# Influence of Unknown Exterior Samples on Interpolated Values for Band-limited Images Additional material 

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This document is a supplementary file where we present some technical proofs and an extended bibliographic study.

## 1 Additional proofs

In this section, we provide the proofs of some of the results presented in the main article. To facilitate their identification, we recap the statement of all the original results and use the same numbers as in the main article.

### 1.1 Proofs from section 2

Proposition 2.3. Let $X_{t}$ be a weakly stationary $R P$. Then the following assertions are equivalent:

1. $X_{t}$ is continuous in quadratic mean
2. $R_{X}$ is continuous at $t=0$
3. $R_{X}$ is uniformly continuous on $\mathbb{R}$

Proof. Stationarity implies the following identity

$$
\mathbb{E}\left[\left|X_{t}-X_{s}\right|^{2}\right]=2\left(R_{X}(0)-R_{X}(t-s)\right)
$$

which easily proves that the first two assertions are indeed equivalent. To complete the proof, we need to show that the last two assertions are equivalent. Without loss of generality, one may assume that $R_{X}(0)=1$. We focus on the difficult part of the equivalence and show that the second assertion implies the third one. Let $t, s \in \mathbb{R}^{d}$. Given that $R_{X}$ is positive semi-definite, we can consider the following positive semi-definite matrix:

$$
R:=\left[\begin{array}{ccc}
R_{X}(t-t) & R_{X}(t-s) & R_{X}(t-0) \\
R_{X}(s-t) & R_{X}(s-s) & R_{X}(s-0) \\
R_{X}(0-t) & R_{X}(0-s) & R_{X}(0-0)
\end{array}\right]=\left[\begin{array}{ccc}
1 & R_{X}(t-s) & R_{X}(t) \\
R_{X}(t-s) & 1 & R_{X}(s) \\
R_{X}(t) & R_{X}(s) & 1
\end{array}\right]
$$

Therefore

$$
\begin{aligned}
0 \leq \operatorname{det}(R)= & 1+R_{X}(t-s) R_{X}(t) R_{X}(s)+R_{X}(t) R_{X}(t-s) R_{X}(s) \\
& -R_{X}(s)^{2}-R_{X}(t)^{2}-R_{X}(t-s)^{2} \\
= & 1+2 R_{X}(t-s) R_{X}(t) R_{X}(s)-R_{X}(s)^{2}-R_{X}(t)^{2}-R_{X}(t-s)^{2}
\end{aligned}
$$

Now using that $\left|R_{X}(t)-R_{X}(s)\right|^{2}=\left|R_{X}(t)\right|^{2}+\left|R_{X}(s)\right|^{2}-2 R_{X}(t) R_{X}(s)$, we get

$$
0 \leq 1-R_{X}(t-s)^{2}+2 R_{X}(t-s) R_{X}(t) R_{X}(s)-\left|R_{X}(t)-R_{X}(s)\right|^{2}-2 R_{X}(t) R_{X}(s)
$$

By using $\left|R_{X}(t)\right| \leq 1$, this implies in turn that

$$
\begin{aligned}
\left|R_{X}(t)-R_{X}(s)\right|^{2} & \leq 1-R_{X}(t-s)^{2}+2 R_{X}(t) R_{X}(s)\left(R_{X}(t-s)-1\right) \\
& \leq 1-R_{X}(t-s)^{2}+2\left|R_{X}(t) R_{X}(s)\right|\left|R_{X}(t-s)-1\right| \\
& \leq 1+2\left|R_{X}(t-s)-1\right|-R_{X}(t-s)^{2}
\end{aligned}
$$

The fact that the right-hand-side converges uniformly to 0 completes the proof.
Definition 2.7 (Linear shift-invariant reconstruction). Let $X_{t}$ be a random process, ( $h_{K}$ ) a reconstruction system and $0<K<\infty$. We define the reconstruction of $X_{t}$ from its samples $X_{k}, k \in\{-K, \ldots, K\}$ by

$$
\tilde{X}_{t}^{\left[h_{K}, K\right]}:=\left(X . \Delta_{K}\right) * h_{K}(t)=\sum_{|k| \leq K} X_{k} h_{K}(t-k)
$$

Then, if the system verifies the reconstruction systems dominated condition, we can extend the previous definition for $K=\infty$

$$
\tilde{X}_{t}^{\left[h_{\infty}, \infty\right]}:=\left(X . \Delta_{\infty}\right) * h_{\infty}(t)=\lim _{K \rightarrow \infty}\left(X . \Delta_{K}\right) * h_{K}(t)=\lim _{K \rightarrow \infty} \sum_{|k| \leq K} X_{k} h_{K}(t-k)
$$

where convergence holds in quadratic mean (q.m.), that is to say

$$
\lim _{K \rightarrow \infty} \mathbb{E}\left[\left(\tilde{X}_{t}^{\left[h_{\infty}, \infty\right]}-\sum_{|k| \leq K} X_{k} h_{K}(t-k)\right)^{2}\right]=0
$$

Proof. In the limit case $(K=\infty)$, we should show that under the reconstruction systems dominated convergence, then the previous limit exists in quadratic mean. In other words, we want to show that

$$
\lim _{K \rightarrow \infty} \mathbb{E}\left[\left(\tilde{X}_{t}^{\left[h_{\infty}, \infty\right]}-\tilde{X}_{t}^{\left[h_{K}, K\right]}\right)^{2}\right]=0
$$

Indeed,

$$
\begin{aligned}
\mathbb{E}\left[\left(\tilde{X}_{t}^{\left[h_{\infty}, \infty\right]}-\tilde{X}_{t}^{\left[h_{K}, K\right]}\right)^{2}\right]= & \mathbb{E}\left[\left(\sum_{k \in \mathbb{Z}^{d}} X_{k} h_{\infty}(t-k)-\sum_{|k| \leq K} X_{k} h_{K}(t-k)\right)^{2}\right] \\
= & \sum_{k \in \mathbb{Z}^{d}, m \in \mathbb{Z}^{d}} \mathbb{E}\left[X_{k} X_{m}\right] h_{\infty}(t-k) h_{\infty}(t-m) \\
& -2 \sum_{k \in \mathbb{Z}^{d},|m| \leq K} \mathbb{E}\left[X_{k} X_{m}\right] h_{\infty}(t-k) h_{K}(t-m) \\
& +\sum_{|k| \leq K,|m| \leq K} \mathbb{E}\left[X_{k} X_{m}\right] h_{K}(t-k) h_{K}(t-m) .
\end{aligned}
$$

Letting $Y_{t}=X_{t}-\mu$, we have

$$
\begin{aligned}
\mathbb{E}\left[X_{k} X_{m}\right] & \left.=\mu^{2}+\mathbb{E}\left[Y_{k} Y_{m}\right]=\mu^{2}+R_{X}(k-m)\right) \\
& =\mu^{2}+\frac{1}{(2 \pi)^{d}} \int e^{i \omega(k-m)} d \Psi_{X}(\omega)
\end{aligned}
$$

