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Competing conceptual systems and their impact on generating mathematical models

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Using a models and modelling perspective, this paper examines instances of competing conceptual systems within a problem situation for an engineering undergraduate. The goal is to explore how conflict impacts the framing of a mathematical model and to present evidence that the assumptions the modeller makes are influenced by his mathematical framing of the problem context.

Keywords: Mathematical modelling, tertiary education.

Research on mathematical thinking often focuses on pointing out or classifying student misconceptions. Fewer studies have examined how an individual makes sense of the problem context, though the student's interpretation may differ from what the task writer intends (Stillman, 2000). Sense making in modelling tasks is worth examining explicitly because student work leading to errors or incorrect models may be attributed to mathematical misconceptions rather than to the student's interpretation of the task.

It is unclear how an individual's framing of a modelling task is tied to the generation of a conventional mathematical representation, especially in a case where there are multiple possible framings. Each framing comes with its own set of assumptions, variables, parameters, and even constraints against which the model and its predictions must be checked. In a group work setting, Lesh & Doerr (2003) identified the production and resolution of "competing conceptual systems" as part of model creation and Lesh, Doerr, Carmona, & Hjalmarson (2003) classified the cognitive conflicts that accompany competing conceptual systems. But what is the interplay between mathematical framing and conceptual systems?

Drawing on a models and modelling perspective, this paper closely examines instances of competing

conceptual systems for a problem situation within a single individual in order to explore how the mathematical framing impacts the creation of the mathematical model. I argue that model construction is influenced by the mathematical framing adapted or derived from the individual's interactions with the problem context and that additional sources of knowledge, beyond mathematical and contextual, are necessary for resolving the conflicts that arise from competing conceptual systems.

THEORETICAL PERSPECTIVE AND DEFINITIONS

The theoretical perspective adopted here draws on the models and modelling perspective (MMP) (Lesh et al., 2003) and a characterization of modelling as a process (the mathematical modelling cycle or MMC) (Blum & Leiß, 2007). The theories are taken together in order to operationalize *mathematical framing* (defined below) and examine its connection to the conventional mathematical representation constructed to represent the real life situation being modelled.

In the most general terms, a *model* is a simplified representation of a system. A *mathematical model* has three components: a situation in the real world, a mathematical representation, and an invertible relationship between the two constructed by the modeller (Blum & Niss, 1991) that preserves structural characteristics as mathematical properties. A key component is identifying appropriate structural characteristics that can be put into correspondence with appropriate mathematical structures and concepts.

The MMP emphasizes the usefulness of mathematical concepts, that models (and not solutions) are the important products of modelling tasks, and that an individual approaches a task with an initial interpretation of the task. The process of formulating a well-posed mathematical problem can be summarized

as *framing* – how an individual renders a problem setting, giving it context, determining which facts and relationships are relevant, and which rules are usable for reasoning (Schwarzkopf, 2007). The framing process can be operationalized in terms of the MMC, which is briefly introduced below.

Blum & Leiß (2007) decomposed the mathematical modelling process into six stages of modelling construction supported by six transitional activities. The first three stages of model construction are: the situation model (an understanding of the problem; a conceptual model of the problem), the real model (an idealized version of the problem with simplifications and assumptions), and the mathematical representation (in conventional mathematical terms). The stages are connected by activities. The situation model is brought about by an individual forming an *understanding* of the problem. During *simplifying/structuring* the modeller identifies assumptions, variables, parameters, and conditions that reduce the potentially messy problem to an idealized real model. For example, an object falling from a height might be thought of as an object in free fall and a freebody diagram could be drawn where the relevant forces from Newtonian mechanics are identified. Next, *mathematizing* occurs where the individual represents the real model (in essence, a collection of assumptions and constraints) in conventional mathematical terms. This is the mathematical representation from Blum & Niss's (1991) definition.

