At the core of modelling: Connecting, coordinating and integrating models

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This theoretical paper introduces the notions of connecting, coordinating and integrating models to analyse and reflect on how models are created and developed. We define, discuss and apply these constructs to some theoretical perspectives in the present modelling discourse. We draw on an example from a model application activity within a model development sequence to illustrate these constructs. Our hope is to spark a discussion that will enhance our understanding about the nature of mathematical modelling and the teaching and learning of, and through, modelling.

Keywords: Modelling, theory, model development sequences, variation theory.

There are many different perspectives on modelling that researchers can adapt when studying the teaching and learning of mathematics (Kaiser & Sriraman, 2006). One commonly made distinction in the literature is the one between modelling and application. Often this characterisation is based on a philosophical stance about the nature and relation between 'mathematical knowledge' (mathematics) and 'knowledge about the rest of the experienced world' (reality) (Blum, Galbraith, & Niss, 2007). Stillman (2012, p. 903) describes this distinction as follows:

> With applications the direction (mathematics → reality) is the focus. "Where can I use this particular piece of mathematical knowledge?" The model is already learnt and built. With mathematical modelling the reverse direction (reality→mathematics) becomes the focus. "Where can I find some mathematics to help me with this problem?" The model has to be built through idealising, specifying and mathematising the real world situation. Both types of task have their place in school classrooms.

Research on modelling perceived in the latter characterisation often uses, or is based on, an idealized conceptualization of modelling as a cyclic process (e.g., Blum & Leiβ, 2007; Borromeo Ferri, 2006).

The role and function of applications and modelling in the teaching and learning of mathematics is also an important dimension in the on-going discussion – is teaching modelling a goal in itself, or is modelling a vehicle for teaching and learning mathematics? Both modelling as described by Stillman as well as most modelling perspectives represented by a cycle diagram of the modelling process include, or at least point out, the potential driver in modelling for teaching and learning mathematics.

Another perspective on modelling is the models and modelling perspective (Lesh & Doerr, 2003b), which provides a coherent framework to think about multitude aspects involved in teaching and learning. Central in the models and modelling perspective (MMP) are the students' previous experiences and knowledge, and how contexts are chosen and used in modelling tasks. Whereas perspectives aligning with the cyclic view on modelling (c.f. Blum & Leiβ (2007) and others) clearly are modelling according to Stillman's distinction, it is not in our view possible to situate the MMP in this dichotomy. Rather, MMP blends applications and modelling to form a modelling-based pedagogy, and through this use of modelling and applications students arguably learn both mathematics and mathematical modelling.

Although our overall research interest aims to better understand the teaching of mathematical modelling, and the teaching of mathematics through modelling, in this paper we ask more fundamental questions about the nature of modelling. We also wish to initiate a discussion about the nature of mathematical

modelling, as well as teaching and learning mathematics through modelling, by examining the constructs of connecting, coordinating, and integrating models.

Before addressing the notions of connecting, coordinating and integrating models, we will discuss some of the central ideas in the models and modelling perspective. Our initial thinking is based on our work within this perspective, and the examples used for illustrational purposes are from this context.

THEORETICAL FRAMEWORK

In the models and modelling perspective, "*[m]odels* are conceptual systems (consisting of elements, relations, operations, and rules governing interactions) that are expressed using external notation systems, and that are used to construct, describe, or explain the behaviours of other system(s) – perhaps so that the other system can be manipulated or predicted intelligently. A *mathematical* model focuses on structural characteristics ... of the relevant systems" (Lesh & Doerr, 2003a, p. 10, italics in original). It is by engaging in learning activities that students' models are developed, modified, extended and revised through "multiple cycles of interpretations, descriptions, conjectures, explanations and justifications that are iteratively refined and reconstructed by the learner" (Doerr & English, 2003, p. 112).