The previous expression becomes

$$
\begin{aligned}
\mathbb{E}\left[\left(\tilde{X}_{t}^{\left[h_{\infty}, \infty\right]}-\tilde{X}_{t}^{\left[h_{K}, K\right]}\right)^{2}\right]= & \mu^{2} \sum_{k \in \mathbb{Z}^{d}, m \in \mathbb{Z}^{d}} h_{\infty}(t-k) h_{\infty}(t-m) \\
& -2 \mu^{2} \sum_{k \in \mathbb{Z}^{d},|m| \leq K} h_{\infty}(t-k) h_{K}(t-m) \\
& +\mu^{2} \sum_{|k| \leq K,|m| \leq K} h_{K}(t-k) h_{K}(t-m) \\
& +\frac{1}{(2 \pi)^{d}} \int \sum_{k \in \mathbb{Z}^{d}, m \in \mathbb{Z}^{d}} h_{\infty}(t-k) h_{\infty}(t-m) e^{i \omega(k-m)} d \Psi_{X}(\omega) \\
& -2 \frac{1}{(2 \pi)^{d}} \int \sum_{k \in \mathbb{Z}^{d},|m| \leq K} h_{\infty}(t-k) h_{K}(t-m) e^{i \omega(k-m)} d \Psi_{X}(\omega) \\
& +\frac{1}{(2 \pi)^{d}} \int \sum_{|k| \leq K,|m| \leq K} h_{K}(t-k) h_{K}(t-m) e^{i \omega(k-m)} d \Psi_{X}(\omega) \\
= & \mu^{2}\left(\sum_{k \in \mathbb{Z}^{d}} h_{\infty}(t-k)-\sum_{|k| \leq K} h_{K}(t-k)\right)^{2}+ \\
& \frac{1}{(2 \pi)^{d}} \int\left|\sum_{k \in \mathbb{Z}^{d}} e^{i \omega k} h_{\infty}(t-k)-\sum_{|k| \leq K} e^{i \omega k} h_{K}(t-k)\right|^{2} d \Psi_{X}(\omega)
\end{aligned}
$$

Considering $K \rightarrow \infty$ and given the convergence part of the dominated condition of Definition 2.6, we note that $\forall t, \forall|\omega| \leq \pi$

$$
\lim _{K \rightarrow \infty} \sum_{k \in \mathbb{Z}^{d}} e^{i \omega k} h_{\infty}(t-k)-\sum_{|k| \leq K} e^{i \omega k} h_{K}(t-k)=0
$$

The previous identity and the domination part of the condition are trivially extended to any $\omega$ by periodicity. Together, they ensure that

$$
\lim _{K \rightarrow \infty} \int\left|\sum_{k \in \mathbb{Z}^{d}} e^{i \omega k} h_{\infty}(t-k)-\sum_{|k| \leq K} e^{i \omega k} h_{K}(t-k)\right|^{2} d \Psi_{X}(\omega)=0
$$

Note that the dominated convergence was used also to interchange infinite sums and integrals with respect to $d \Psi_{X}$. For instance, one can claim that

$$
\begin{aligned}
& \int \sum_{k \in \mathbb{Z}^{d}} \sum_{m \in \mathbb{Z}^{d}} h_{\infty}(t-k) h_{\infty}(t-m) e^{i \omega(k-m)} d \Psi_{X}(\omega)= \\
& \sum_{k \in \mathbb{Z}^{d}} \sum_{m \in \mathbb{Z}^{d}} \int h_{\infty}(t-k) h_{\infty}(t-m) e^{i \omega(k-m)} d \Psi_{X}(\omega)
\end{aligned}
$$

because we have the following domination $\forall K>0, \forall M>0$ :

$$
\begin{aligned}
\left|\sum_{|k| \leq K} \sum_{|m| \leq M} h_{\infty}(t-k) h_{\infty}(t-m) e^{i \omega(k-m)}\right| & =\left|\sum_{|k| \leq K} h_{\infty}(t-k) e^{i \omega k}\right| \times\left|\sum_{|m| \leq M} h_{\infty}(t-m) e^{-i \omega m}\right| \\
& \leq M(t)^{2}
\end{aligned}
$$

Proposition 2.9. The Shannon-Whittaker reconstruction system $\left(h_{K}(t)=\operatorname{sinc}(t):=\frac{\sin (\pi t)}{\pi t}\right)$ is interpolating and verifies the reconstruction systems dominated condition. Note that it reproduces constants only asymptotically as $K \rightarrow \infty$.

Proof.

- Convergence: Given that the reconstruction kernel is the same for every $K$ ( $h_{K}=$ sinc $)$, the partial sum in the definition 2.6

$$
s_{K}(\omega):=\sum_{|k| \leq K} e^{i \omega k} \operatorname{sinc}(t-k)
$$

is the partial symmetric Fourier series of the $2 \pi$-periodic function defined on $]-\pi, \pi[$ by

$$
e_{t}(\omega):=e^{i \omega t}
$$

This function is piecewise $\mathcal{C}^{1}$ and as such its Fourier series converges to its regularized version, that is to $e_{t}(\omega), \forall|\omega|<\pi$ and to $\frac{e_{t}(\pi)+e_{t}(-\pi)}{2}=\cos (\pi t)$ for $|\omega|=\pi$.

- Dominated condition: One can define the Cesaro sum of this series as

$$
\begin{aligned}
\sigma_{K}(\omega) & :=\frac{1}{K+1} \sum_{0 \leq l \leq K} s_{l}(\omega) \\
& =\int_{|\nu| \leq \pi} e_{t}(\omega+\nu) \mathcal{K}_{K}(\nu) d \nu
\end{aligned}
$$

where $\mathcal{K}_{K}:=\frac{1}{2 \pi(K+1)} \frac{\sin ^{2}\left((K+1) \frac{\nu}{2}\right)}{\sin ^{2}\left(\frac{\nu}{2}\right)}$ is the Fejér Kernel. This kernel having a unit $L^{1}$-norm, we obtain that $\left|\sigma_{K}(\omega)\right| \leq 1$.
It is a general (and easily verifiable) fact that if the Fourier coefficients are $\mathcal{O}\left(\frac{1}{k}\right)$ (the case here for $e_{t}$, which coefficients are $c_{k}(t):=\operatorname{sinc}(t-k)$ ), then $\left|s_{K}(\omega)-\sigma_{K}(\omega)\right|$ is uniformly bounded. Indeed,

$$
\begin{aligned}
\left|s_{K}(\omega)-\sigma_{K}(\omega)\right| & =\left|s_{K}(\omega)-\frac{1}{K+1} \sum_{0 \leq l \leq K} s_{l}(\omega)\right| \\
& =\left|\frac{1}{K+1} \sum_{0 \leq l \leq K} s_{K}(\omega)-\frac{1}{K+1} \sum_{l \leq K} s_{l}(\omega)\right| \\
& =\frac{1}{K+1}\left|\sum_{0 \leq l \leq K}\left(s_{K}(\omega)-s_{l}(\omega)\right)\right| \\
& =\frac{1}{K+1}\left|\sum_{0 \leq l \leq K} \sum_{l<|k| \leq K} c_{k} e^{i k \omega}\right|=\frac{1}{K+1}\left|\sum_{|k| \leq K} \sum_{0 \leq l<|k|} c_{k} e^{i k \omega}\right| \\
& =\frac{1}{K+1}\left|\sum_{|k| \leq K}\right| k\left|c_{k} e^{i k \omega}\right|=\frac{1}{K+1}\left|\sum_{k \leq K}\right| k\left|\mathcal{O}\left(\frac{1}{k}\right)\right|=\mathcal{O}(1)
\end{aligned}
$$

Finally, it results that $s_{K}(\omega)$ is uniformly bounded. This completes the proof concerning the reconstruction systems dominated condition. The fact that the sinc kernel reproduces constants asymptotically derives from the Shannon-Whittaker theorem in the context of deterministic tempered distributions (see [19] §IV-G).

