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Assessing the best staircase: Students' modelling based on experimentation with real objects

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In this study, we look at the use of experimental activities for developing mathematical modelling, considering those as consistent with the Models and Modelling Perspective and Realistic Mathematics Education. The experimentation with several staircases elicited a central model based on linearization. Such model stems from the key idea of average step and constant slope and allows mathematical generalization.

Keywords: Mathematical models, realistic mathematics education, models and modelling perspective, experimental activities.

INTRODUCTION

The use of experimental activities in mathematics is a way of providing students a practical approach to real world problems with a particular emphasis on mathematical modelling. It encourages students to collect data, to interpret them and to develop reasoning and mathematical communication through discussion and presentation of findings. Experimentation with real and material daily objects in mathematics learning puts the real phenomenon as a central part of students' work, pushing them to understand how it works, to address it from a mathematical point of view, and to look for ways in which it can be "driven" mathematically, while incorporating real experience and everyday-life knowledge.

Unlike what happens in experimental sciences, it is far from common the inclusion of activities in mathematics classes where students are invited to examine and study material objects and artefacts from the real world in search for mathematical ways of interpreting the surrounding everyday reality. However, this

may represent an interesting possibility to address mathematical modelling from a distinctive perspective, namely by representing an opportunity to investigate the role of experimentation in the development of mathematical models in school learning (Bonotto, 2003; Halverscheid, 2008).

It is possible to explore the mathematics involved in everyday life by asking about the appropriate size of the steps of a staircase. In the image beside (Figure 1), one can see a diagram of a staircase profile retrieved from a website on housing and accessibility where the 'step geometry' for best preventing falls, especially from elderly people, is suggested.

The activity of studying a number of different staircases in the surrounding environment was seen a good context to engage 9th graders in significant mathematical modelling, namely entailing mathematical ideas such as gradient, slope, similarity of triangles, trigonometric ratios and others. Our research aim is to understand the effects of experimental work with artefacts on students' development of mathematical ideas and concepts through modelling tasks. In particular, we intend to see how the idea of slope is con-

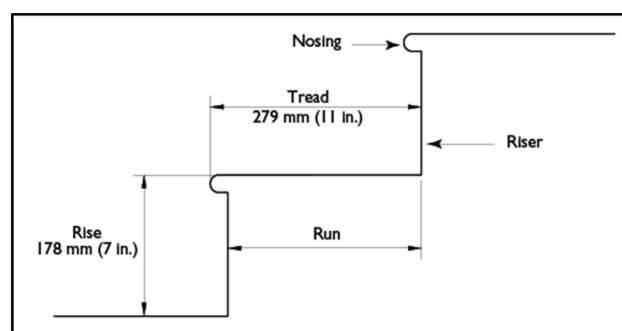


Figure 1: Suggested dimensions for the steps

(Source: http://www.cmhc-schl.gc.ca/en/co/acho/acho_012.cfm)

ceptualised throughout the activity of examining and assessing staircases.

THEORETICAL FRAMEWORK

The theoretical framework draws on both the theory of Realistic Mathematics Education and the Models and Modelling Perspective, by taking experimentation with real artefacts as a hinge which connects and articulates the two theories. This is carried out under the perspective of networking theories that bear significant intersecting concepts in spite of their different roots and foundations: in particular, both theories conceptualize learning in mathematics in terms of constructing and developing mathematical models of the real world.

The model development perspective

In the Models and Modelling Perspective (MMP), the term *model* is the key to explain students' learning of mathematics. A model is conceived as a system of elements, relationships between elements, and operations that describe or explain how the elements interact (Lesh & Doerr, 2003; Lesh & Harel, 2003; Lesh & Zawojewski, 2007). To become a model, the system has to be useful for describing, representing, interpreting, explaining or making predictions about a phenomenon and its behaviour.

