limited the research to this school where access was easily granted. However, the interviews revealed such a variety of beliefs that including other schools in the sample was not judged necessary.

The analysis is still in progress. All interviews have been transcribed and the two interviews of each student have been paired. The second interview is regarded as a continuation of the first one and each pair is analysed as a whole. So far I have worked with the paired interviews of 18 students in a chronological order. As a first step each of them was read as a story trying to identify the main factor or factors which marked the student’s relationship with mathematics. This initial reading revealed that the main points of each interview could be organised as a cohesive narrative around these factors. The factors were very diverse (e.g. doubt, theory, mistakes, fiction). However there were broad themes which appeared repeatedly in most of the narratives. The factors may be seen as different ways to colour such themes.

One of the themes is the ontological status of mathematics. This paper focuses on it in connection to epistemological issues of mathematical truth and

certainty, and to meaningfulness of mathematics for the students. The results that follow are organised around the concepts of discovery and invention. They are based on the interviews of eighteen students, who here have been given pseudonyms.

## FINDINGS

### Discovery

Some students maintained that mathematics exists. For example, Platonas, maintained

When in the past, they tried to interpret a phenomenon . . . they needed mathematics, in a sense they, not created it, in a sense mathematics was there, but they, that is, they discovered it, yes.

Of course, most students had a difficulty explaining how mathematics exists. Nonetheless, their belief was usually not shaken and even when it was, they still found it hard to coordinate this with their experience.

Yes, mathematics isn't something ordinary that you can say you discover, it is a way of reasoning. . . It's invented, now that you mention it, but it isn't that we came up with mathematics, now you'll ask me who did? (Aspasia)

A dubious concept was imaginary numbers. However, although most of them admitted that they are invented, they retained their Platonistic beliefs.

Yes, imaginary numbers are called imaginary exactly because we invented them. However, in general mathematics is discovered. (Xenofontas)

But mathematics hasn't been created. It's been discovered in the sense that, okay apart from some things which we have made in order to help us, in general mathematics is something that exists. (Foivos)

Mathematics was perceived to exist around us. It started from observing objects around us and it ends in explaining phenomena around us.

It's just that based on . . . numbers, humans defined that a certain object, this is the 1, this is the 2, and so slowly they discovered that around them there are groups of identical objects. So then they started doing operations, and this led after many

years in the invention<sup>3</sup> of theorems in order to justify phenomena that occurred around them. (Filia)

The paradox is that although discovery implies that mathematics is independent of human beings it also brings mathematics close to human beings. If mathematics is out there in the physical world then it is something quite intimate and not just some weird figment of imagination.

I know that it isn't impersonal and that everything is based on it. . . I've thought about it. In order to construct something the mathematics which made it is needed . . . so I'm grateful to mathematics. (Foivos)

None of the Platonists doubted that mathematics has applications in our lives.

the exercises, for example, they have applications on things that we want to find. . . for example, we have an, an equation and we want to know the result . . . for something that will help in our daily lives. (Filia)

Mathematics was important exactly because it explains our world and otherwise it wouldn't have been so developed.

No, [mathematics] would exist, but . . . we wouldn't have discovered it to the extent that we have discovered it now. (Ermis)

In all, mathematics was meaningful. Moreover, human agency was not absent with respect to mathematical discovery. After all, it is people, mathematicians, who produce mathematics. This could justify why students, who generally endorsed Platonism, sometimes utilised phrases which would hint at invention while describing mathematics as a human activity. Further justification is provided by the fact that invention succeeds anyway in penetrating mathematical activity. At least we did not find symbols in the world; we only agreed to use them in order to denote what we did find in the world.

As you go backwards you'll eventually reach the basis, an axiom of the kind  $1+1=2$ . . . This is so because *you have defined it so*. (Foivos)

I believe that it was *an initiative and an inspiration* of those who started all this. (Patonas)

Other common beliefs were that mathematics may change, but the change is incremental. Essentially, change is better perceived as development, an enlargement of mathematics when new data are discovered.

Yes I believe that if some needs lead to an extension of mathematics, then new rules will be discovered . . . on the basis of the old ones, of course. (Platonas)

No, this is a development . . . and complex numbers, which they didn't know, they discovered them. And it emerged through, now I remember. . . I think through physics, the issue of light. (Ermis)

New propositions complement the old ones. All of them believed that mathematics essentially comprises one system.

I don't know [if we could have defined things differently] because whatever we have defined we have defined it based on our universe, based on some things that we observe. (Foivos).

No, [it can't be different]. Mathematics is in a way the explanation of what we see. It's something natural, that is, you have one apple and another apple, so you have two apples, it can't be something else. (Xenofontas)

Different sub-systems may exist but they do not cancel each other; they co-exist as different models of the same reality. The old models suffice for certain cases, while the new ones explain new data which cannot fit the old model.

