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Abstract. We construct a topological invariant of algebraic plane curves, which is in some sense an adaptation of the linking number of knot theory. This invariant is shown to be a generalisation of the $\mathcal{I}$-invariant of line arrangements developed by the first author with Artal and Florens. We give a practical tool for computing this invariant, using a modification of the usual braid monodromy. As an application, we show that this invariant distinguishes a new Zariski pair of curves, i.e. a pair of curves having same combinatorics, yet different topology. These curves are composed of a smooth cubic with 5 tangent lines at its inflexion points. As in the historical example of Zariski, this pair can be geometrically characterized by the mutual position of their singular points.

1. Introduction

The topological study of algebraic plane curves was initiated at the beginning of the 20th century by F Klein and H Poincaré. One of the main questions is to understand the relationship between the combinatorics and the topology of a curve. It is known, since the seminal work of O Zariski [18, 19, 20], that the topological type of the embedding of an algebraic curve in the complex projective plane is not determined by the combinatorics. Indeed, Zariski constructed two sextics with 6 cusps having same combinatorics, and proved that the fundamental group of their complements are not isomorphic. Geometrically, these two curves are distinguished by the fact that the cusps in the first curve lie on a conic, while they do not in the second one. Since this historical example, using various methods, numerous examples of pairs of algebraic curves having same combinatorics but different topology have been found, see for example E Artal, J I Cogolludo and H Tokunaga [3], P Cassou-Noguès, C Eyral and M Oka [8], A Degtyarev [9], M Oka [14], I Shimada [15], or the first author [11]. E Artal suggests in [1] to call such examples *Zariski pairs*.

The topology of curves in $\mathbb{CP}^2$ is intimately connected to the topology of knots and links in S^3. Several tools are indeed shared by these two domains, such as the homology or the fundamental group of the complement, the Alexander polynomial or module, etc, although they usually have rather different behaviors.

Recently, E Artal, V Florens and the first author defined a topological invariant of line arrangements (i.e. algebraic plane curves with only irreducibles of degree 1) which is in some sense modeled on the linking number of knot theory [4]. This invariant was then successifully used in [11] to distinguish a new Zariski pair. In the present paper, we construct another invariant adapting the linking number to the more general case of algebraic plane curves. In the case of a line arrangement, this invariant is shown to be equivalent to the invariant of [4], thus providing a generalization of this earlier work through a different adaptation of the linking number.

The construction of our linking invariant can be roughly outlined as follows. The basic idea is to consider a cycle γ in an algebraic plane curve $\mathcal{C}$, which in some sense intersects a minimal number of components of $\mathcal{C}$, and take its image in the first homology group of the complement $E_{\mathcal{C}^c_\gamma}$ of the curve $\mathcal{C}^c_\gamma$ formed by the components which do not intersect the cycle γ. Unfortunately such cycles – and thus their homology classes – are not well defined in general. One of the issues here is the non existence of a canonical basis of the first integral homology group of the components with nonzero genus, hence the definition of cycles traversing such components. In order to remove these indeterminacies, we consider the *linking set* $\{\gamma\}$, which is defined as the set of all cycles combinatorially equivalent to γ (in the sense that they have same projection on the incidence graph of $\mathcal{C}$), regarded in the quotient of $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})$ by an appropriate *indeterminacy subgroup* $\mathfrak{I}_\gamma$. Roughly speaking, this indeterminacy subgroup measures the combinatorial difference between various embeddings of the cycle in the complement $E_{\mathcal{C}^c_\gamma}$. The result is a topological invariant of ordered and oriented algebraic plane curves

This linking invariant has a nice behaviour for some particular choice of curve or cycle. In the case of rational curves, that is, curves whose irreducible components have all genus 0, we indeed observe that the linking set is simply a singleton. This allows us to prove the equivalence with the $\mathcal{I}$-invariant of [4] in the case of line arrangements. We also obtain a stronger version of the invariance theorem in the case of a cycle contained in a single irreducible component.