In this perspective, modelling activities are important for students to reveal their various modes of reasoning, as well as for the development of conceptual systems and for creating efficient ways of representing structural aspects of the situation.

The focus of model exploration activities is on the underlying structure of the elicited model and especially on the strengths of various representations as well as on how to productively use these different representations (Årlebäck, Doerr, & O'Neil, 2013, p. 941).

An essential aspect of MMP is the recognition that solutions to problems usually involve several modelling cycles where the descriptions, explanations and predictions are gradually refined while solutions are being revised or rejected based on their interpretation in the real context.

Model eliciting activities are the kind of activities that provide students with real-life situations where math-

ematical thinking is necessary for successfully dealing with problems and achieving solutions (Lesh & Lehrer, 2003). The kind of problems proposed is meant to engage students in bringing forth their personal experience and informal knowledge. It is expected that more than one adequate solution is proposed and the use of tools, resources and collaboration are highly endorsed.

Students are presented a real situation from which a problem is originated. For solving it, they usually start to simulate the real situation through experimentation, then they have to model the situation (mathematizing it through diagrams, tables, symbols, relationships between variables, equations...), and finally they generate mathematical results and obtain one or more solutions to the initial problem. Throughout this process students perform micro-cycles of mathematization, i.e., within the modelling cycle, the steps are not all sequentially performed but rather there are forward and backward movements within the cycle itself.

The problem of finding a real model of a convenient staircase seems to comply with this perspective. The real situation of going up and down a ladder is something very common in daily life and it incorporates empirical knowledge and personal experience. There are staircases in many places, both indoors and outdoors, which makes experimentation, simulation, and actual experience with the artefact easy to accomplish. In addition, there is a clear purpose in electing a convenient staircase: everyone has felt, at some point, that there are stairs which aren't easy to climb or descend. Finally, a staircase is an object intended to join by successive steps different levels of a site and the mathematical concept of slope is immediately present in the relationship between the run and the rise of a step. Deciding on the best staircase involves forms of representing and expressing the real world situation, it comprises problem solving (or decision making), brings a specific purpose to modelling, and the resulting model can be developed, modified, and generalised.

The realistic mathematics education theory

Two main ideas that are central to the theory of Realistic Mathematics Education are *mathematization* and *guided reinvention*. Contextualized situations aim to generate meaningful experiences for students that will bring forth the implicit mathematics through

a mathematization process. By working on contextualized situations to achieve contextualized solutions, students gradually develop mathematical tools that progressively lead to higher levels of mathematical thinking.

The principle of guided reinvention requires contextualized problems to be well chosen, in order to engage students in the development of strategies that generate informal solutions. The informal way to get solutions is a starting point for formalization and generalization, which is referred to as progressive mathematization (Gravemeijer, 1994; 1999). The process of reinvention is set in motion when students use everyday language (informal description of phenomena) to develop informal or formal mathematical ways of conceptualizing solutions to real problems.

In RME, the starting point of mathematics teaching should be experientially real to the student. The mathematization process will lead to: explore the situation; locate and identify relevant mathematical elements; schematize and visualize patterns, and develop a model that integrates mathematical concepts. It is expected that students subsequently apply these to other realistic situations, and in doing so, reinforce and strengthen their mathematical knowledge.

The models allow students to work at different levels of abstraction, making possible that even those who have difficulty with more formal notions make progresses and create strategies for solving problems (Gravemeijer & Stephan, 2002). The term model refers to models of situations and to mathematical models that are developed by the students at different abstraction levels.

Four levels of models are described in the design of experiences based on the RME theory. At the situational level, domain-specific knowledge and situational strategies are used within the context of the situation, taking into account the knowledge and experience often obtained out of school; at the referential level or in the production of *models of*, the models are closely related to the situation described in the problem; at the general level or in the production of *models for*, mathematical strategies dominate over the reference to the context; and at the formal level, one works with conventional procedures and notations (formal mathematical language) without the support of the context of the situation (initial model).

