# Understanding issues in teaching mathematical modelling: Lessons from lesson study

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*This paper challenges the conceptualisation of the OECD's PISA model for assessment of mathematical processes and questions common approaches to modelling in the classroom. Drawing on evidence from research using a lesson study model, we argue that the crucial formulation phase of modelling, in which bridging between context and mathematics takes place, is undervalued. Consequently, we conclude that teaching towards assessment such as, or modelled on, PISA items could provide students with an impoverished experience of modelling and leave them inadequately prepared to engage in this important mathematical practice.*

**Keywords**: Modelling, problem solving processes, lesson study, PISA.

## INTRODUCTION

> *"The formulation of a problem is often more essential than its solution"*
>
> (Einstein & Infeld, 1938, p. 92)

Cai and Howson (2013) argue that there is evidence of some convergence of mathematics curricula around the world due to the TIMSS and PISA series of international comparative studies. In particular, as a result of the PISA series, there has been a noticeable increase in interest in mathematical modelling and problem solving. In this article, therefore, we focus on these important aspects of mathematics in schools. We consider this a particularly pertinent time to raise issues in relation to the PISA conceptualisation of problem solving and the validity of the framework that is used to define the domain of mathematics and the test items that result.

At the heart of our concern, and research, has been developing greater understanding of how, in classroom learning, students might develop a mathematical literacy that will better prepare them to be able to apply mathematics effectively in modelling problems so that they are able to make sense of situations that arise from a range of different contexts. There has been considerable theorising and research in areas that might inform our concerns. For example, in relation to mathematical literacy see Steen (2001), for mathematical modelling see the 14th ICMI Study (Blum, Galbraith, Henn, & Niss, 2007) and for problem solving see Schoenfeld (1992). However, as a mathematics education community, our detailed understanding of teachers' classroom practices and students' actions in problem solving and mathematical modelling is much less well developed than our understanding of conceptual development. We report here results from the first year of our ongoing research in relation to the teaching of problem solving and modelling.

## CONCEPTUALISATIONS OF PROBLEM SOLVING, MODELLING AND MATHEMATICAL PROCESSES

Acknowledging the important influence that the PISA series of international assessments play in informing the development of curricula, and by implication teaching and learning, around the world, we consider the PISA definition of mathematical literacy as an important starting point in understanding acknowledged conceptualisations of the field:

> Mathematical literacy is an individual's capacity to formulate, employ, and interpret mathematics in a variety of contexts. It includes reasoning mathematically and using mathematical concepts, procedures, facts and tools to describe, explain and predict phenomena. It assists individuals to recognise the role that mathematics plays in the world and to make the well-founded judgments

> and decisions needed by constructive, engaged and reflective citizens. (OECD, 2013, p. 25)

Fundamental to the goal of PISA is a quest to measure the ability of students to be able to use mathematics to make sense of different contexts that have relevance and authenticity. This has implications for the age appropriateness of problems/tasks used. In developing a framework or vision of mathematical literacy in practice PISA has at its core a modelling cycle (Figure 1). Such cycles are well known (for example, see Blum & Leiβ, 2007; Kaiser & Sriraman, 2006; Maaß, 2006; etc.). Although there are many variations both in the detail of the conceptualisation of the practice and its diagrammatic representation, the main the PISA diagram captures the essence of all.

Here the important processes as one moves from a contextual to a mathematical world and back again are:

- formulating – in which relevant mathematics that can lead to a solution, or sense-making, of the problem is identified. An appropriate mathematical structure and representation(s) are developed by making simplifying assumptions and identifying variables.
- employing – involves mathematical reasoning that draws on a range of concepts, procedures, facts and tools to provide a mathematical solution.
- interpreting and evaluating – involves making sense, and considering the validity, of the mathematical results/solution obtained in terms of the context in which the problem situation arises.

The cyclical representation of this overall process provides for the expectation that a refinement, or complete rethink, of the mathematical structure representing the real world situation may be desired, or even necessary. It is also important to bear in mind that progression around the cycle is not necessarily entirely one way, as there may be the need to refine thinking at any stage as the potential effects of decisions being taken become apparent and need modification as one proceeds.