Cognitive conflicts, in the Piagetian sense, often arise as the modeller attempts to frame the problem. Lesh, Doerr, Carmona, & Hjalmarson (2003) described three kinds of cognitive conflicts that arise as conceptual models develop: within-model mismatches (incongruence among aspects of representational media), model-reality mismatches (when predictions do not match reality), and between-model mismatches (incongruence between ways of thinking about a problem). For example an object falling from a great height could be modelled using the algebraically based kinematics equations. We know from mechanics that these equations hold only when air resistance is negligible. In a situation where an extended body is falling a great distance the effect of air resistance is not negligible. If we used kinematics equations to predict the velocity of the body at time t , the prediction might differ significantly from an actual measurement. This is an example of a model-reality mismatch. Debating

whether to use kinematics equations or more accurate differential equations to model the situation would be an example of a between-model mismatch.

The three mismatches are not mutually exclusive but are evidence of competing conceptual systems and highlight where the individual is considering alternative framings of the task. In this paper, I refer to the mismatches with regard to local mathematical model construction rather than global mathematical knowledge construction. The results are presented as a set of illustrative vignettes purposefully selected for their ability to demonstrate and explain the impact of framing on model construction.

METHODOLOGY

This study followed a case study logic found in social sciences (Walton, 1992) where the guiding principle was to provide evidence that challenged dominant ideas about the modelling process. Specifically, the goal was to provide insight into how competing conceptual models in order to argue that framing is idiosyncratically tied to the modeller's experiences. The phenomenon to be illustrated, *competing conceptual systems* “coalesced in the course of the research through a systematic dialogue of ideas and evidence” (Ragin, 2004, p. 127). The case is presented through a set of illustrative vignettes purposefully selected for their ability to demonstrate and explain the impact of framing on model construction.

Data were collected from a series of seven task-based interviews with four undergraduate engineering students enrolled in differential equations. Tasks were designed in order to evoke the mathematical modelling cycle (Blum & Leiß, 2007) and in accordance with guiding principles on openness of the problem statements (Lesh & Zawojewski, 2007; Maaß, 2010). The participants were selected in order to maximize variety in their approaches to modelling tasks and so they had a range of mathematical strength. The participant whose work is reported in this paper, Trystane, did not score top marks in his mathematics or engineering classes. His work on the Falling Body Problem (described below) was selected to share because within one task he exhibited all three kinds of cognitive conflict.

The interviews were video recorded and transcribed. Cognitive conflicts were identified in two ways. First,

the transcripts were coded according to the transitions in the MMC and the occurrences of the first three transitions were examined to be sure that framing was taking place. Second, transcripts were summarized as thick descriptions (Geertz, 1973) of the mathematics used by each student during each task. Introduction, changes, or adjustments in mathematical framings were tracked. Throughout the reduction, my focus was on the student's mathematics and on its structural ties to the real world situation presented by the problem statement. This analysis through writing produced a second-order model (Steffe, 2013) of the student's mathematics during modelling. Instances were interpreted as competing conceptual frameworks when the student was debating among mathematical framings or among assumptions that would simplify the problem but would require different mathematical framings.

The Falling Body Problem, for which Trystane's work is presented below, is a first-year physics or calculus problem solvable by kinematics if one assumes that there is no wind resistance. Otherwise, a differential equation is necessary to model the falling body's velocity.

On November 20, 2011, Willie Harris, 42, a man living on the west side of Austin, TX died from injuries sustained after jumping from a second floor window to escape a fire at his home. What was his impact speed?

ANALYSIS AND RESULTS

The vignettes presented below are based on the thick descriptions but reference the transcripts from a problem common to a first year kinematics course, *The Falling Body Problem*. Trystane's work on this task was selected to illustrate cognitive conflict and competing conceptual models within a single individual because it offers a sense of tension as he debated the merits of each. The ongoing tension is not conveyed when examining model revision among multiple individuals.

Trystane explicitly considers three different mathematical framings of the task that eventually lead to model revision. The framings were identified because he returned to understanding and simplifying/structuring the problem, according to the MMC. He first considers using algebraic kinematics equations, which wouldn't account for air resistance. He then

considers algebraic energy equations, until he realizes he doesn't have information about the man's weight. He then tries to derive a differential equation in order to account for air resistance. These distinct framings account for different variables, require different sets of assumptions, and rely on different mathematical concepts and structures.