Theorem 2.12 (Spectral representation of the generalized aliasing RMSE). Let $X_{t}$ be a random process of average $\mu$ and power spectrum $d \Psi_{X}, K<\infty$ and $\left(h_{K}\right)$ a reconstruction system ( $h_{K}$ is not required to be interpolating nor to preserve constants). Then the mean squared error of the reconstruction $\tilde{X}_{t}^{\left[h_{K}, K\right]}=$ $\left(X . \Delta_{K}\right) * h_{K}(t)$ is

$$
\begin{equation*}
R M S E_{\left[X . \Delta_{K} * h_{K}\right]}(t)^{2}=\left|\mu-\mu \Delta_{K} * h_{K}(t)\right|^{2}+\frac{1}{(2 \pi)^{d}} \int\left|e^{i \omega t}-\left[\left(e^{i \omega \cdot} \Delta_{K}\right) * h_{K}\right](t)\right|^{2} d \Psi_{X}(\omega) \tag{1}
\end{equation*}
$$

The formula remains true if $K=\infty$ under the reconstruction systems dominated condition (given in Definition 2.6.).
Proof. We start with $K<\infty$. Let $Y_{t}:=X_{t}-\mu$.

$$
\begin{aligned}
R M S E_{\left[X . \Delta_{K} * h_{K}\right]}(t)^{2} & =\mathbb{E}\left[\left(X_{t}-\sum_{|k| \leq K} X_{k} h_{K}(t-k)\right)^{2}\right] \\
& =\mathbb{E}\left[\left(Y_{t}+\mu-\sum_{|k| \leq K}\left(Y_{k}+\mu\right) h_{K}(t-k)\right)^{2}\right] \\
& =\left|\mu-\mu \Delta_{K} * h_{K}(t)\right|^{2}+\mathbb{E}\left[\left(Y_{t}-\sum_{|k| \leq K} Y_{k} h_{K}(t-k)\right)^{2}\right] \\
& =\left|\mu-\mu \Delta_{K} * h_{K}(t)\right|^{2}+R M S E_{\left[Y . \Delta_{K} * h_{K}\right]}(t)^{2}
\end{aligned}
$$

At this point, the first error contribution in Equation (1) is clearly established. Let us deal with the second one. Using that $\mathbb{E}\left(Y_{t} Y_{s}\right)=\frac{1}{(2 \pi)^{d}} \int e^{i \omega(t-s)} d \Psi_{X}(\omega)$ we have

$$
\begin{aligned}
R M S E_{\left[Y . \Delta_{K} * h_{K}\right]}(t)^{2}= & \mathbb{E}\left[Y_{t}^{2}\right]-\sum_{|k| \leq K}\left(\mathbb{E}\left[Y_{t} Y_{k}\right]+\mathbb{E}\left[Y_{k} Y_{t}\right]\right) h_{K}(t-k) \\
& +\sum_{|k|,|m| \leq K} \mathbb{E}\left[Y_{k} Y_{m}\right] h_{K}(t-k) h_{K}(t-m) \\
= & \frac{1}{(2 \pi)^{d}} \int 1-\sum_{|k| \leq K}\left(e^{i \omega(t-k)}+e^{-i \omega(t-k)}\right) h_{K}(t-k) d \Psi_{X}(\omega) \\
& +\frac{1}{(2 \pi)^{d}} \int \sum_{|k|,|m| \leq K} e^{i \omega(m-k)} h_{K}(t-k) h_{K}(t-m) d \Psi_{X}(\omega) \\
= & \frac{1}{(2 \pi)^{d}} \int\left|e^{i \omega t}-\sum_{|k| \leq K} e^{i \omega k} h_{K}(t-k)\right|^{2} d \Psi_{X}(\omega),
\end{aligned}
$$

which proves Equation (1).
Considering $K \rightarrow \infty$, we note that by the convergence part of the dominated condition of Definition 2.6 by taking $\omega=0$,

$$
\mu \Delta_{K} * h_{K}(t) \rightarrow \mu \Delta_{\infty} * h_{\infty}(t)
$$

Recall that the dominated condition of Definition 2.6 is formulated for all $|\omega| \leq \pi$ but is actually valid for all $\omega$ by periodicity. Therefore, thanks to the dominated convergence theorem

$$
\int\left|e^{i \omega t}-\sum_{|k| \leq K} e^{i \omega k} h_{K}(t-k)\right|^{2} d \Psi_{X}(\omega) \rightarrow \int\left|e^{i \omega t}-\sum_{k} e^{i \omega k} h_{\infty}(t-k)\right|^{2} d \Psi_{X}(\omega)
$$

### 1.2 Proofs from section 5

Lemma 5.8. Let $W_{t}$ be a Nyquist band-limited white-noise of unit variance. Then, following the notations as in Definitions 4.2 and 5.7

$$
\begin{aligned}
& A_{d \Psi_{W}}(t)=\sum_{|k| \leq K} \frac{\pi^{2}}{N^{2} \sin ^{2}\left(\frac{\pi}{N}(t-k)\right)}\left(1-\operatorname{sinc}^{2}\left(\frac{1}{N}(t-k)\right)\right) \\
& B_{d \Psi_{W}}(t)=\sum_{|k| \leq K} \frac{2 \pi^{2}}{N^{2} \sin ^{2}\left(\frac{\pi}{N}(t-k)\right)}\left(1-\operatorname{sinc}\left(\frac{1}{N}(t-k)\right)\right)
\end{aligned}
$$

Proof. Without loss of generality, we can assume that $\mu=\mathbb{E}\left[W_{t}\right]=0$. It is useful to notice that, under this assumption,

$$
\begin{aligned}
& A_{d \Psi_{W}}(t) \frac{\sin ^{2}(\pi t)}{\pi^{2}}=\mathbb{E}\left[\left(W_{t}-W \cdot \Delta_{K} * \operatorname{sinc}(t)\right)^{2}\right] \\
& B_{d \Psi_{W}}(t) \frac{\sin ^{2}(\pi t)}{\pi^{2}}=\mathbb{E}\left[\left(W_{t}-W \cdot \Delta_{K} * \operatorname{sincd}{ }_{K}(t)\right)^{2}\right]
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A_{d \Psi_{W}}(t) \frac{\sin ^{2}(\pi t)}{\pi^{2}}= & \mathbb{E}\left[\left(W_{t}-\sum_{|k| \leq K} W_{k} \operatorname{sinc}(t-k)\right)\left(W_{t}-\sum_{|m| \leq K} W_{m} \operatorname{sinc}(t-m)\right)\right] \\
= & R_{W}(0)-2 \sum_{|k| \leq K} R_{W}(t-k) \operatorname{sinc}(t-k)+ \\
& \sum_{|k|,|m| \leq K} R_{W}(m-k) \operatorname{sinc}(t-k) \operatorname{sinc}(t-m)
\end{aligned}
$$