One of the features that stands out from activities grounded on the RME theory is the mathematical concept(s) that the student is expected to develop. This is usually apparent when reading and interpreting the situations presented, which sometimes include a request to find a generalization. This generalization is not the only goal but it is surely a goal to be reached throughout continued work on this type of activities.

From the point of view of RME, the situation of deciding on a convenient staircase motivates students to work actively on the context. The mathematization process is expected to start with realistically considering the actual situation of climbing and descending staircases. A referential model would probably start to grow from sketches of different staircases, leading to the idea of step as a covariation between horizontal and vertical distance. A model of the covariation might be developed in close relationship with the particular features of a staircase and in terms of how it fits human body (the size of the foot and the range of a person's step). Ultimately a general model of a staircase may be developed in terms of relevant dimensions and this entails a mathematical model for the slope of a straight line.

Experimentation and artefacts use in mathematics

In this study, we have in mind the rationale offered by Bonotto (2003, 2010, 2013) and by other researchers (e.g., Alsina, 2007; Halverscheid, 2008) for the use of experiments in school mathematics. The main idea is that handling and manipulating objects and cultural artefacts existing in students' daily lives is an important way to connect mathematics with reality.

... an extensive use of suitable artifacts could be a useful instrument in creating a new link between school mathematics and everyday-life with its incorporated mathematics, by bringing students' everyday-life experiences and reasoning into play (Bonotto, 2010, p. 21).

It can be argued that mathematical modelling involves a comprehensive cycle that stands on its own as a mathematical approach to solving real problems. However, both MMP and RME are theories that seek to connect modelling and mathematizing with the learning of mathematical ideas and the development of mathematical meaning. They both promote activities where realistic situations require some kind of informal

knowledge and the unfolding of some mathematics by acting upon and representing real phenomena. They both place mathematics as the ultimate target of the modelling activity but they both acknowledge that models have to be evaluated as to their agreement with the real world and revised if needed.

Experiences with cultural artefacts are more than a context for the formulation of problems; they are the realistic contexts in which data, informal constructions, conjectures and testing occurs, that is, they become the real world.

Experiments related to mathematics find their natural place in the framework of mathematical modelling because they represent the 'rest of the world' for which mathematical models are built (Halverscheid, 2008, p. 226).

Experimental activities with real objects as the basis for mathematical modelling are grounded on the following facts: (1) students have the opportunity to "learn by doing" as they manipulate and experiment, conjecture and validate; (2) working with concrete materials is a form of questioning mathematically the properties of objects; and (3) inquiry through experimentation is reflected in thinking and mental models and becomes a means to develop an understanding of mathematical models.

Quite often the argument that manipulatives are only useful in elementary schooling is set forth. In contrast, we claim that mathematical modelling with experiments and artefacts may synthesize the connection between two relevant theories – RME and MMP – for the inclusion of mathematical modelling in mathematics learning.

METHODOLOGY

Our study concerns a classroom activity implemented in two classes of 9th grade students (14–15 year-olds),

one with 20 students and the other with 23 students. Throughout the teaching intervention several other activities that involved experimentation with cultural and everyday objects were proposed. The activities were performed in groups of 4 to 5 students in each class, totalling ten groups. The groups were video-recorded during the classes; the transcribed dialogues and the written reports of the groups were collected and analysed.

Qualitative content analysis of the data was developed and episodes were selected to document students' models and the conceptual structures underlying such models.

The task

The real problem consists of establishing the convenient dimensions of a staircase for a house building. To initiate the activity, the examination of particular cases was proposed as a means to define the convenience criteria; measuring, recording and sketching were also conducted, which entailed the identification of variables and relationships between them (Figure 2). Therefore, students went out of the school and walked in the city to experience many of the stairs that are part of the outdoor architecture, from the oldest to the newest ones.