The PISA framework, as in Figure 1, draws our attention to how problems may arise in a range of different contexts that can be classified as being *personal*, *societal*, *occupational* or *scientific*. Also of major importance are the mathematical content domains that in problem solving and modelling interact in symbiotic relationship with the problem-solving/modelling processes: PISA identifies these as being *quantity*, *uncertainty and data*, *change and relationships* and *shape and space*.

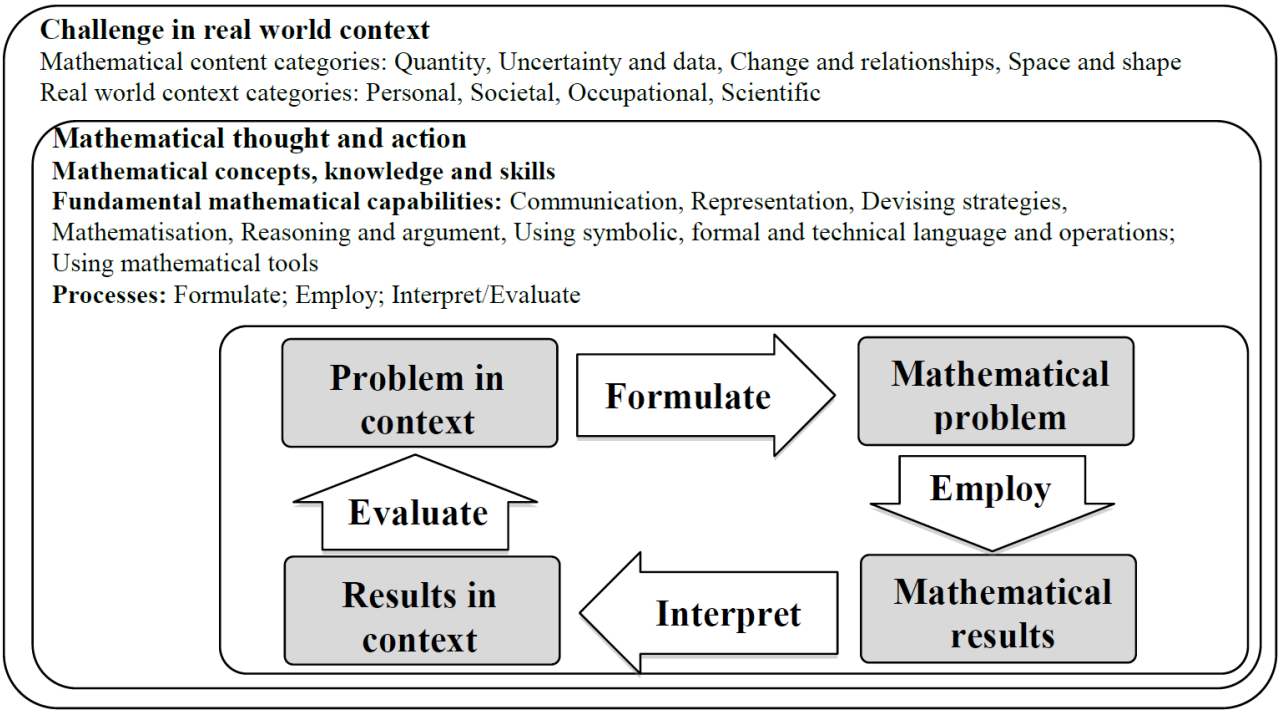


**Figure 1:** "A model of mathematical literacy in practice" (OECD, 2013)

In developing assessment items, it is acknowledged that it is not necessary that students engage with the whole modelling cycle: particular items may focus on only parts of modelling as a mathematical practice. This reflects the way in which adults engage with mathematics in practice, for example, in the workplace (Wake, 2014), where it is more usual to work with, or develop further, the mathematical models of others rather than start from scratch.