A well-established line of research within this perspective has focused on model eliciting activities (MEAs) in multiple contexts with learners from primary school through university (see references in Ärlebäck, Doerr, and O'Neil (2013)). MEAs are activities where students are confronted with a problem situation in which they need to construct a model in order to make sense of the situation. There are six well-established principles for designing MEAs. The six design principles are: the reality (or sense-making) principle; the construction principle; the self-evaluation principle; the documentation principle; the simple prototype; the generalization principle (Lesh & Doerr, 2003a; Lesh, Hoover, Hole, Kelly, & Post, 2000). However, isolated MEAs can fall short of supporting students developing a generalized model that can be used and re-used in a range of contexts (Doerr & English, 2003). What is needed are multiple structurally related modelling activities offering multiple opportunities for the students to explore, apply and test relevant mathematical constructs in different situations and contexts. This is the idea and function of model development sequences (Doerr & English, 2003; Lesh, Cramer, Doerr, Post, & Zawojewski, 2003).

Model development sequences begin with a MEA to confront the student with the need to construct a model to make sense of a problem situation. The MEA is then followed by one or more model exploration activities and model application activities (see Figure 1). Model exploration activities (MXA) focus on the underlying structure of the elicited model in the MEA with special attention to the use and function of different ways to represent the elicited model. The initially elicited model is further developed by examining the strengths of various representations and ways of using representations productively. Model application activities (MAA) engage students in applying their model to new situations and contexts, thereby refining their language for interpreting and describing the context.

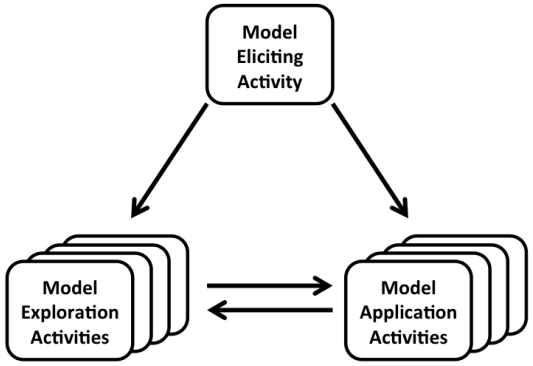

Figure 1: The general structure of a model development sequence

When students work through the model development sequence, they engage in multiple cycles of descriptions, interpretations, conjectures and explanations, resulting in iteratively refining and developing their models. In this process, interacting with other students and participating in teacher-led class discussions are key practices for facilitating this development.

A model development sequence focusing on the average rate of change

We now turn to briefly describe a model development sequence focusing on average rate of change consisting of one MEA, two MXAs, and two MAAs (see Figure 2). For a more detailed description see Ärlebäck, Doerr and O'Neil (2013). From this point an onwards, references to the particular activities

in this sequence will be indicated with an asterisk (*). In the MEA* of this sequence, students analysed their own bodily motion along a straight line. They experimented with motion detectors attached to graphing calculators generating position vs. time graphs, created linear graphs based on written specifications, replicated the motion behind given position vs. time graphs, and generated written descriptions of how they moved. Within this context, students' initial thinking and models about function values (position), average rate of change (average velocity), sequences of differing values of average rate of change (sequences of differing average velocities) and the relationship between these quantitates were elicited.

In the two MXA*s the students explored various representations to describe and interpret changing phenomena using their emerging model of average rate of change. Using two different computer environments, the students generated motion of animated characters by creating velocity vs. time graphs, created position graphs from velocity information, investigated how the average rate of change of a function is represented as table values, graphs, and equations, and explored representations of exponential growth and decay.

In the two MAA*s that followed, the students used their models to make explicit interpretations, descriptions and predictions about the behaviour of, first, the intensity of light with respect to the distance from a light source, and second, the voltage drop over a fully charged discharging capacitor in an simple resistor-capacitor circuit. The MAA*s gave the students opportunities to use their models in different contexts (one having distance rather than time as the independent variable), and to work with phenomena with negative rates of change.

A model application activity – the light lab

In this section, we will briefly describe the Light Lab model application activity (MAA). The overall structure of the six tasks in this MAA is shown in Figure 2.

The first pre-lab task focused on the students' intuitive and initial models about how intensity varies depending on the distance from a point light source. In a one-dimensional scenario of an approaching car the students sketched graphs of how the intensity of the car's headlights varied depending on the distance to the car, and described how light disperses from a point source in terms of light rays.