The activity was divided into four parts. The first is the introduction to the topic under study. The second, called "From experience..." is a hands-on experimental phase performed with everyday objects (staircases) aiming at developing informal models and collecting data. The third, called "... to the model" consists of a mathematical study of the data obtained in the experimental stage, and one of its goals is the creation of a local mathematical model and possibly a more general model that can apply to wide-ranging situations. Finally, a written report is required to document the experimental situation, the assumptions made, the strategy used, the results obtained, and the evaluation of the proposed solution.



Figure 2: Staircases in the city and students collecting data

DATA AND RESULTS

The model of the average step and constant slope

During the fieldwork, all students experienced up and down five staircases with different characteristics, in order to classify them from 1 to 5, corresponding to a scale from less convenient (1) to more convenient (5). They also measured the run and the rise of several steps in each of the staircases and recorded them on their worksheets. The five staircases were proposed by the teacher within a relatively limited area around the school and fitting the time available for working on the outside.

One observation made by several groups was that the steps in certain staircases did not maintain the same dimensions, with variations of a few centimetres on the treads and on the mirrors. The subsequent discussion was on the possibility of standardizing the sizes of the steps on each staircase considered. The idea of taking the average of the lengths of the treads and the lengths of the mirrors was one of the suggestions. One group realised that the sum of the treads corresponded to the total run the same way as the sum of the mirrors was equal to the total rise. Then, dividing by the number of steps they got the average step dimensions and started to analyse the ratio between them. Another group preferred to measure the pitch line (the hypotenuse) for a sequence of k steps and found the total rise by adding up the mirrors of the k steps. Then using Pythagoras obtained the total run and finally divided the run and the rise by k to get the average step dimensions.

Therefore the model used by most of the groups for the staircase was a model of constant slope regardless the variability of the steps. By considering homoge-

nous steps, students were making the assumption that each step corresponds to a unit of change. No matter the steps were irregular, the inclination was consistent in every step. Based on the ideas that students proposed, which evolved around the steepness of the staircase, the notion of constant slope was being developed as schematized in Figure 3. Even though they have not mentioned this ratio as the slope of the pitch line, their reasoning involved the idea of inclination of a straight line.

In their reports, students showed their representations of the staircases and their evaluations in terms of convenience for climbing and descending. To their qualitative evaluation they associated the ratio of the vertical change (mirror) to the horizontal change (tread), even though they did not define this ratio as the slope of the pitch line. In their discussions it became consensual that there was a distinctive ratio for the most convenient stairs, as in the following dialogue:

(After reviewing the classification made and the relations found)

Student M: So the conclusion is that the tread should be twice the mirror so that the stairs are convenient!

Teacher: But can the tread be of any length as long as it's twice the mirror?

Student D: A bit larger than the size of the foot.

Student M: About 30 cm.

An extract of this group's report shows how they assessed a few staircases (Figure 4).