## METHODOLOGY

Fundamental to the research project that informs this paper is a concept of professional learning that is focused on practitioner enquiry into teaching, learning and classroom practice. The project aims, therefore, to develop and research professional learning communities in which teachers work together and learn from each other, informed by 'knowledgeable others', who have a role of stimulating the community by drawing on a range of expertise that is research-informed. An adaptation of the Japanese lesson-study model (Fernandez & Yoshida, 2004) has been used and continues to evolve. This involves a community of teachers and a 'knowledgeable other' collaborating in a cyclical process that involves planning a 'research lesson', joint observation of the lesson and critical reflection in a detailed post-lesson discussion (Wake, Foster, & Swan, 2013). Lesson Study is perhaps particularly attractive, as it has the potential to meet the requirements that we know facilitate effective professional learning (Joubert & Sutherland, 2009); namely, that it is:

— sustained over substantial periods of time;

— collaborative within mathematics departments/teams;

— informed by outside expertise;

— evidence-based/research-informed;

— attentive to the development of the mathematics itself.

Here we draw on our research which has involved 3–4 teachers at each of nine schools organised in two geographically located clusters (of 5 and 4 schools), with teachers collaborating and involved in research lessons across their cluster. We adopted a case-study methodology in order to obtain rich, contextual data. This data consists of video recordings of the planning meetings, research lessons and post-lesson discussions, researcher records of students' working in research lessons, and audio recordings of interviews with teachers and other participants.

In this paper we present, as indicative of the outcomes we observed across the 30 research lessons that have been carried out within the project to date, examples from the work of two students that encapsulate their mathematical activity and learning in relation to mathematical modelling.

## CASE STUDY

### The task and lesson

The research lesson was with a class of 13 to 14 year-olds with little, but some, experience of working on problem-solving/modelling tasks. The students had worked on the task in the lesson prior to the research lesson, providing the teacher with insight into the

**110 years on**

This photograph was taken about 110 years ago.
The girl on the left was about the same age as you. As she got older, she had children, grandchildren, great grandchildren and so on.
Now, 110 years later, all this girl's descendants are meeting for a family party.
How many descendants would you expect there to be altogether?

| **Twentieth Century facts** | |
|---|---|
| At the beginning of the 20th century the average number of children per family was 3.5.<br>By the end of the century this number had fallen to 1.7 | In 1900, life expectancy of new born children was 45 years for boys and 49 years for girls.<br>By the end of the century it was 75 years for boys and 80 years for girls. |

**Figure 2:** Task '110 years on'. (Source: Bowland Maths Assessment tasks: http://www.bowlandmaths.org.uk/)

different ways in which they were beginning to understand the context and problem and the ways in which they were formulating a mathematical model of the problem. The inquiry focus of the lesson-study group in this particular research lesson was to better understand how mathematical representations may assist structuring and supporting mathematical thinking.

The chosen task is one from a collection of assessment tasks to be found at http://www.bowlandmaths.org.uk/. These differ from PISA tasks in that they are open-ended modelling tasks; albeit with guidance for teachers about how they might observe progression in each of the process skills such as formulating, employing and so on. In this way they are much less structured than PISA tasks leaving students with considerably more scope to explore the context. Due to restrictions of space here, we illustrate student outcomes by reference to the work of only two students, these being chosen to provide some evidence of the diversity in student thinking in this particular lesson.

### Student A

Student A presented a list of key information that he considered relevant to the problem and also a list of assumptions which included the quantified factors presented in the formulation of the task. He also listed other factors that are not quantified in any way but are factors that could affect his solution. None of this is illustrated here due to space limitations. Student A's visualisation (Figure 3) of the situation effectively includes a timeline showing key years following the taking of the photograph. For example, 1903 is taken as the year in which the girl in the photograph is 13 years old. He assumed that after 3 years, in 1906, the girl married and after another 4 years, at age 20 she gave birth to 4 children. Throughout the period Student A assumed that people marry at age 16 and have children at age 20. He attempted to take account of the information that the average number of children per family was initially 3.5 by rounding to give 4 children in the first generation and then allocating a total of 14 children in the next generation (that is, that each of the girl's four children have 3.5 children). A similar logic underpinned his calculations to give the number of children in the next generation, although it is unclear how subsequent values in his diagram were calculated. Finally the student made some decisions about who from the early generations would have died in advance of the party.