In the second task, students used a point source of light to collect 15 measurements of light intensity data at one cm intervals from the source. In part one of the lab, students made scatter plots of their data and wrote descriptions of how the intensity of the light changed with respect to distance from the light source, and compared this relationship to their predictions from the first pre-lab. The students also calculated, described and plotted the average rate of change of the data in one cm intervals, and created rate graphs of the calculated average rates of change.

The second pre-lab introduced an inverse square model for how the light intensity varies with distance from a light source. Using four images representing light intensity indicated by number of dots per square inch at different distances, students determined the intensity at given distances from the light source.

In the second part of the lab, students determined a function fitting their collected data, explained their work, and analyzed the average rates of change of their function using the difference quotient; they calculated and graphed the average rates of change of the function, and described and interpreted the graphs of the average rates of change values. We will use these tasks from within this model application activity to examine our notions of connecting, coordinating and integrating models.

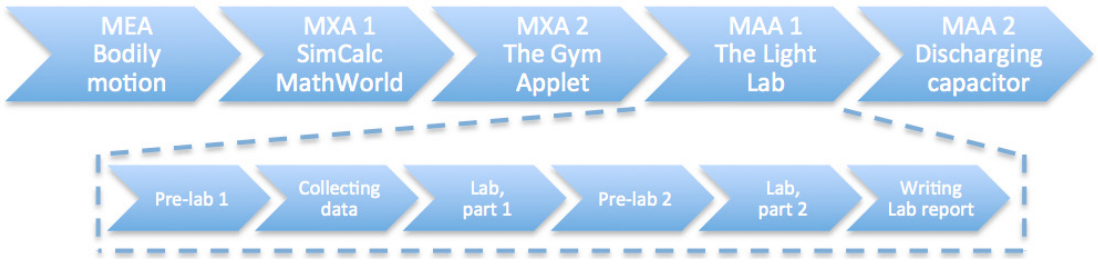

Figure 2: The model development sequence (top) and the six Light Lab tasks (bottom)

CONNECTING, COORDINATING AND INTEGRATING MODELS

We take *connecting models* to mean the establishing of a relationship between two or more previously unrelated models. As used here, connecting models captures the realization that previously isolated and unattached models, potentially from different disciplines or subject matters, partially overlap or have points of interest in common that in a given problem solving situation seem productive to explore. Connecting models also means taking the initial steps in identifying and delimiting the models being connected in the first place. Connecting models can thus be thought of as a united set of models together with some initial ideas and rationale for why these models might productively be considered together (see Figure 3). The reasons and rationale for connecting the models can be intuitive, tentative or speculative in nature, and to a large extent will depend on the modeller's previous experiences and knowledge. An example of an activity connecting models is the first pre-lab in the Light lab MAA*. Here, students' intuitive and initial models about how intensity varies depending on the distance from a point source of light in terms of previous experiences of car's headlights and how light disperses from a point source in terms of light rays (knowledge from previous courses in physics) are elicited. The juxtaposition of these questions implicitly suggests that it might be productive to consider these conceptual systems (or models) together. Another example of an activity connecting models from the Light lab MAA* comes from the second pre-lab which introduces a new representation (density of dots at varying distances) to be examined and considered along with the set of models the students are currently working with.

When a set of models has been connected, the *coordinating of the models* is the successively more systematic exploration and pursuit of the overlaps and points of interest in common across the models. The coordination of the models transforms the set containing the previously disjoint models into a set of more aligned models. This is a process entailing gradually mapping out where, and becoming clearer about how, the models partially overlap. The goal for moving beyond having a connected set of models and striving towards a coordinated sets of models is to facilitate the latter set of coordinated models as a whole to function in a more intertwined and concerted way when put into action and applied to the problem solving situation at hand. This is achieved by creating one or multiple bonds between the models, and successively making these clearer and stronger (see Figure 3). The coordinating of models is a multi-dialectic process where the specifics of all of the models in the set are important. The process of coordination results in the set of models becoming more aligned or (perhaps) more alienated. We note that the process of iterative refinement in a models and modelling perspective includes the sorting out and filtering of ideas, where some constructs (or models) are rejected as others become more aligned and subsequently integrated. A consequence of an unsuccessful coordination can give rise to the possible exclusion of the alienated model(s) or the rejection of the whole set altogether. An example of an activity facilitating students to coordinate models is the first part of the Light lab MAA* (Lab, part 1). Here the students are urged to keep thinking about their initial ideas and models elicited in the first pre-lab alongside representations (scatter plots, descriptions, rate graphs) of their collected real data and calculated values of the average rate of change over one centimeter intervals.