From a practical viewpoint, we also show how this invariant can be computed on concrete examples, using an adaptation of the braid monodromy. This makes a concrete connection between our invariant and the usual linking number of knot theory.

To illustrate this adaptation of the linking number to algebraic curves, we use it to distinguish a family of new Zariski pairs. These new examples are formed by curves of degree 8 and are composed of a smooth cubic and 5 tangents lines at the inflexion points of the cubic. These examples are noteworthy in that they are characterized by a similar geometric property as Zariski's historical example; more precisely, these curves can be distinguished by the conic passing through the 5 tangent points, which intersects the cubic in exactly 5 points in the first curve and at 6 points in the second one. It would be interesting to investigate the behavior of the linking invariant on other known examples of Zariski pairs, which are in some sense close to the above ones. Very recently, T Shirane succeeded in showing that a series of curves proposed by Shimada [16] indeed form π_1-equivalent Zariski tuples [17]; the simplest of these examples is given by a cubic and a quartic intersecting along 6 double points. We expect that our invariant might be able to detect (some of) these examples, and that it might be related to Shirane's construction. Furthermore, establishing such a relation may allow to determine whether our invariant is independent of the fundamental group of the complement. Other close examples are to be found in the work of Artal and Tokunaga [6]. Note that in both situations, the quotient by the indeterminacy subgroup involved in our construction turns out to be non trivial, thus suggesting that the linking invariant could detect the different topologies.

The rest of this paper is organized as follows.

In Section 2, we recall some basic definitions and results on the topology and combinatorics of algebraic curves. In particular, we define the incidence graph, which encodes parts of the combinatorics of the curve, and recall the Zariski-van Kampen presentation of the fundamental group of the curve complement, using braid monodromy.

Section 3 contains the construction/definition of the linking set and of the indeterminacy subgroup associated to a cycle of the curve. In the latter part of this section, we state and prove the main invariance theorem (Theorem 3.13).

Some particular cases of the invariance theorem are studied in Section 4, corresponding to the cases of rational curves and of a cycle sitting in a single irreducible component. The equivalence of our linking invariant with the $\mathcal{I}$-invariant of [4] for line arrangements is given in this section.

Section 5 deals with the computation of the linking set. We slightly modify the usual notion of braid monodromy to compute the image of a cycle in the first homology group of the complement $E_{\mathcal{C}^c_\gamma}$. This is illustrated with the example of a smooth cubic and two tangent lines.

We give in Section 6 the main application of our construction, namely a set of new Zariski pairs. This is done by an explicit definition and computation, using the above mentioned method, together with an additional argument that allows to remove a restrictive hypothesis of the invariance theorem.

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Notations and conventions. We will make use of the following throughout the paper.

- Unless otherwise specified, $\mathcal{C}$ denotes a reducible curve in $\mathbb{CP}^2$ of degree d, with irreducible components: $\{C_1, \cdots, C_n\}$.
- A *cycle* in the curve $\mathcal{C}$ is a closed loop in $\mathcal{C}$ with non trivial homology class in $\mathrm{H}_1(\mathcal{C}, \mathbb{Z})$, which may contain singular points of arbitrary multiplicity.
- All homology groups are to be understood with integral coefficients, and this will often be omitted in the notation.

2. Preliminary definitions

This section contains some well-known results on the combinatorics of curves (see [3] for more details), and on their topology, such as braid monodromy or the Zariski-van Kampen presentation of the fundamental group of the complement.

2.1. Combinatorial data.

2.1.1. *Combinatorics.* We review here the combinatorial data naturally associated with a plane curve. Formally, we have

Definition 2.1. The *combinatorics* of $\mathcal{C}$ is the homeomorphism type of the pair $(\mathrm{Tub}(\mathcal{C}), \mathcal{C})$, where $\mathrm{Tub}(\mathcal{C})$ is a tubular neighbouhood of $\mathcal{C}$.