Since $R_{W}(t)=\operatorname{sinc}(t)$ and in particular since $R_{W}(k-m)=\mathbb{1}_{k=m}$, we get

$$
\begin{aligned}
A_{d \Psi_{W}}(t) \frac{\sin ^{2}(\pi t)}{\pi^{2}} & =1-2 \sum_{|k| \leq K} \operatorname{sinc}^{2}(t-k)+\sum_{|k| \leq K} \operatorname{sinc}^{2}(t-k) \\
& =1-\sum_{|k| \leq K} \operatorname{sinc}^{2}(t-k)
\end{aligned}
$$

Similarly we have,

$$
\begin{aligned}
B_{d \Psi_{W}}(t) \frac{\sin ^{2}(\pi t)}{\pi^{2}}= & \mathbb{E}\left[\left(W_{t}-\sum_{|k| \leq K} W_{k} \operatorname{sincd}_{K}(t-k)\right) *\left(W_{t}-\sum_{|m| \leq K} W_{m} \operatorname{sincd}{ }_{K}(t-m)\right)\right] \\
= & R_{W}(0)-2 \sum_{|k| \leq K} R_{W}(t-k) \operatorname{sincd}_{K}(t-k)+ \\
& \sum_{|k|,|m| \leq K} R_{W}(m-k) \operatorname{sincd}_{K}(t-k) \operatorname{sincd}_{K}(t-m) \\
= & 1-2 \sum_{|k| \leq K} \operatorname{sinc}(t-k) \operatorname{sincd}_{K}(t-k)+\sum_{|k| \leq K} \operatorname{sincd}_{K}^{2}(t-k)
\end{aligned}
$$

Besides, using Parseval's identity for $\operatorname{sincd}_{K}(t-k)=\mathcal{D} \mathcal{F} \mathcal{T}^{-1}\left(e^{2 i \pi t \frac{l}{N}}\right)$ we obtain

$$
\begin{equation*}
\sum_{|k| \leq K} \operatorname{sincd}_{K}^{2}(t-k)=\frac{1}{N} \sum_{|l| \leq K}\left|e^{2 i \pi t \frac{l}{N}}\right|^{2}=1 \tag{2}
\end{equation*}
$$

So that, at this point we have shown

$$
\begin{aligned}
& A_{d \Psi_{W}}(t) \frac{\sin ^{2}(\pi t)}{\pi^{2}}=1-\sum_{|k| \leq K} \operatorname{sinc}^{2}(t-k) \\
& B_{d \Psi_{W}}(t) \frac{\sin ^{2}(\pi t)}{\pi^{2}}=2\left(1-\sum_{|k| \leq K} \operatorname{sinc}(t-k) \operatorname{sincd}_{K}(t-k)\right)
\end{aligned}
$$

And finally, thanks to Equation (2),

$$
\begin{aligned}
& A_{d \Psi_{W}}(t) \frac{\sin ^{2}(\pi t)}{\pi^{2}}=\sum_{|k| \leq K}\left(\operatorname{sincd}_{K}^{2}(t-k)-\operatorname{sinc}^{2}(t-k)\right) \\
& B_{d \Psi_{W}}(t) \frac{\sin ^{2}(\pi t)}{\pi^{2}}=2 \sum_{|k| \leq K}\left(\operatorname{sincd}_{K}^{2}(t-k)-\operatorname{sinc}(t-k) \operatorname{sincd}{ }_{K}(t-k)\right)
\end{aligned}
$$

which leads to

$$
\begin{aligned}
A_{d \Psi_{W}}(t) & =\pi^{2} \sum_{|k| \leq K}\left(\frac{1}{N^{2} \sin ^{2}\left(\frac{\pi}{N}(t-k)\right)}-\frac{1}{\pi^{2}(t-k)^{2}}\right) \\
& =\sum_{|k| \leq K} \frac{\pi^{2}}{N^{2} \sin ^{2}\left(\frac{\pi}{N}(t-k)\right)}\left(1-\operatorname{sinc}^{2}\left(\frac{1}{N}(t-k)\right)\right)
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
B_{d \Psi_{W}}(t) & =2 \pi^{2} \sum_{|k| \leq K}\left(\frac{1}{N^{2} \sin ^{2}\left(\frac{\pi}{N}(t-k)\right)}-\frac{1}{\pi N(t-k) \sin \left(\frac{\pi}{N}(t-k)\right)}\right) \\
& =\sum_{|k| \leq K} \frac{2 \pi^{2}}{N^{2} \sin ^{2}\left(\frac{\pi}{N}(t-k)\right)}\left(1-\operatorname{sinc}\left(\frac{1}{N}(t-k)\right)\right)
\end{aligned}
$$

Proposition 5.11. Let $W_{t}$ be a Nyquist band-limited white-noise of unit variance. Then, $\forall\left|t_{K}\right| \leq \beta K$ with $\beta<1$,

$$
B_{d \Psi_{W}}\left(t_{K}\right)_{N} \stackrel{\sim}{\rightarrow} \frac{2 \pi^{2}}{N} \int_{-\frac{1}{2}-\frac{t_{K}}{N}}^{\frac{1}{2}-\frac{t_{K}}{N}} \frac{1}{\sin ^{2} \pi(u)}(1-\operatorname{sinc}(u)) d u
$$

Proof. We start from the expression in Lemma 5.8

$$
B_{d \Psi_{W}}\left(t_{K}\right)=\sum_{|k| \leq K} \frac{2 \pi^{2}}{N^{2} \sin ^{2}\left(\frac{\pi}{N}\left(t_{K}-k\right)\right)}\left(1-\operatorname{sinc}\left(\frac{1}{N}\left(t_{K}-k\right)\right)\right)
$$

We are going to develop some kind of Euler-Maclaurin formula (of order 1) to express the previous sum as an integral. Let $f_{N}$ a continuously differentiable function on $]-\left(K+\frac{1}{2}\right), K+\frac{1}{2}\left[\right.$. And let $g(y)=y-\lfloor y\rfloor-\frac{1}{2}$,
which is differentiable on $\left[k, k+\frac{1}{2}[\right.$ and on $] k-\frac{1}{2}, k\left[\right.$ with derivative is $g^{\prime}(y)=1$. Then,

$$
\begin{aligned}
\int_{k}^{k+\frac{1}{2}} f_{N}(y)+g(y) f_{N}^{\prime}(y) d y & =\int_{k}^{k+\frac{1}{2}}\left(g f_{N}\right)^{\prime}(y) d y \\
& =g\left(k+\frac{1}{2}\right) f_{N}\left(k+\frac{1}{2}\right)-g(k) f_{N}(k)
\end{aligned}
$$

Then, since $g\left(k+\frac{1}{2}\right)=0$ and $g(k)=-\frac{1}{2}$, one obtains

$$
\int_{k}^{k+\frac{1}{2}} f_{N}(y)+g(y) f_{N}^{\prime}(y) d y=\frac{1}{2} f_{N}(k)
$$