### Student B

Student B's representation of the situation (Figure 4) appears to show *fewer* descendants in each subsequent generation, which is contrary to what we would expect. On closer inspection of the diagram other features appear equally strange: for example, Student B distinguishes between males and females in the diagram, using a different marker for each. This leads to brothers and sisters, as well as sisters and sisters or brothers and brothers, being the parents of offspring. The calculations presented by Student B (not illus-

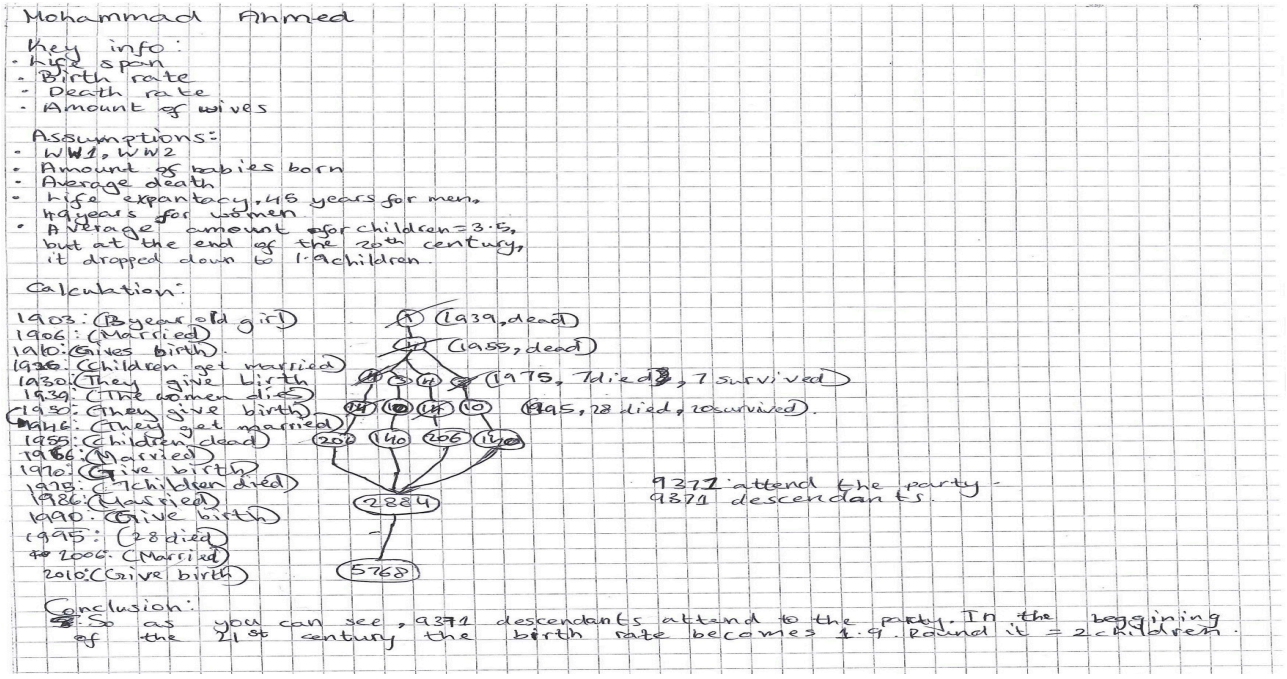


**Figure 3:** Extract from the response to the task '110 years on' by Student A

trated) provide insight into some of the thinking that underpins his diagram. It appears that Student B fell into the trap of selecting values given in the posing of the task and operating with these to calculate values he believed he needed to make progress. For example, he divided the information that there is a total time of 110 years between the photograph and the party by the average number of children per family (3.5) to calculate that altogether there should be 32 children. He also calculated that the girl in the photograph had 9 children by dividing 110 by 13 (the girl's age at the time of the photograph).

## ANALYSIS AND DISCUSSION

As can be seen from the examples of the work of the two students illustrated here, the students not only arrived at very different solutions, they have very different understandings of the context/situation. Indeed, it appears that student B did not have an appropriate understanding of the situation at all. This was also true of some of the other students in the class, as evidenced by their representations and calculations. Of course we would expect varied approaches by students working on a modelling task, but here it is clear that some of the models being *formulated* were invalid.