Following [5], this definition can be given a more 'combinatorial' flavour as follows. The combinatorics of $\mathcal{C}$ is given by the data

$$\left(\mathrm{Irr}(\mathcal{C}), \deg, \mathrm{Sing}(\mathcal{C}), \sigma_{\mathrm{top}}, \{\beta_P\}_{P\in\mathrm{Sing}(\mathcal{C})}\right),$$

where:

- $\mathrm{Irr}(\mathcal{C})$ is the set of all irreducible components of $\mathcal{C}$,
- deg assigns to each irreducible component its degree,
- $\mathrm{Sing}(\mathcal{C})$ is the set of all singular points of $\mathcal{C}$,
- σ_{top} assigns to each singular point its topological type,
- for each singular point $P \in \mathrm{Sing}(\mathcal{C})$, β_P assigns to each local branch of $\mathcal{C}$ at P the global irreducible component containing it.

Two curves have the same combinatorics if there exist bijections between their sets Irr and Sing of irreducible components and singular points, which are compatible with the assignments deg, σ_{top} and $\{\beta_P\}$ in the natural way; see [5, Rem. 3] for details.[1]

The combinatorics is *ordered* if we add an order on the set $\mathrm{Irr}(C) = \{C_1, \cdots, C_n\}$. In this case, two curves have the same ordered combinatorics if the bijection between the sets Irr also preserves the order.

[1]Here, we adopt a somewhat condensed definition of the combinatorics compared to that of [5, Rem. 3], where the assignment σ_{top} (resp. $\{\beta_P\}$) implicitly contains the data of the set of topological types in $\mathrm{Sing}(\mathcal{C})$ (resp. the set of local branches of $\mathcal{C}$ at P).

2.1.2. *Incidence graph.* A part of the combinatorics of $\mathcal{C}$ can be encoded graphically as follows.

Definition 2.2. The *incidence graph* $\Sigma_\mathcal{C}$ of $\mathcal{C}$ is a bipartite graph, where the vertex set is decomposed into

- $V_{\mathrm{Irr}} = \{v_C \mid C \in \mathrm{Irr}(\mathcal{C})\}$, the component vertices;
- $V_{\mathrm{Sing}} = \{v_P \mid P \in \mathrm{Sing}(\mathcal{C})\}$, the point vertices.

An edge of $\Sigma_\mathcal{C}$ joins $v_C \in V_{\mathrm{Irr}}$ to $v_P \in V_{\mathrm{Sing}}$ if and only if $P \in C$.

So, roughly speaking, this graph contains the information of the 'handles' formed by the intersection of the irreducible components.

Given a cycle γ in $\mathcal{C}$, there is a natural *projection* $\Sigma(\gamma)$ of γ on the incidence graph $\Sigma_\mathcal{C}$, which we define below. To do so, we first recall some basic vocabulary.

Let Σ be a finite graph with no multiple edges, i.e. no pair of edges with same endpoints. A *walk* on Σ is a finite sequence $(v_0, v_1, \cdots, v_k)$ of vertices in Σ, possibly with repetitions, such that v_i and v_{i+1} are connected by an edge e_i for each $i \in \{0, \cdots, k-1\}$. The composition of two walks $(v_0, \cdots, v_k)$ and $(v'_0, v'_1, \cdots, v'_k)$ such that $v_k = v'_0$ is simply defined as the walk $(v_0, \cdots, v_k, v'_1, \cdots, v'_k)$. A *closed walk* is a walk where the first and last vertices v_0 and v_k coincide. In this paper, we will consider closed walks up to cyclic permutations. Finally, a walk is called *contractible* if the path obtained by following the edges e_i when i runs from 1 to $k-1$ is a contractible path in $\Sigma_\mathcal{C}$.