Similarly, one can obtain a similar formula on $\left[k-\frac{1}{2}, k\right]$ and combine it with the previous one. This yields

$$
\begin{aligned}
f_{N}(k) & =\left(\int_{k-\frac{1}{2}}^{k} f_{N}(y)+g(y) f_{N}^{\prime}(y) d y+\int_{k}^{k+\frac{1}{2}} f_{N}(y)+g(y) f_{N}^{\prime}(y) d y\right) \\
& =\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} f_{N}(y)+g(y) f_{N}^{\prime}(y) d y
\end{aligned}
$$

Using this equality we obtain,

$$
\begin{aligned}
\sum_{|k| \leq K} f_{N}(k) & =\int_{-\left(K+\frac{1}{2}\right)}^{K+\frac{1}{2}} f_{N}(y)+g(y) f_{N}^{\prime}(y) d y \\
& =\int_{-\left(K+\frac{1}{2}\right)}^{K+\frac{1}{2}} f_{N}(y)+\left(y-\lfloor y\rfloor-\frac{1}{2}\right) f_{N}^{\prime}(y) d y
\end{aligned}
$$

Letting $f_{N}(y)=\frac{1}{N} f\left(\frac{y-t_{K}}{N}\right)\left(\right.$ with $\left.f(u)=\frac{1}{\sin ^{2}(\pi u)}(1-\operatorname{sinc}(u))\right)$ yields

$$
\begin{aligned}
\frac{N}{2 \pi^{2}} B_{d \Psi_{W}}\left(t_{K}\right) & =\sum_{|k| \leq K} f_{N}(k)=\int_{-\left(K+\frac{1}{2}\right)}^{K+\frac{1}{2}} f_{N}(y)+\left(y-\lfloor y\rfloor-\frac{1}{2}\right) f_{N}^{\prime}(y) d y \\
& =\int_{-\frac{1}{2}-\frac{t_{K}}{N}}^{\frac{1}{2}-\frac{t_{K}}{N}} f(u) d u+\frac{1}{N} \int_{-\frac{1}{2}-\frac{t_{K}}{N}}^{\frac{1}{2}-\frac{t_{K}}{N}}\left(N u+t_{K}-\left(\left\lfloor N u+t_{K}\right\rfloor+\frac{1}{2}\right)\right) f^{\prime}(u) d u
\end{aligned}
$$

The last equality relies on the substitution $u=\frac{y-t_{K}}{N}$ and the link between $f_{N}$ and $f$, in particular $f_{N}^{\prime}(y)=$ $\frac{1}{N^{2}} f^{\prime}(u)$. Since, $\left|t_{K}\right| \leq \beta K$, the bounds in the second integrals belong to the range $\left[-\frac{1}{2}(1+\beta), \frac{1}{2}(1+\beta)\right]$ where the integrand is bounded. Note that without this assumption (for instance if $t_{K}=K$ ), the second integral could explode.

Proposition 5.12, Let $W_{t}$ be a Nyquist band-limited white-noise of unit variance. Then, $\forall\left|t_{K}\right| \leq K$ with $\frac{\delta\left(t_{K}\right)}{K} \rightarrow q^{1}$.

$$
\frac{2}{\delta\left(t_{K}\right)+\frac{1}{2}}+\mathcal{O}\left(\frac{\log (N)}{N}\right) \leq B_{d \Psi_{W}}\left(t_{K}\right) \leq \frac{2}{\delta\left(t_{K}\right)}+\mathcal{O}\left(\frac{1}{N}\right)
$$

Proof. First, note that since $\Delta\left(t_{K}\right) \geq N / 2$ then $\frac{2}{\Delta\left(t_{K}\right)}=\mathcal{O}\left(\frac{1}{N}\right)$. It results that the global upper bound that we have derived in Corollary 5.10 implies the one of this theorem. Let us then focus only on the lower

[^0]bound. Without loss of generality, let us assume that $t_{K}>0$. First, we notice that
\[

$$
\begin{aligned}
\frac{N}{2 \pi^{2}} B_{d \Psi_{W}}\left(t_{K}\right) & =\sum_{k=-K}^{K} f_{N}(k) \\
& =\sum_{k=-K}^{-1} f_{N}(k)+\mathcal{O}(1) .
\end{aligned}
$$
\]

Therefore we need to evaluate bounds for $S_{K}:=\sum_{k=-K}^{-1} f_{N}(k)$. A monotonicity argument shows that

$$
\begin{aligned}
& S_{K} \geq \int_{-K}^{0} f_{N}(y) d y \\
& S_{K} \geq \int_{\frac{-K-t_{K}}{N}}^{-\frac{t}{N}} f(u) d u \\
& S_{K} \geq \int_{-1+\frac{\delta\left(t_{K}\right)+\frac{1}{2}}{N}}^{0} f(u) d u+\mathcal{O}(1)
\end{aligned}
$$

Using that $f(u)=\frac{1}{\pi^{2}(1+u)^{2}}-\frac{1}{\pi^{2}(1+u)}+O_{u \in]-1,-1 / 2]}(1)$ (where $\mathcal{O}_{u \in]-1,-1 / 2]}(1)$ means a bounded term on $u \in]-1,-1 / 2]$ ), we obtain that, when $x \rightarrow 0^{+}$

$$
\begin{aligned}
\int_{-1+x}^{0} f(u) d u & =\frac{1}{\pi^{2} x}+\frac{1}{\pi^{2}} \log (x)+\mathcal{O}(1) \\
& \geq \frac{1}{\pi^{2} x}+\mathcal{O}(\log (x))
\end{aligned}
$$

Therefore, we have

$$
\frac{N}{2 \pi^{2}} B_{d \Psi_{W}}\left(t_{K}\right) \geq \frac{N}{\pi^{2}\left(\delta\left(t_{K}\right)+\frac{1}{2}\right)}+\mathcal{O}\left(\log \left(\frac{\delta\left(t_{K}\right)+\frac{1}{2}}{N}\right)\right)
$$

And since $\forall\left|t_{K}\right| \leq K, \delta\left(t_{K}\right) \geq \frac{1}{2}$, then $\mathcal{O}\left(\log \left(\frac{\delta\left(t_{K}\right)+\frac{1}{2}}{N}\right)\right)=\mathcal{O}\left(\log \left(\frac{1}{N}\right)\right)$.
Lemma 5.13. $\forall|\omega|<\pi$ and $\forall|t|<K+1 / 2$,

$$
\left|\sum_{|k|>K}\left(e^{i \omega k}-e^{i \omega k \% N}\right) \frac{(-1)^{k}}{t-k}\right| \leq \frac{4}{\left|1+e^{i \omega}\right|}\left(\frac{1}{\delta(t)+\frac{1}{2}}+\frac{1}{\Delta(t)+\frac{1}{2}}\right) .
$$