Within-model mismatch

Trystane began by reading the task silently to himself and then stated that he would need to know how big the man was in order to account for air resistance. He indicated that mass was not an important factor and that his approach would be to use a "bunch of kinematics formulas." The kinematics formulas would not account for air resistance. He then indicated that he would like to use energy equations but that he'd need to know the man's weight. Since that information was unavailable, he'd have to "do it a different way and [he'd] have to know how high he fell from." Trystane's conflict arose due to a between-model mismatch. He was considering the merits of adopting different mathematical framings. Note that this is distinct from adopting different mathematical representations because both framings would be expressed in terms of an algebraic formula. There was a mismatch between the information perceived as available (note that he could have used m for mass) and his preferred mathematical framing (energy equations) leading him to adopt the kinematics equations. This framing subsequently influenced the variables and assumptions that could be made, prompting him to seek the height the body fell from.

Model-reality mismatch

Next, Trystane estimated that the window was 16 feet above ground. He attempted to use kinematics equations to calculate the impact velocity but encountered difficulty since "it's [the equation] got time in it and I don't know how long he fell." He resolved this issue and obtained an impact velocity of 32 ft/sec. He validated his answer: "At 32 feet, is actually the height of my house, is how I would think of it. He would fall, he would go that distance in 1 second. I'd say that's reasonably fast for a human to go." The interviewer then asked whether the size of the man mattered. Trystane responded, "Technically, yeah. If he's a really big guy, he's gonna have more wind resistance falling down. Other than that it wouldn't." Trystane indicated that he wanted to take wind resistance into account, but didn't know how to. This suggests a model-reality mis-

match because he knows that the extended body will experience air resistance but his model does not account for it. The competing conceptual systems were acknowledged (desire to include wind resistance and the fact that the kinematics equations did not incorporate it) and led to Trystane rejecting the kinematics equations and seeking to use a differential equation to describe the body's velocity throughout the fall.

Trystane began speaking about changing rate-of-descent and the interviewer responded by challenging his choice of model.

Trystane: Air resistance is more the faster you go, rather than the slower you go. So in [the man's] case, the higher up he fell from, the more wind resistance until he reached his terminal velocity. Which is when wind resistance is pushing up as much as gravity is pulling down.

Interviewer: What I'm hearing you say is that in very few cases wind resistance is actually negligible. You have to have a very short fall or be in a vacuum. Does it bother you at all that you don't learn how to take that into account or that the models you use all assume no air resistance?

Trystane responded that he felt that including air resistance in the models early in his physics studies would have been "needlessly complicated." He explained that wind resistance is negligible for most applications and offered an empirical demonstration by dropping a pencil and stated that "the pencil falling this high [placed pencil lower, near the desk surface] and this high [placed pencil higher, further from the desk surface] is so close to the same that it's not worth taking the effort to figure out what it is." This explanation demonstrates that negligibility of wind resist-

ance has to do with height rather than other factors he had mentioned previously such as mass or the value or change in value of the force due to wind, or size of the object. In general, negligibility of a variable is an assumption that is related to the estimated sensitivity of predictions to that variable and also to available information.

Between-model mismatch

Trystane's treatment of wind resistance demonstrates that he weighed expended effort against improvement of results when determining which variables to include and therefore which mathematical model to adopt. Though not in competition, Trystane entertained two conceptual models of the problem situation, one with wind resistance and one without. The source of the dispute was a between-model mismatch. The resolution was to choose the less-complex model because the extra effort necessary to build a more accurate model was not worthwhile.