Our experience across, and analysis of data from, lessons confirms that this is a common occurrence and leads us to contend that there are essential aspects of both problem solving and modelling that are under-emphasised in school practices when compared to similar practices in out-of-school settings. Crucially, in the process of *formulating*, the difficulty associated with understanding a complex context, and how this needs to be simplified so that it can be represented mathematically, is underestimated. This process requires an understanding of how a range of mathematics (concepts, procedures, facts and tools) might best be marshalled to provide a mathematical structure to represent the problem context/situation. Such understanding is important and needs to inform the simplification of the context/situation. Pollack (1969), Borromeo Ferri (2006) and Treilibs (1979), among others, draw attention to this important aspect of mathematical modelling. Across our case studies we note that students have particular difficulties with this crucial first step in modelling.

This leads us to emphasise that the mathematical model being developed to provide a mathematical structure that maps to a *simplified* structure of the reality it represents. In the initial phase of developing a mathematical model, this simplification of reality to provide what is in effect a model of reality or 'real model' is not necessarily a simple matter. The structure that is proposed has to be a realistic representation, capturing essential elements of what might in fact be a complex situation, *and* it has to be such that the modeller can use mathematics that they know and are comfortable to work with. Borromeo Ferri (2006) discusses how

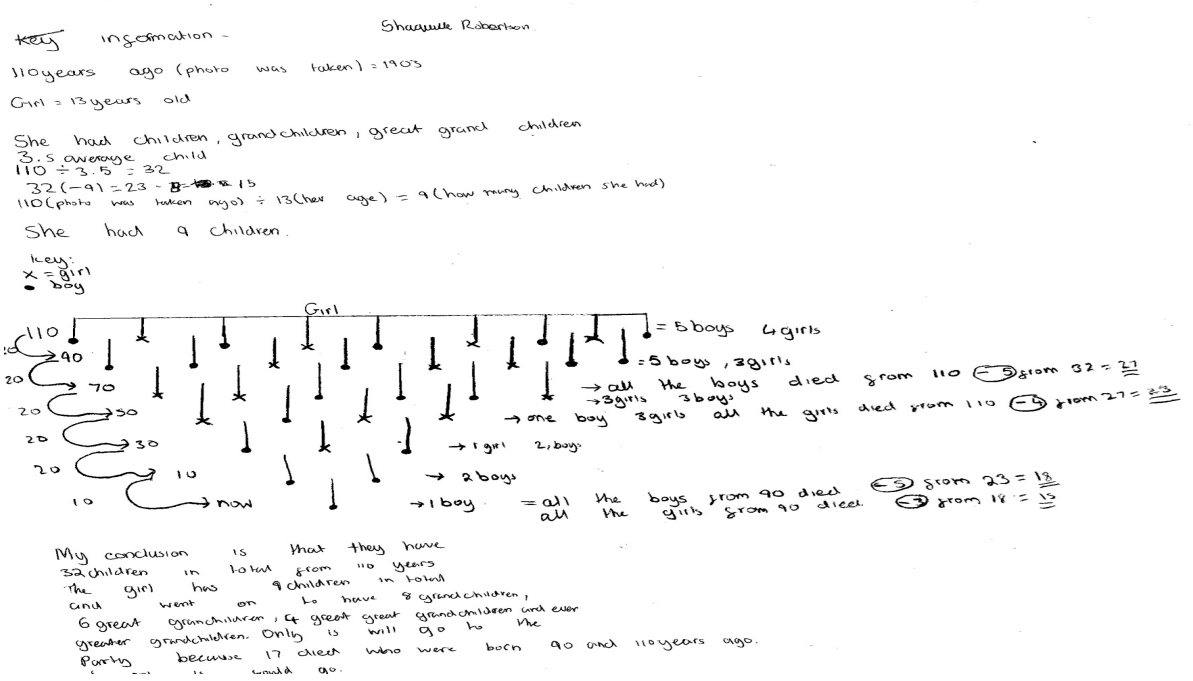


**Figure 4:** Response to the task '110 years on' by Student B

different researchers have attempted to characterise this early stage in the modelling cycle and identifies the terms 'situation model', 'mental representation of the situation' and 'real model' as being of significance here. Integral to this stage of mathematical modelling is the development of an understanding of the task: this should not necessarily be assumed as unproblematic. Blomhøj and Højgaard Jensen (2003) in their description of the modelling process and associated schema identify early processes as involving 'formulation of task' and 'systematisation', with the latter relating to making sense of the reality of the problem situation. Treilibs (1979) breaks down the formulation phase into subtasks, such as: modelling the situation by making simplifying assumptions, identifying relevant variables, generating relationships, and so on. Whatever terms we use to describe these initial steps towards being able to work mathematically, we have found that for many students the formulating stage can prove problematic and generate a lot of questions and discussions before a mathematical model can be developed.