We can now define the projection $\Sigma(\gamma)$ on the incidence graph $\Sigma_\mathcal{C}$ of a cycle γ in $\mathcal{C}$, as follows. If γ avoids $\mathrm{Sing}(\mathcal{C})$, then it is contained in a single irreducible component $C \in \mathrm{Irr}(\mathcal{C})$, and $\Sigma(\gamma)$ is simply the corresponding vertex v_C. Otherwise, γ decomposes as a union of paths with endpoints in $\mathrm{Sing}(\mathcal{C})$; for each such path connecting, say, the singular point p to p' in the irreducible component c, consider the corresponding walk in $\Sigma_\mathcal{C}$, given by the sequence $(v_p, v_c, v_{p'})$; the projection $\Sigma(\gamma)$ is then given by composing the walks associated with each path, in the order of passage when running along γ. (Since we consider walks up to cyclic permutation, this does not depend on a choice of base point on the cycle.)

2.2. **Topology of the complement.**

We now recall some well-known facts on the topology of algebraic curves and their complement.

Definition 2.3. The *topological type* of a curve $\mathcal{C}$ is the homeomorphism type of the pair $(\mathbb{CP}^2, \mathcal{C})$.

This definition is classically refined in two compatible ways. On one hand, the topological type is *oriented* if we further require that the homeomorphism preserves the global and the local orientation (around the irreducible components of $\mathcal{C}$). On the other hand, it is *ordered* if the homeomorphism preserves a fixed order on the irreducible components of $\mathcal{C}$.

The fundamental group of the complement is finitely presented, and an explicit presentation can be given using the so-called Zariski-van Kampen method [13, 10], which we briefly outline below.

Let $*$ be a point in the complement $E_\mathcal{C} = \mathbb{CP}^2 \setminus \mathcal{C}$ of the curve $\mathcal{C}$, and L_∞ be a line containing $*$ and intersecting $\mathcal{C}$ at d points. Pick in L_∞ a system of meridians $\{m_1, \cdots, m_d\}$ around these d intersection points, based at $*$. The fundamental group $\pi_1(L_\infty \setminus \mathcal{C})$ is generated by $m_1, \cdots, m_d$, with (only) relation $m_1 \cdots m_d = 1$.
Now, consider the projection $\mathbb{CP}^2 \setminus \{*\} \to \mathbb{CP}^1$ centered at $*$. Pick in this $\mathbb{CP}^1$ a system of meridians $\{\sigma_1, \cdots, \sigma_r\}$ around each point a_j such that the fiber L_{a_j} above a_j is either tangent to $\mathcal{C}$ or passes through a singular point of $\mathcal{C}$ ($j = 1, \cdots, r$).
For any based simple closed loop $\sigma \subset \mathbb{CP}^1$ which is disjoint from the points a_j ($j = 1, \cdots, r$), the fiber over each point of σ intersects $\mathcal{C}$ at d distinct points, so that the loop σ defines naturally

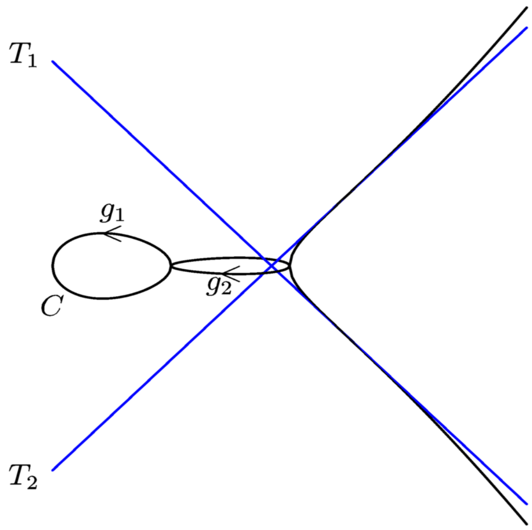

FIGURE 1. Real picture of the curve $\mathcal{T}$.

an element of the fundamental group of the configuration space of d points, which is the braid group on d strands. This braid associated with σ acts naturally on $\pi_1(L_\infty \setminus \mathcal{C})$, and this action is called the *braid monodromy* of σ.
In particular, each meridian σ_j induces a braid monodromy action $\rho_j \in \mathrm{Aut}\,(\pi_1(L_\infty \setminus \mathcal{C}))$.