Later, the interviewer suggested that Trystane use a differential equation to model the velocity of the falling body. He concurred and then began by indentifying the man's movement rate as a function of time as an important variable. He then wrote the equation (in Figure 1), which is a first-order, linear, homogeneous equation in standard form. He then wrote the generic solution where Q represents position and dQ/dt represents velocity. He wrote the solution with the intention of determining the value of λ . Assuming the initial positions and velocity were both zero, he substituted the general solution into the equation and obtained the expression and thus the result

Within-model mismatch

Trystane pondered the correctness of the model:

I'm not sure that that's right because I'm not sure if there should be some sort of constant increase

$$\frac{dQ(t)}{dt} + \beta Q(t) = 0$$

$$Q(t) = C e^{-\beta t}$$

$$Q(t) = C e^{\lambda t} \quad Q(t) = 0$$

$$Q(0) = 0$$

$$\lambda C e^{\lambda t} + C e^{\lambda t} = 0$$

$$\lambda C e^0 + \beta C e^0 = 0$$

$$\lambda = -\beta$$

Figure 1: First order linear differential equation with initial conditions

as you get faster, um, I guess that just stems from fluid mechanics. For instance, I don't know if it's a linear graph [draws left graph in Figure 2] or if as you're going faster it gets [traces figure on the right].

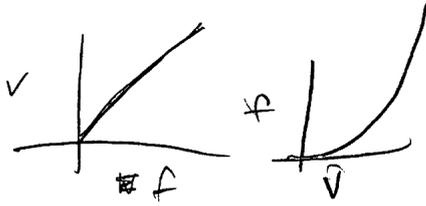


Figure 2: Exponential and linear relationships between air resistance and velocity

The graphs represented force due to wind vs velocity. Trystane knew that there should be an increasing relationship between the two quantities, but didn't know what the relationship would look like and he was debating which representation best matched the situation. He concluded that the model was incorrect because velocity should not increase exponentially with position as but that it was the best he could do without a fluid dynamics book.

Here, Trystane struggled to validate the model he selected because his conceptual model of the rate of change of the body's velocity was not well defined. The within-model mismatch surfaced as he tried to use the two distinct graphical representations of force versus velocity. The consequence was that he rejected the differential equations model because it was incompatible with his conceptual interpretation of ever-increasing velocity described by an exponential solution.

Next, Trystane noted that wind resistance was

sort of like friction where it's an opposing force. Actually, that kinda sparks an idea. If the wind is always just an opposing force, it could be treated like the force of friction. I dunno. Maybe in a certain density wind you would multiply the coefficient of wind friction or air friction times the surface area of the body moving.

Here Trystane reintroduced the size of the man as an important variable and spent the remainder of the session attempting to include it in his differential equations representation. His final mathematical model (after he "waive[d] the white flag") is the nonlinear differential equation shown in Figure 3.

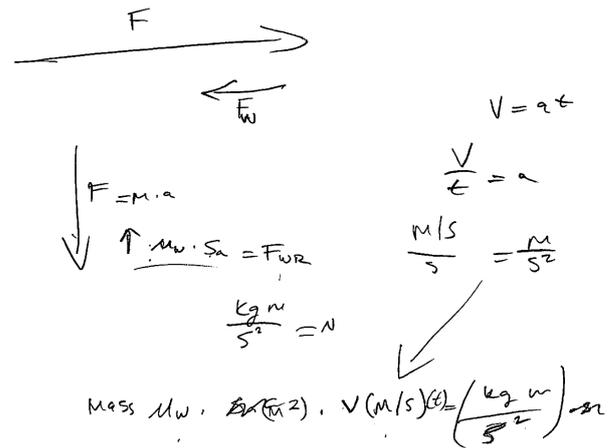


Figure 3: Nonlinear differential equation including wind resistance and surface area

Between-model mismatch

The transition from a first order linear equation with an incorrect solution to the final version of Trystane's model was provoked by competing conceptual systems that had been acknowledged at the start of the modelling task. One conceptualization of the physical system included wind resistance, which depended on the man's size, as a variable. The other conceptualizations ignored wind resistance in favour of algebraic equations with either known or easily-estimated parameters. This ongoing conflict, which was set aside while Trystane explored other possible models, was a between-model mismatch. Trystane's insistence on including wind resistance is a testament to how competing conceptual systems can provoke model development. Moreover, this look at competing conceptual systems suggests that the conceptual system determined the framing of the problem. That is, his selection of which mathematics to use was interactive-ly determined by the data he had available.