No, [Euclid] wasn't wrong. It's just that when they examined it deeper and with more cases . . . they suggested that other things may also happen. (Platonas)

The belief in one system and incremental change of mathematics is also reflected in their belief that there is a unique absolute truth which we may not be able to find, but which we slowly approach. Mathematical conclusions are part of this truth.

Truth is one-sided. . . I believe that new things are continually discovered. That is, soon we'll have learned much more; now we're still in the darkness. (Aspasia)

[The proof] is essentially the tangible evidence that a proposition that you have assumed is true. (Platonas)

Interestingly though, Platonism did not exclude verification of mathematics through fallible social processes.

Somebody says an idea, 500 people agree, 600 disagree and in the end one of the 600 finds something else or they simply agree because one of the 500 proves that it holds for additional reasons which the first one had not found. (Foivos)

This is reminiscent of Lakatos' *Proofs and Refutations* rather than Plato. However, it is not in opposition with Platonism per se. If mathematics is external to humans it can remain infallible even though their attempts to discover it are not. So, Platonism allows for certainty in mathematics even if people are not entirely certain about it.

When I think about mathematics and somebody shows me something, that this must be done, [then] I'll think why it mustn't, I will examine it. . . Therefore, so far: yes, I'll accept the results of mathematics, but always having also in mind the doubt that something else may hold. (Ermis)

## Invention

Most students suggested that mathematics is invented.

[Mathematical conclusions] are unshakable because they are stable, that is, they don't change. You'll tell me that some of them change, but they have been checked, as I mentioned before. It has been supported that they are unchangeable, that is, their value is permanent. (Platonas)

Generally, I don't believe that mathematics exists as a material idea, that is, you can't touch it. (Diomidis)

In mathematics there is 'if this holds then it's done so'. That's all there is. Or 'let', 'let this be' . . . Assumptions of the mind. (Evyenia).

It's a human creation. . . I think that when you prove something, you essentially make the rule. (Pelopidas)

The paradox in this case is that although invention implies that mathematics is part of the human intellect it may also create a gap between mathematics and the individual. This depends on whether the invention of mathematics is meaningful to the student. There were students for whom mathematics was deeply meaningful, students for whom mathematics had some worthwhile meaning and students who struggled to find any meaning in mathematics.

Yes [mathematics] is standardised . . . but this has another beauty. (Loukianos)

Yes, I belong to the couples who though separated I still love [mathematics]. (Litha)

Mathematics is completely theoretical, that is, the logic that it has, it won't produce . . . something crazy, that is, it won't be something that I can use in my everyday life, that's why I don't hold mathematics in great estimation. (Kosmas)

In the first case students had at least a feeble idea of axioms and perceived mathematics as something that humans have invented based on initial assumptions in order to suit their needs.

It doesn't mean that they hold necessarily, we just have created things so that they . . . improve our everyday life. (Lysimahos)

the world of mathematics is as we define it, that's why there are different geometries . . . And geometries, all that exist, they were created with the intention of solving some problems. (Kleomenis)

In the second case mathematical invention was perceived as some sort of experimentation. Mathematics was invented as applications corroborated some assumptions.

basically everything has an experiment. Because in order to find something new, for example, you must try it out. This is called experiment. (Lida)

I think that they solved many times an exercise or type of exercise . . . that they were reaching at

the same conclusion repeatedly, so . . . then they said to make it a rule . . . Not that they deliberately tried to make a rule, I believe that it just appeared. (Diomidis)

Finally, in the third case invention appeared to be the result of the lack of meaning.

I'd say pre-existed, pre-existed? It didn't pre-exist, it's all human investigation, I believe. (Kosmas)

That is, someone would have imagined all these, to someone all these came; it can't be just like this. (Evyenia)

Some students in the third group seemed to perceive mathematics as some people's personal views. These were students who held a highly relativistic view about life.

They should ask Pythagoras. . . [Me having an opinion on his theorem], essentially it's like me going and saying something with respect to a view of Socrates. (Klio)

Okay now, it would be somehow [strange], if we said for each [person] that they don't think correctly (Evyenia)

In all, only two students who chose invention believed in a unique truth, and even these did not believe that we had access to it. Moreover, they were both students who did not find mathematics meaningful.

we are just people, each of us is just a unit, If we could see the world from above then we would be able to judge that this is a definite truth, this is a definite lie. (Kosmas)

Certainty was much more moderate among students who maintained that mathematics is invented. However, it was present especially in the cases when mathematics was also meaningful – even moderately – to them. Some of them found certainty in the exact process of invention, but almost all of them grounded it on social reasons too. Nevertheless, the process of invention itself was excluded from certainty.