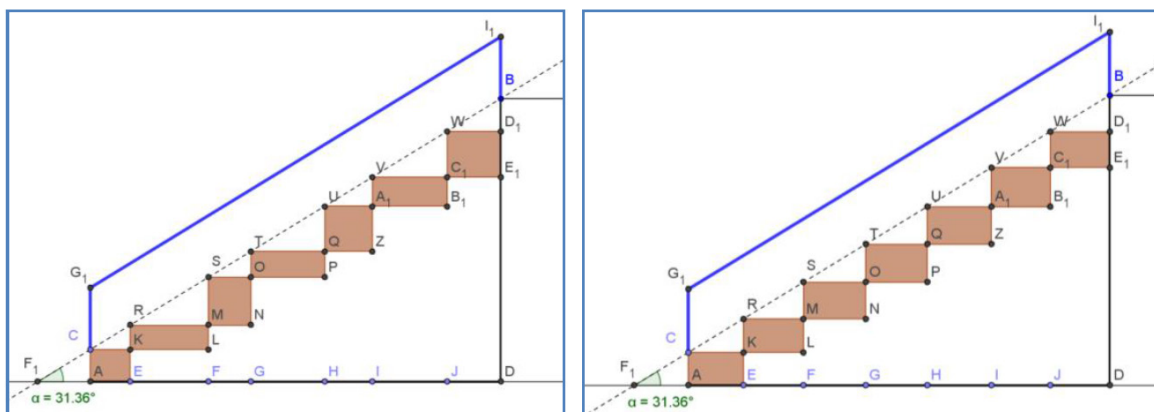


Figure 3: The variation of the steps doesn't change the slope of the staircase

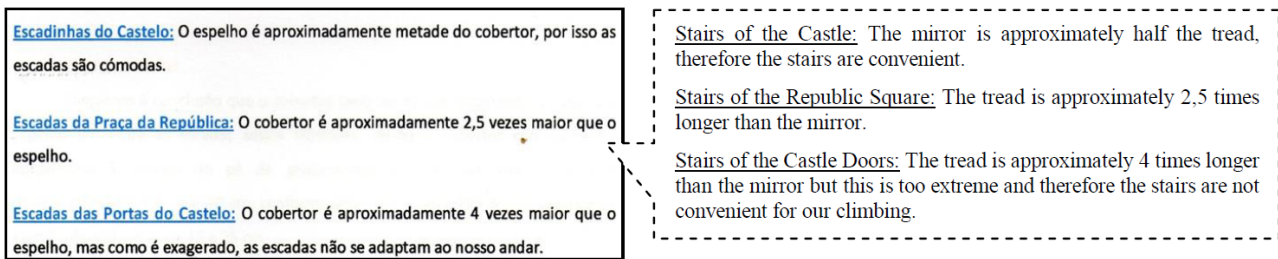


Figure 4: Excerpt of a report with the slope as a criterion for convenience

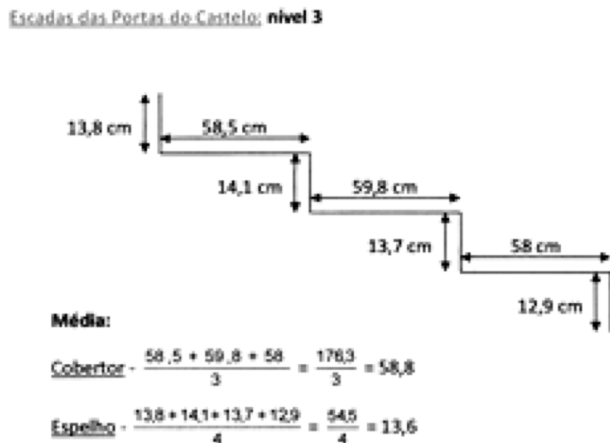


Figure 5: Model of the stairs of the Castle Doors (level 3) and the average step

Another piece of the report shows the idea of using the mean of the dimensions of a sample of steps to find out the dimensions of the average step by assuming a constant slope for the staircase (Figure 5). This was a common strategy in other groups, as a way of dealing with the irregular steps.

The conclusion of this particular group about a convenient staircase was summarised as follows: “We have found that the tread should be approximately twice the mirror so that the stairs are convenient, and the tread would have the approximate length of the foot of most people, therefore measuring between 30 and 40 cm, depending on being indoors or outdoors stairs. As to the mirror, it should measure between 15 and 25 cm”.

DISCUSSION AND CONCLUSIONS

The mathematics curriculum for grades 7–9 covers the study of the linear function and stresses the meaning of the parameters in the equation $y = ax + b$, in particular through its graphical representation. The meaning of the parameter a is handled qualitatively in terms of the slope of the line. The linear function is also related to the idea of constant rate of change. However, this conceptual model brings the difficulty

of associating a rate between quantities to a geometric notion.