DISCUSSION AND CONCLUSIONS

Analysis illustrated three kinds of mismatches that arise as cognitive conflict during mathematical model construction. The mismatches were symptoms of competing conceptual systems each of which came with their own sets of assumptions, variables, and expectations. The conflicts were identified during validating activity when the modeller noted mismatches between expectations of how the mathematical model should be constructed and its congruence with his idealized conceptualization of the situation to be modelled. Thus, expectations were based on both real world experiences, such as Trystane's claim that

mass does not affect the velocity of a falling body, and the mathematical structure selected underlying the model, such as when Trystane rejected an exponential growth model for velocity.

Both mathematical and nonmathematical knowledge have been identified as important to model construction. Indeed, modelling is often characterized as bringing together both bodies of knowledge. In the vignettes above, Trystane generated and referred to ideas not explicitly in the problem statement. He relied on additional resources beyond mathematical and contextual knowledge. His spontaneous demonstration with the falling pencils and his resolution of the within-model mismatch between the two graphs of force against velocity suggest that he appeals to common sense and to thought experiments as resources for generating missing information. At times, Trystane had to abandon a particular framing because of a conflict where he did not have the relevant information to use the framing. In another case he was able to generate the missing information via a thought experiment. For example, the task did not give the height the body fell from but he was able to validate his answer (32 feet per second) by using the corresponding length of time it would take to fall 32 feet. In such a case, Trystane used a thought experiment to obtain and validate an appropriate estimate of the height.

Conflict resolution may depend certainly depends on student knowledge about the context and student characteristics (such as persistence). It may also depend on the modeler's values in model construction, and these values may be discipline specific. For example, Trystane demonstrated that he valued economy of effort. He valued avoiding "needlessly complicated" problem idealizations. He also valued avoiding revisions that made the model overly complex for only a marginal gain in accuracy or predictive power.

Trystane used knowledge both of plausible alternative mathematical representations and structures as well as sufficient knowledge of the problem context in order to resolve the conflict productively. Trystane demonstrated reasoning based on a blend of mathematical and nonmathematical knowledge: his value judgments about effort and worthwhileness and his ability to willingness missing information. This suggests that mathematical and contextual knowledge alone do not account for how the conceptual

model or subsequent mathematical model are revised. Future research should investigate criteria or factors students use to decide how to resolve competing conceptual systems.

Two related aspects that merit further examination are *how* the conflict resolution is executed and how conflict recognition can be promoted. When cognitive conflicts arose, at times Trystane rejected the initial model (such as his rejection of the energy equations) or to the revision of a model (such as his decision to include air resistance in the differential equation). One resolution required an additional assumption (neglect air resistance) while another required a change in mathematical structure (regarding net force as a function of time).

Mismatches between the individual's expectations and the model produce the cognitive conflict and the mismatches point to the presence of competing conceptual systems. When conflict is absent, the task may be too easy or familiar or the student may not recognize a particular kind of incongruence that is noticeable to the teacher or researcher. It should be noted that Trystane, as an engineering student was trained to look for such mismatches.

The conflicts arose when Trystane tried to fit available data to selected framings, such as setting initial velocity to zero in the differential equation. He changed representation and structure as information became available rather than undertaking derivations. He made progress when he had a mathematical frame that fit his personal (or scientific) experiences. This suggests that at least in some cases, framing precedes (or even determines) the relevant assumptions and variables sets.

Aside from theoretical consequences to perceptions of mathematical modelling as a cyclic, linear process, these observations have practical consequences. First, tasks should be selected that are amenable to multiple possible framings. Second, if a goal of using modelling tasks is to help students learn to make simplifying assumptions, it may be beneficial to use modelling tasks where potential framings are not obvious.

Mathematical modelling cycles have been long offered as descriptions of the mathematical modelling process, but the community still does not have an adequate explanation for how mathematical and non-

mathematical knowledge are blended to render a real world problem as a mathematical one. The models and modelling perspective adopted here, along with the cognitive conflict framework (Lesh et al., 2003) revealed that competing conceptual systems play an important role in the selection of appropriate mathematical structure, mathematical representation and the subsequent fitting of available data into the selected model. The cognitive conflicts framework is a promising avenue to reveal how validation of the mathematical model leads to resolution (or lack of resolution) of the competing conceptual systems.

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