Because it's theory . . . basically there's no chance. . . in life, something may hold or may not hold . . . Well, no [it isn't strange that you don't find this in

mathematics] because mathematics is theoretical. (Kleomenis)

[We accept the first assumptions] because we get used to them . . . I think that there haven't been attempts to change . . . the foundations . . . So since they have results and validity in everyday life [we] continue using them. (Lysimahos)

[What's proven] usually doesn't change . . . all the mathematicians have seen them, and they have been considered . . . but I think that within the university context . . . I think that there is more room to doubt them and to be demolished by someone. (Diomidis)

Certainty was absent only in cases when mathematics was not meaningful to the students. This could simply be due to under-confidence, but sometimes was inherent of a subjective view of mathematics. If certainty persists in this group then it is genuinely social.

I wouldn't say that something said by mathematics is always true. . . you take cases and you assume, essentially, as we said before, 'let this be' or 'if that'. (Evyenia)

I haven't seen anything different, only what I have been taught . . . they haven't shown to me something else in order to believe that it may not be this way. (Pelopidas)

What is special about social certainty is that it can remain intact even in the face of change because each time it includes exactly these truths which are believed to be certain.

So until someone demolishes it, it's right, it's true. If it's demolished, then it's wrong. . . because it's truth, we accept the truth, but truth may many times be reversed with the presentation of new evidence. (Kosmas)

Nevertheless, certainty was not absolute, but the result of the scarcity of change or of the lack for necessity of change. Moreover, although past content was generally viewed as stable, it was not entirely safeguard against invention.

Okay, there is a chance of mistakes, but I believe that most of them won't change. (Diomidis)

If it changes then all the rest should change too . . . I'm not absolute about this not happening. I just don't think that it's possible to happen. (Danai)

Therefore, change is not necessarily incremental. Nevertheless, mathematics remained a unified system apart from the cases of utter subjectivity and of one student whose knowledge of axioms was more developed. Otherwise the system was one: what they have been taught in school.

The most typical example is geometry. Euclid organised it anyway, but afterwards Riemann? Who was it? He didn't like it; he wanted to use, to show other things, and so he changed it. (Kleomenis)

[Definitions may] not have the exact same words, they simply have the same sense . . . It can't be [that they don't have the same sense]. (Diomidis)

## CONCLUSION

Although the students had learned mathematics within a traditional setting, most of them believed that mathematics was invented. However, it was within the context of invention that mathematics could appear meaningless to students. Contrary to what would be expected according to the literature (e.g. Simon et al., 2000), students who believed that mathematics is discovered also viewed it as a human activity. Their account of the discovery was given in social terms and echoed *Proofs and Refutations* (1976). Most importantly, the fact that mathematics existed was coupled with mathematics' ability to explain the natural world and it made mathematics meaningful. On the other hand, some of the students who saw mathematics as a human invention failed to find meaning in it. Moreover, it seemed that this failure almost forced the idea of mathematics as an invention; it was just somebody else's invention and they could not see themselves in it.

Furthermore, Platonism is also associated with the belief that mathematics is a static body of knowledge (Charalambous et al., 2009). Nevertheless, all students regarded mathematics as something that evolves. A static element appeared indeed among Platonists, but referred to past knowledge and it did not prevent new data amending this knowledge. Additionally, this belief was not restricted to students who believed in discovery of mathematics. It was generally endorsed by most students though it was socially tinged when

mathematics was seen as an invention. Mathematics brings results so there is no reason to change it; old claims have been already checked by myriads of mathematicians and so on. Doubt, when present, seemed to be a general trait of the student's personality and the greatest doubter among my sample was a Platonist.

This picture of discovery or invention of mathematics as painted of the students of this study is quite different from the one usually forwarded by mathematics education. However, what seems more important is not what students believe about the being of mathematics, but whether they find meaning in it. If they do, then they will be willing to engage with it. It seems that Platonism may help towards this goal. It would also be interesting though to find the reasons which lie behind the divergent views of invention of mathematics. Some students do not find the invention meaningful. However, invention appeared to allow for a clearer view of the organisation of mathematical knowledge into axiomatic systems and thus a better understanding of mathematical epistemology.

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## ENDNOTES

1. Invention and discovery do not represent a strict dichotomy neither in the literature (e.g. Livio, 2011) nor in my interviews. However, for issues of space I will focus on the two extremes and on the predominant view in each student's interview.
2. I use 'Platonism' as an umbrella term for all theories which postulate that mathematics somehow exists.
3. Filia used invention meaning 'we were aware of it, that we wanted to find something'.