Proof. We can use the exact same structure of proof as for the truncated Shannon-Whittaker case (Lemma 4.7). Only now, we need to find an upper bound for $\left|S(\tilde{b})_{K}^{k}\right|$ with $\tilde{b}_{k}=\left(-e^{i \omega}\right)^{k}-e^{i \omega k \% N}(-1)^{k}=b_{k}-c_{k}$ where $c_{k}=e^{i \omega k \% N}(-1)^{k}$. Therefore, it is sufficient to find a bound for $\left|S(c)_{K}^{k}\right|$ since

$$
S(\tilde{b})_{K}^{k}=S(b)_{K}^{k}-S(c)_{K}^{k}
$$

To do so, we shall note that since $c_{k+q N}=(-1)^{q} c_{k}$, we can cluster the terms of $S(c)_{K}^{k}=\sum_{K<k^{\prime} \leq k} c_{k^{\prime}}$ in groups of $N$-terms, and each group will cancel the previous one. At the end it will remain only fewer than $N$ terms. The clustering is based on the fact that, $K<k^{\prime} \leq k$ is equivalent to

$$
q_{N}\left(k^{\prime}\right) \geq 1 \text { and }\left\{\begin{array}{c}
q_{N}\left(k^{\prime}\right)<q_{N}(k) \\
\text { or } \\
\left(q_{N}\left(k^{\prime}\right)=q_{N}(k) \quad \text { and } \quad k^{\prime} \% N \leq k \% N\right)
\end{array}\right.
$$

This leads us to derive the following new expression for $S(c)_{K}^{k}$, where $q^{\prime}$ and $r^{\prime}$ stand respectively for $q_{N}\left(k^{\prime}\right)$ and $k^{\prime} \% N$

$$
\begin{aligned}
S(c)_{K}^{k} & =\sum_{1 \leq q^{\prime}<q_{N}(k)} \sum_{\left|r^{\prime}\right| \leq K} c_{q^{\prime} N+r^{\prime}}+\sum_{-K \leq r^{\prime} \leq k \% N} c_{q_{N}(k) N+r^{\prime}} \\
& =\left(\sum_{1 \leq q^{\prime}<q_{N}(k)}(-1)^{q^{\prime}}\right) \sum_{\left|r^{\prime}\right| \leq K} c_{r^{\prime}}+(-1)^{q_{N}(k)} \sum_{-K \leq r^{\prime} \leq k \% N} c_{r^{\prime}} .
\end{aligned}
$$

If $q_{N}(k)$ is odd, then the term on the left vanishes and

$$
\left|S(c)_{K}^{k}\right|=\left|\sum_{-K \leq r^{\prime} \leq k \% N} c_{r^{\prime}}\right| \leq \frac{2}{\left|1+e^{i \omega}\right|}
$$

If $q_{N}(k)$ is even, then we need to include one more group of $N$ terms in the clustering as follows

$$
\begin{aligned}
S(c)_{K}^{k} & =\sum_{1 \leq q^{\prime} \leq q_{N}(k)} \sum_{\left|r^{\prime}\right| \leq K} c_{q^{\prime} N+r^{\prime}}-\sum_{k \% N<r^{\prime} \leq K} c_{q_{N}(k) N+r^{\prime}} \\
& =\left(\sum_{1 \leq q^{\prime} \leq q_{N}(k)}(-1)^{q^{\prime}}\right) \sum_{\left|r^{\prime}\right| \leq K} c_{r^{\prime}}+(-1)^{q_{N}(k)+1} \sum_{k \% N<r^{\prime} \leq K} c_{r^{\prime}} .
\end{aligned}
$$

Here again, the first term vanishes and

$$
\left|S(c)_{K}^{k}\right|=\left|\sum_{k \% N<r^{\prime} \leq K} c_{r^{\prime}}\right| \leq \frac{2}{\left|1+e^{i \omega}\right|}
$$

Then, whatever the parity of $q_{N}(k)$,

$$
S(\tilde{b})_{K}^{k} \leq \frac{4}{\left|1+e^{i \omega}\right|}
$$

## 2 Relevant literature

It is close to impossible to cover the entire literature that was dedicated to truncation error or more generally to approximation errors during nearly six decades. Nonetheless we will attempt to give a broad overview of the corresponding literature. That will hopefully complete the short bibliographic discussion that was sketched in the introduction of this paper.

### 2.1 Classification criteria

In this part, we focus on different criteria that can help classifying the variety of related works. Those criteria may concern one or both of the two following aspects

- the properties of the signal (or more rigorously of the signal model)
- and the way the continuous signal is sampled/recovered.

Signal model Published works can be separated first according to whether they deal with truncation errors or with generalised aliasing. We recall that by generalised aliasing we mean both spectral aliasing due to an actually non-band-limited signal and to truncation error. Examples of works that deal with not necessarily band-limited signals include [8, 9, 32, 28, 3, 39]. Those works often consider the Shannon sampling theorem under the angle developed by de la Vallée Poussin in [27] instead of the one of Whittaker [36]. The difference lies in the fact that the former consists in letting the sampling rate $\Omega$ grow to infinity while the latter prefers to increase the extent $K$ of the sampling domain. Notation-wise, if one considers truncation error only, one may assume that the sampling rate is fixed and set it as we did to 1 by default.

Assuming now that the signal is indeed Nyquist band-limited and therefore that we have to cope with truncation error only, then strongly different estimates are obtained if the signal is over-sampled (existence of a guard-band) or not. On the one hand, works such as [38, 5, 11] consider the over-sampled case while the general case is dealt with in [6, 20]. In agreement with our own developments, all published bounds validate the fundamental difference between the two cases with respect to the error decay. As we stated it before, the oversampling assumption is not satisfactory for natural images.

Another important difference resides in the modeling of signals as deterministic (e.g. in 38, 5] ) or random processes $([2,6,10])$. Both approaches have their pros and cons. On the one hand, the deterministic approach leads to estimates that are always true. But that goes with a price since such results always rely on signal properties that cannot be estimated from a finite number of samples. On the other hand, relying on the RP model leads to weaker estimates that involve expectation or probabilities. But that allows to formulate much weaker assumptions, for instance that the signals simply remains statistically identical if we consider stationarity. In this specific case, we argue that the required properties such as the signal variance can be fairly well estimated from the samples. As a matter of fact, we have taken advantage of this fact to design an experimental protocol adapted to natural images. From the mathematical perspective, the deterministic sampling theorem almost directly provides a mean square equivalent for random processes, by simply applying the deterministic one to the autocorrelation function $R_{X}$. The first application of this process is certainly in [2], where the classical Shannon-Whittaker result was adapted to stationary random processes. It was also applied in different settings, for example the result of [16] concerning reconstruction from local averages was adapted in 31. This context will be discussed hereafter. Quite on the contrary, when considering error estimates, no such process was found, and theorems concerning random processes, although close in spirit to the deterministic ones require a specific proof.

Sampling and reconstruction We discuss now several points in sampling (or more generally signal analysis) and reconstruction (a.k.a. synthesis) where the published works may differ. First, some works consider non uniform sampling (often referred as interpolation from scattered data). It is known that the Shannon sampling theorem generalizes to these settings [13, 29, 18, Truncation error estimates in this case were considered for example in [4] in order to demonstrate the convergence of non-uniform sampling series. Although non-uniform sampling may seem irrelevant for images that are uniformly sampled, it implicitly arises when resampling operations are pipelined.