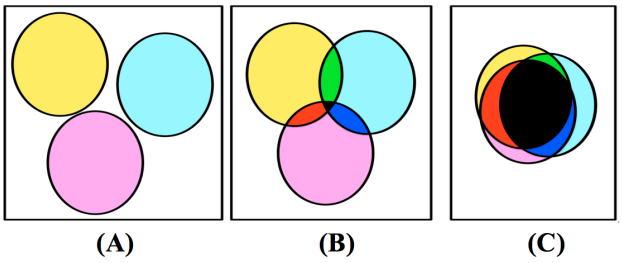

Figure 3: A conceptualization of connecting (A), coordinating (B), and integrating (C) models

Integrating models means merging the models so that the set now is conceived as one model in its own right (see Figure 3). Coordinated models are said to be *integrated* when the level of coordination is so high that further coordinating of the models in principle does not change the understanding and function of the newly merged models. A central feature of an integrated model is that it is self-contained. In the Light Lab MAA*, the students write a final lab report. The goal for the student is to present an integrated model of how intensity varies depending on the distance from a point light source. The purpose of the report is to reflect the students' integrated understanding of light dispersion on a qualitative and quantitative level using their collected data, average rate of change,

rate of change graphs, the difference quotient, and a spherical light emission model.

Having introduced and exemplified the notions of connecting, coordinating and integrating models, we now turn to discuss how these constructs might be used and applied to current discussions about the nature of mathematical modelling, as well as the teaching of, and through, modelling.

MEAs, MXAs, and MAAs

Generally within a model development sequence, an MEA elicits the students' initial thinking about a problem situation, as that thinking is externally represented. This means that the students start to structure the problem solving situation by trying to identify and connect models that can work productively in the specific situation. An example of an MEA functioning to connect models is the MEA* where the students are working with motion detectors to make sense of bodily motion along a straight path. In this activity, the students are placed in a situation that exposes them to multiple representations through which to make sense of the situation. Centred on their own bodily motion, students are offered an opportunity to form and connect an initial set of models, consisting of their previous ideas and models together with the representations introduced in the MEA*, which then can be furthered explored and applied.

Generally within a model development sequence, the MXAs focus on supporting the students in developing the models elicited in the MEA by examining different representations: graphs, symbols and algebraic representations, tables, everyday language, manipulatives, embodied and animated motions as well as students' self-constructed representations. Regarding representations as models in their own right (Lesh, Post, & Behr, 1987), MXAs can then primarily be seen to be about the coordination of models with occasional elements of connecting models if new representations or models are introduced during the activity. Typically, this is done by either exploring communalities of the models connected in the MEA, but the emphasis of a MXA is really on connecting, coordinating and using representations. The two MXA*s in the model development sequence on average rate of change are good illustrations of this. In these two activities students use computer programs that provide access to digital environments and work on tasks that explore different representations of key ideas in their own right as well as the relationships among these.

The main purpose of MAAs is to provide students with new contexts and situations where they can apply their developing or previously developed models. However, to do this some sort of connection has to be established mapping the models of the students to the context of the problem situation at hand. In other words, the context of the problem situation and the models of the students first have to be connected. Then, in order to ensure the adequacy, legitimacy and the proficiency of this connection, it has to be coordinated, integrating the specifics of the new situation and context with the model being applied.

Applications, modelling, and the modelling cycle

Returning to Stillman's (2012) distinction between applications and modelling, it is now possible to argue analogously that connecting, coordinating and integrating models are necessary and fundamental processes both for applying models "already built and learned" (p. 903) and when "model has to be built through idealising, specifying and mathematising the real world situation (p. 903). The important difference between applications and modelling is not the point of departure (the mathematics vs. the real world) per se; rather, the difference manifests itself in the amount of effort and time spent on coordinating the context of problem situation and the models of the modeller before one ends up with an integrated model for adequately make sense of, and use for, the situation.