Our consideration of this issue and how we might present tasks that involve students in effective mathematical modelling practices has raised the question of distinguishing between problem solving and modelling. We consider that mathematical models are mathematical structures that map to, or represent, a simplification of a real context, and as such they are useful when they have repeated use so as to consider variation of (a) factor(s) in the reality. For example, in the students' response to the task illustrated in this article assumptions made about the age at which people have children could be varied and the impact on results considered. In relation to this aspect of a mathematical model, we recognise that there are canonical models, such as exponential functions, inverse square laws, etc. that can, with adaptation, be used across numerous contexts and situations. In such cases, features of the mathematical model relate to factors and structure of the reality. For example, the growth of a population can be considered exponential when the rate of growth is proportional to the size, $P$, of the population, with the value of $k$ in the equation $P=P_0e^{kt}$ being related to the time it takes the population to double in size. In mathematics teaching we are concerned with such adaptations as well as with models that are bespoke to particular contexts and situations, such as in the task presented in this article.

In either a canonical model or a bespoke model, change of an identified factor in the reality results in change of a variable in the mathematical structure (model), and vice versa: that is, varying factors in the mathematical model has implications for the reality it represents. This, we use to distinguish between solving a problem using mathematics (which also has a mathematical structure that maps to a simplified reality) and modelling. In the case of problem solving, there is a single solution, albeit dependent on decisions taken to simplify the reality (which may eventually be modified/refined); on the other hand, in the case of modelling, the expectation is that there will be variation of important factors and repeated use of the model. We consider drawing attention to this characteristic of a mathematical model as having importance as a potential pedagogical vehicle that can be used to focus students' attention on the aspects of modelling that we identify as being under-emphasised in school mathematics. These aspects are (i) the simplification of reality and (ii) the development of a mathematical structure that represents, or maps onto, the simplified reality, with each of these needing to be informed by detailed understanding of the implications and potential of each other.

In the examples of student working that we present here we note that it is clear that the students' representations, and by implication their simplification of reality, do not adequately capture the essence of the situation in order to allow them to successfully arrive at a valid solution. In this particular lesson the variation in student approach to the problem and consequently their understanding was perhaps the most significant and immediate observation made by the observers of the lesson. Although developing a family tree was the most typical approach each student's manifestation of this suggested at least minor differences in their understanding of the context leading to significant differences in the assumptions being made, and consequently variables and fixed values decided upon. Some students took very different approaches with at least one student providing an entirely textual solution with no apparent diagrammatic visualisation or calculations evident. This variation in understanding resulted in solutions that were also very different with the number of descendants varying from values that were less than fifty to values greater than 1000. The focus for the students appears to have been on arriving at a single solution rather than developing an appropriate mathematical structure for the situa-

tion. This is particularly starkly visible in the work of Student B. It seems likely that a pedagogical approach that focused on developing a model that could have repeated application, allowing for variation of a key factor, such as the time between generations, has the potential to force this issue in the classroom. In considering the design of tasks that might be appropriate to bring to the surface the important aspects of modelling that we have identified, we have found that introducing the requirement for students to work towards a product, such as an explanation to a particular audience about the effects of varying a particular factor in the reality, is potentially helpful in generating awareness of this aspect of modelling/using models. For example, in the case of '110 years on' we suggest that students might be required to write advice and explain to a caterer possible maximum and minimum numbers of guests at the party.

We suggest that greater emphasis should be given to model development in mathematics lessons; this seems crucial if we are to support student learning of effective modelling. We note that this appears under-emphasised in the PISA organising framework for the mathematics domain and, as a consequence, and importantly, in their assessment items (OECD, 2013). We therefore urge that there needs to be careful thought about how best to support students' learning in curricula based on PISA's conceptualisation of the mathematics domain. As it currently stands, we view that teaching towards a PISA notion of problem-solving/modelling may not support the mathematically literate students we seek. We also recommend further research in this area of modelling as a classroom practice in mathematics so that we are better informed about student learning, and what it means to make progress in learning, in this important area.

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