Besides, many authors rightly argue that actual sensors do not collect samples but local averages [16]. Error estimates in this context are presented in [22, 30]. Other similar results presented in [31] were simplified and extended in [14] including results for the non-oversampled case. Besides, local averages and classical samples can also be regarded as special cases of scalar product measurements with frames. Frames were first introduced in a seminal paper by Duffin and Schaeffer [13] and have played a key role ever since in many applications of signal processing. An in-depth survey on the topic can be found in [1]. Because of the redundancy they may introduce, frames are in several ways more robust than orthonormal bases. In particular, the noise present in the frame coefficients is reduced by projecting it on a subspace of $l^{2}$ before reconstructing the signal. It is to be noted that many previously discussed aspects can be reinterpreted in the context of frames. For instance the frame redundancy is related to oversampling. Similarly, non-uniform sampling theorems can be obtained from non-harmonic Fourier frames. This relationship simply stems from the identity $f\left(t_{n}\right)=\left\langle f, \operatorname{sinc}\left(.-t_{n}\right)\right\rangle=\left\langle\hat{f}, e^{-i t_{n} \cdot \mathbb{1}_{[-\pi, \pi]}}\right\rangle$, valid if $f \in L^{2}$ is Nyquist band-limited. Therefore,
if the sequence $\left(e^{-i t_{n} \omega}\right)_{n}$ is a fram ${ }^{2}$ on $L^{2}([-\pi, \pi])$ then $\hat{f}$ can be reconstructed from its frame coefficients, and so can $f$. A thorough study of various error estimates (quantization, jitter, truncation and aliasing) in the case of frame reconstruction for irregularly sampled signals is presented in [15.

On the reconstruction side, most of the work, where truncation errors are estimated, focuses on the sinc kernel, but other kernels were also considered. Often the point is to take advantage of oversampling which introduces the so-called guard-band where the signal spectrum vanishes. This guard-band allows to design low-pass filters that are smooth in the frequency domain and that therefore vanish more quickly than the sinc (in time or space domain). This is the case for [17, 11] and more recently [34]. To us, a more interesting extension would concern kernels that are of common use in image processing. This includes finite-length kernels (nearest-neighbor, linear, cubic interpolations), fast-decay kernels such as B-splines and DFT interpolation. To our knowledge, the latter was only considered in 37 and under the oversampling assumption. On the other hand, errors associated with B-Splines were studied in [35] using the Strang-Fix condition [33], a powerful tool developed in the field of approximation theory. It must be noted that the truncation error is completely ignored in [35] where the emphasis is put on aliasing.

On a more technical stance, few articles consider error estimates for multi-dimensional signals i.e. $t=$ $\left(t_{1}, \ldots, t_{d}\right) \in \mathbb{R}^{d}$. In the context of truncation error, this extension was first considered in 38 in a rather succinct paragraph and without proof. In fact, we believe that the proposed upper-bound is erroneous since it implies that the error vanishes as soon as one component $t_{i}$ is an integer. The approach that we have proposed in Remark 6.1 was actually inspired by [26] where it is used to extend a one-dimensional upperbound published in [25]. Unfortunately, their one-dimensional bound itself is erroneous. Indeed, letting their free parameter $q \rightarrow \infty$ makes the error vanish everywhere. We believe that the issue comes from an invalid use of the Bernstein lemma, applied to a function that is only differentiable on ] $0,2 \pi$ [ while it is only valid under differentiability on the whole torus. As a side note, we would like to emphasise that despite several attempts, we could not find any published erratum concerning 38. On the contrary, the incorrect result of [25] was pointed out in 21.

Let us end this discussion on another technical comment. Several articles [24, 39] claim to provide bounds that are uniform on the whole signal domain, and their bounds appear to be independent from the distance to the image border. Such results may seem quite an achievement compared to the bounds derived in other works and to ours in particular. In fact, most of the time, uniform bounds are only an illusion since they are in fact obtained by using a moving sampling window $|k-t| \leq K^{\prime}$ which is a rather favorable situation compared to the more realistic situation of a fixed sampling window $|k| \leq K$ considered herein. In an attempt to obtain a fair comparison, one may adapt results obtained in a moving window context into a fixed window situation. To do so, one must simply extract the largest symmetric window $|k-t| \leq K^{\prime}$ that is included into a prescribed fixed window $|k| \leq K$. It turns out that this corresponds to setting $K^{\prime}=K-t$, and replaces seemingly uniform bounds into ones that do depend on the distance to the nearest border. Other works such as [20] obtain uniform bounds in the fixed window settings by relying on strong decay conditions.

### 2.2 Article summaries

It would be useful to establish a taxonomy of the literature based on the criteria described previously. This task goes beyond the purpose of this article. We will instead focus on selected articles that deal specifically with bounds on the truncation error and that are the most closely related to the present study. We provide short summaries of these selected works and explain qualitatively where they fail to answer some of our concerns.

In [39, the authors consider centered Whittaker-Kotelnikov-Shannon truncated series,

$$
\left(S_{\Omega, N}\right) f(t):=\sum_{-N-2 \Omega t \leq k \leq N+2 \Omega t} f\left(\frac{k}{2 \Omega}\right) \operatorname{sinc}(2 \Omega t-k)
$$

[^1]and consider the assumption that $f$ belongs to the unit ball of the Sobolev space $W_{p}^{r}(\mathbb{R}), r \in \mathbb{N}, p \geq 1$. They first estimate the aliasing absolute error for Sobolev spaces, namely
$$
\left|f(t)-\left(S_{\Omega, \infty} f\right)(t)\right| \leq C . \Omega^{-r+\frac{1}{p}} \ln \Omega
$$
and obtain as a corollary a truncation error, with exactly the same bound when $N=\Omega^{r p}$. This result is quite interesting but actually does not meet our requirement of obtaining a tight bound from the decay of the truncation error near the boundary. In truth, the above estimate only holds near the center of the known part of the signal. The decay at the center of the signal for $r=1$ and $p=2$ is of the order of $|\Omega|^{-1}$, which is consistent with our estimates for oversampled signals. Effectively, these values yield $-r+\frac{1}{p}=-\frac{1}{2}$ and $r p=2$. On the other hand, it is difficult to draw an analogy between our results and the results of this paper. Indeed, the above estimates are obtained for $f$ in a unit ball of the Sobolev space. But this assumption implies that the average value of the samples $f(k)$ is square integrable and therefore becomes very small when $\Omega \rightarrow \infty$. In contrast, our typical assumption of an outer image being made of white noise allows for an image with infinite Sobolev energy. It seems to us that for the scopes of image processing the Sobolev assumption is therefore not valid and should be replaced by the assumption that the image belongs to a local stationary Sobolev space, for example by assuming that $\frac{1}{|\Omega|} \int_{[-\Omega, \Omega]}\left(|f(t)|^{2}+\left|f^{\prime}(t)\right|^{2}\right) d t \leq C$. This last assumption would be compatible with white noise.

Another paper [34] considers the approximation of bounded uniformly continuous (BUC) functions by a generalized Shannon-Whittaker formula of the form

$$
\begin{equation*}
\sum_{k} f\left(\frac{k}{W}\right) s(W t-k) \tag{3}
\end{equation*}
$$

where the function $s$ is a generalization of the sinc function that retains the two essential features for being a Gibbs-effect free good interpolation kernel:

1. it is band limited and even;
2. its spectrum $\hat{s}$ is continuous and vanishes outside the interval $[-1,1]$.