Regarding modelling ideally conceived as a cyclic process (e.g., Blum & Leiβ, 2007, and others), connecting, coordinating and integrating models provides a more dialectic and dynamic conceptualisation of the processes involved. For example, each of the transitions in cyclic perspectives on modelling, such as the modelling cycle used by Blum and Leiβ (2007) and Borromeo Ferri (2006), e.g., transitions between *real situation – mental representations of the situation – real model – mathematical model*, have to be subjected to sequential connecting, coordinating and finally developed into an integrated model to tackle the prevailing situation. In addition, the different transitions in terms of connecting, coordinating and integration should not be thought of as carried out sequentially, but rather as processes fundamentally evolving simultaneously, nested and organically. From this

perspective, the typically one-way pointing arrows indicating the transitions in cyclic conceptualizations of modelling (e.g., Blum & Leiβ, 2007, and others) are misleading when trying to understand the complexity of, first, the modelling process in its own right, and second, and more importantly, the teaching and learning of modelling as well as teaching and learning mathematics through modelling.

THE SELF-SIMILAR NATURE OF TEACHING MATHEMATICS THROUGH MODELLING

Lesh and Doerr (2003b) discuss the connection between MMP and complexity theoretical ideas. One foundational idea of complexity theory is the principle of self-similarity, that structure and pattern repeat itself at different scales (Davis & Simmt, 2006). We wish to argue that this is also the case for the structure of model development sequences and the activities therein (at least for the MXAs and MAAs). The structure of the model development sequence at large and the activities that constitutes the sequence naturally varies from design to design, but using the notions of connecting, coordinating and integrating, the self-similarity of the two levels starts to stand out. In this paper we have tried to illustrate this point by providing examples from the model development sequence on average rate of change and the tasks in the Light Lab MAA*. We have argued that the two pre-labs in the Light Lab MAA* are activities connecting models, a characteristic of an MEA. In other words, Pre-lab 1, focusing on the students' initial ideas in the Light Lab MAA* (see Figure 3), has the function of an MEA. The elicited models and ideas are then subsequently coordinated in part one and part two of the lab under the support of an activity of data collection and a connecting activity (Pre-lab 2). However, the two parts of the lab (Lab, part 1&2) both function as MXAs with the primary goal of supporting the students in developing their model based on work on representations. The last task of the Lab, when students write a report, really aims at pulling the whole Light Lab MAA* together in an integrating sense. Looking at any MXA in a model development sequence, a similar argument applies: models need to be connected to set up the activity before the coordinating of the models can start. This self-similar aspect of model development sequences and the activities within occurs because when teaching mathematics through modelling, it is a necessity (by definition!) to bring in extra-mathematical contexts and situations.

DISCUSSION

One of the main points we have tried to emphasis in this paper is that regardless what theoretical stance on modelling is adapted, the three processes of connecting models, coordinating models, and integrating models are fundamental in all types of modelling situations. In our opinion, these three constructs capture and acknowledge the dialectic and complex nature of creating and developing models. Constructs like this might be the first steps towards a common conceptualization of modelling that bridges the research field so that we might better understand, coordinate and summarize research findings from different research traditions based on different perspectives.

By applying the three notions of connecting, coordinating and integrating models to the model development sequences, MXAs and MAA respectively, the self-similarity between the sequence and the MXAs and MAAs within the sequence came to the fore. The implication of this structural finding for teaching and the design of tasks might be profitably considered in future research.

While thinking about how models really develop and are formed in terms of connecting, coordinating and integrating models, we also found strong similarities to the principles of variation theory. The four types of variation within variation theory discussed by Marton, Runesson and Tsui (2004), *contrast*, *generalization*, *separation*, and *fusion* especially seems to resonate with the notions put forward in this paper: connecting models – contrast variation; coordinating models – generalization and variation of separation; integrating – fusion. To further investigate this, and what more variation theory has to offer, seems a promising way to continue the research initiated in this paper.

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