The task of finding a convenient staircase led to the observation that real stairs in the existing surroundings are not perfectly regular stairs. It also showed several facts clearly important to the concept of linear variation and to its translation into a geometric model. In fact, a plausible real model of a staircase is coherent with the notion of regularity, that is, all the steps having the same dimensions. However, it is not necessarily so in real life. Yet, the idea of a constant slope is a good initial model of a staircase. In other words, the average slope of the steps corresponds to the slope of the staircase, and both are consistent with the rate between the rise and the run.

From the point of view of the MMP, the activity of choosing a convenient staircase elicited a relevant mathematical model. The profile of a staircase is consistent with the constant ratio between the length of the tread and the length of the mirror thus suggesting how steep the staircase is. It is a robust and mathematizable idea which makes the basis of the mathematical model of linear variation (or slope):

$$\frac{(\Delta y)_1}{(\Delta x)_1} = \frac{(\Delta y)_2}{(\Delta x)_2} = \dots = \frac{(\Delta y)_i}{(\Delta x)_i}$$

for any number of steps i , and

$$\frac{\sum_i (\Delta y)_i}{\sum_i (\Delta x)_i} = \frac{(\Delta y)_i}{(\Delta x)_i}$$

Its translation into the geometric constant rate of change is well represented by the variable but proportional dimensions of the steps as in the following graphs, showing different changes in run and rise (Figure 6).

Regarding the theory of RME, the mathematization of the idea of a regular staircase matches an informal model of linearization, leading to calculating the mean of the dimensions of the different steps. Then,

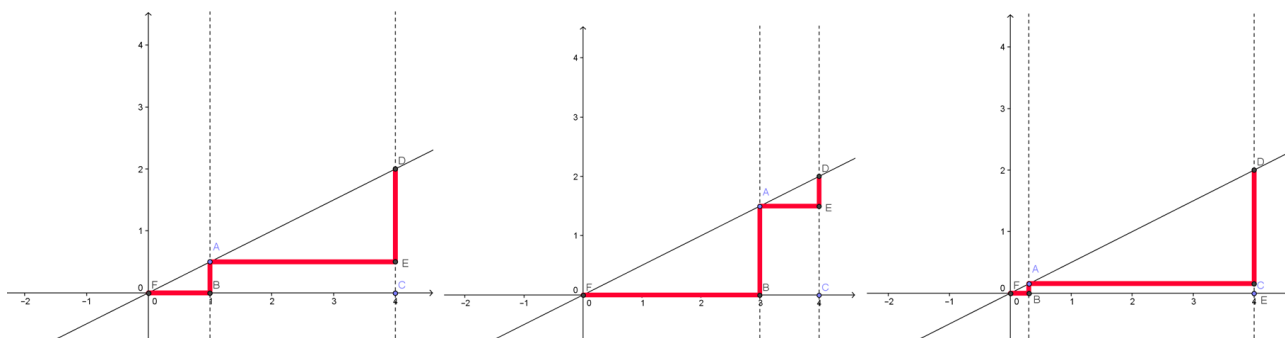


Figure 6: Variable but proportional dimensions of the steps (equal slope)

the relationship between the average rise and the average run develops into a gradient: the constant rate at which the value of the rise changes with respect to the change of the run, or the 'slope of the staircase'. Therefore the slope has a connection to the steepness of the stairs as expressed by the ratio of the mirror to the tread. A model of a convenient staircase (referential model) was characterized by the students as having a ratio of 2:1 for tread to mirror. This is a *model of a convenient staircase* but it is also a model that can be easily transformed into a *model for any staircase* and likewise for the formal constant rate of change.

Both theories allow seeing mathematical modelling as a process that simultaneously creates a simplification of reality, introduces a mathematical point of view and leads to interpreting reality through mathematics and interpreting mathematics through reality.

The experimental, practical and direct way to address the problem through examination of several outdoors real stairs proved to be of vital importance. In the real world stairs are uneven and linearization emerged as a sensible mathematization tool.

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