The sound and practical object of the paper is to give tight estimates for the convergence of the series (3) for several classic interpolation filters satisfying the above two requirements. The conclusion is that for most usual such filters, the decay of the series is so fast that a finite sum does nicely the approximation job. The sinc interpolator itself is of course not included in this framework. The paper deals with a class of functions $f$ that will be consistently approximated by (3) only if their spectrum have a profile similar to that of $\hat{s}$. In short, the study is very relevant for slightly over-sampled signals obtained by anti-aliasing filters. Clearly, this excludes the band-limited white noise considered in our study. As shown in the experimental section, our band limited white noise model for external samples is nevertheless realistic for digital images.

In [24], the authors consider the estimation of a sharp upper bound for functions $f \in L^{2}\left(\mathbb{R}^{d}\right)$ (or $f \in L^{p}$ in [23]) of the form $\epsilon_{K}(t) \leq C\|f\|_{2}$ and then transpose their results to the weak Cramer class of RPs (a generalisation of weakly stationary RPs). The upper bound is sharp in the sense that no other upper bounds of the same form could possibly be sharper since the bound is reached in a worst-case signal. It is therefore a very appealing result. However it is important to notice that other bounds can very well yield finer estimates of the truncation error, even without being sharp in the previously described way. Indeed this may be the case as soon as an upper bound does not follow the same predefined form. Theorem 1 in [24] states that the maximal truncation error for an $L^{2}$ function which is $[-\pi, \pi]$ band-limited satisfies

$$
\left\|f-S_{N} f\right\|_{\infty} \leq\|f\|_{L^{2}} \sqrt{1-\frac{8}{\pi^{2}} \sum_{n=1}^{N} \frac{1}{(2 n-1)^{2}}}
$$

The order of magnitude of the remainder of this series being $O\left(\frac{1}{N}\right)$, it follows that the maximal error is of the order of $O\left(\frac{1}{N^{\frac{1}{2}}}\right)$. This result seems at first directly compatible with our result on band limited white
noise. Nevertheless, taking into account that the trajectory of a white noise on $\mathbb{R}$ has in average an infinite $L^{2}$ norm (i.e. $\mathbb{E}\left[\left\|X_{t}\right\|_{2}^{2}\right]=+\infty$ by virtue of the Fubini-Tonelli theorem) while the function $f$ is bounded in $L^{2}(\mathbb{R})$, one can conclude that the estimate of Theorem 1 does not apply to our purposes. Theorem 2 undergoes the same comments. Theorem 3 gives a similar estimate to Theorem 1. Although we did not succeed to establish whether a white stationary noise, for example, could be interpreted as a weak Cramér random field, it is nevertheless clear that the estimate of Theorem 3, like for Theorem 1, ensures only that the maximal error is of the order of $\frac{1}{\sqrt{N}}$.

In [3], the authors consider the problem of interpolating a $C^{\infty}$ function on an interval $[-X, X]$ by using the Shannon-Whittaker interpolation with a step $h=\frac{X}{N}$. This means that only the $2 N+1$ samples $n h$ for $n \leq N$ are available in the Shannon Whittaker series. The paper provides an exact truncation error $e_{N, f, h}(x)$ at $x \in[-X, X]$ depending on $h, N, X$, namely

$$
e_{N, f, h}(x)=(-1)^{N} \operatorname{sinc}\left(\frac{x}{h}\right) \sum_{k=1}^{\infty} b_{2 k}(x)(2 h)^{2 k-1}
$$

where

$$
b_{2 k}(x)=2\left(1-2^{-2 k}\right) a_{2 k}(x)
$$

and

$$
a_{2 k}(x)=\frac{B_{2 k}}{(2 k)!}\left[\left(\frac{f(y)}{x-y}\right)^{(2 k-1)}(X)-\left(\frac{f(y)}{x-y}\right)^{(2 k-1)}(-X)\right]
$$

where $B_{n}$ are the classic Bernoulli numbers. The formula, being exact, can be applied in our setting, namely with $h=1, X=N$. Furthermore, it is easily seen that the term with highest magnitude in terms of the inverse distance to the boundary is the first one obtained for $k=1$ in the series, namely

$$
\begin{aligned}
e_{N, f}(x) & \equiv \frac{1}{8}(-1)^{N} \operatorname{sinc}(x)\left[\frac{f^{\prime}(N)}{x-N}-\frac{f(X)}{(x-N)^{2}}-\frac{f^{\prime}(-N)}{x+N}+\frac{f(N)}{(x+N)^{2}}\right] \\
& \equiv \frac{1}{8}(-1)^{N} \operatorname{sinc}(x)\left[\frac{f^{\prime}(N)}{x-N}-\frac{f^{\prime}(-N)}{x+N}\right]
\end{aligned}
$$

where the asymptotic equivalence " $\equiv$ " is meant to hold when $x$ tends to the boundaries of the interval, $X$ of $-X$. Here we used $B_{2}=\frac{1}{6}$. Notice that the error is zero when $x$ is an integer in $[-N, N]$ and notice also that the decay of the error from the boundary has an order of magnitude proportional to the inverse of the distance of the sample to the interval boundary. Thus, again, the error estimate is different from the one we promote, the settings being also anyway different. Indeed, first, we have no knowledge of $f^{\prime}(N)$ or $f^{\prime}(-N)$ (or of the successive derivatives) for white noise, since we only know a finite number of samples $f(k)$. Thus the above formula, exact though it is, is not directly applicable to the assumption of an external white noise. Nevertheless, some analogy of the last mentioned equivalence can be drawn, since we obtain a decay in $\frac{1}{x-N}$ from the boundary equivalent to the one occurring with oversampled (and therefore smooth) signals.

In 20, (true) uniform upper bounds are derived for the $L^{2}$ functions. Obviously, in order to obtain uniform estimates, the authors had to rely on additional assumptions on the signal, namely that the functions decay as $\frac{1}{|t|^{r+1}}$ where $r>0$. This assumption is a simplification of the one used in [7] but remains a very strong one. They obtain the following result

$$
\left|\epsilon_{K}(t)\right| \leq \frac{C}{K^{r+\frac{1}{2}}}
$$

One may notice that the behaviour of the error is largely different from the one we have obtained. This is easily explained by the fast decrease of the signal (as $\frac{1}{|t|^{r}+1}$ ) which makes the missing samples importance less significant than under the stationarity assumption. Unfortunately here, it is not possible to find a limit case that is both valid under the $L^{2}$ assumption and that resembles the context of our study. In particular the stationarity is incompatible with the $L^{2}$ assumption.

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[^0]:    ${ }^{1}$ This assumption is useful to ensure that the terms in the big $\mathcal{O}$ notation are indeed negligible compared to the remaining ones.

[^1]:    ${ }^{2}$ The required conditions to actually obtain a frame on the Nyquist-band-limited space are provided in [12] (theorem 3.1) and attributed to Jaffard and Seip. On the one hand, the sample locations $t_{n}$ must be relatively separated in order to comply with the frame upper bound. On the other hand, they need to be sufficiently dense in order to achieve the lower bound.