Theorem 2.4 (Zariski–Van Kampen; [13])**.** *The fundamental group of the complement of $\mathcal{C}$ has the following presentation*

$$\pi_1(E_\mathcal{C}) \simeq \langle m_1, \cdots, m_d \mid m_1 \cdots m_d = 1 \text{ and } \rho_j(m_i) = m_i \ ; i = 1, \cdots, d-1, j = 1, \cdots, r\rangle.$$

The first homology group of $E_\mathcal{C}$ can be deduced from the above. Since any two meridians of each component C_i of $\mathcal{C}$ are conjugate, we can consider their conjugacy class x_i ($i = 1, \cdots, n$) so that, if we denote by d_i the degree of the irreducible component C_i of $\mathcal{C}$, we have:

Theorem 2.5. *The first integral homology group of the complement of $\mathcal{C}$ has the following presentation*

$$\mathrm{H}_1(E_\mathcal{C}) \simeq \langle x_1, \cdots, x_n \mid d_1 x_1 + \cdots + d_n x_n = 0\rangle.$$

In particular, $H_1(E_\mathcal{C})$ is isomorphic to $\mathbb{Z}^{n-1} \times \mathbb{Z}_{\gcd(d_1,\cdots,d_n)}$.

Note that $H_1(E_\mathcal{C})$ is thus free of rank $n-1$ as soon as some component of $\mathcal{C}$ is a line.

3. The linking invariant

In this section, we define the main object of this paper, namely the linking invariant of algebraic curves, and give some of its main properties.

In order to illustrate the definitions given in this section and in Section 5, we will consider the example of the curve $\mathcal{T}$ defined as the cubic $C : x^3 - xz^2 - y^2 z = 0$, with its two real tangents lines (denoted by T_1 and T_2). The part of the curve $\mathcal{T}$ for $x \in (-1; 3)$ is pictured in Figure 1.

Notation. In what follows, $\mathcal{C}$ denotes an algebraic curve as in the previous section.
Although our construction does not involve any choice, it will be convenient for making this construction more explicit to *fix*, once and for all, a homological parametrization of each irreducible

component of our given algebraic curve $\mathcal{C}$. More precisely, we fix here the assignment

$$\{\Gamma_C\}_{C\in\mathrm{Irr}(\mathcal{C})}$$

of an ordered set of oriented simple closed curves on each component $C \in \mathrm{Irr}(\mathcal{C})$, that forms a basis for its first integral homology group. Note that this additional piece of data is vacuous if and only if the curve is rational (*i.e.* all components have genus 0).

3.1. **Some definitions.** We begin with definitions necessary for the construction of our linking invariant.

Definition 3.1. Let γ be a cycle in a curve $\mathcal{C}$. The *support* of γ is the union of the irreducible components of $\mathcal{C}$ that intersect *any* simple closed curve c freely homotopically equivalent to γ:

$$\mathrm{Supp}(\gamma) = \bigcap_{c\sim\gamma} \{C \in \mathrm{Irr}(\mathcal{C}) \mid C \cap c \neq \emptyset\}.$$

Likewise, the *internal support* of γ is the subset of $\mathrm{Supp}(\gamma)$ defined by

$$\overset{\circ}{\mathrm{Supp}}(\gamma) = \bigcap_{c\sim\gamma} \left\{ C \in \mathrm{Supp}(\gamma) \mid (C \overset{\circ}{\cap} c) \neq \emptyset \right\}.$$

In what follows, we will only consider a particular family of cycles, which are those cycles that are contained in their supports.

Definition 3.2. A cycle is *minimal* if it is contained in the components of its internal support.

Let γ be a minimal cycle of $\mathcal{C}$, then we set

$$\mathcal{C}_\gamma = \bigcup_{C\in\mathrm{Supp}(\gamma)} C \qquad \text{and} \qquad \mathcal{C}_\gamma^c = \bigcup_{C\notin\mathrm{Supp}(\gamma)} C.$$

We have then $\mathcal{C} = \mathcal{C}_\gamma \cup \mathcal{C}_\gamma^c$, and $\mathcal{C}_\gamma \cap \mathcal{C}_\gamma^c \subset \mathrm{Sing}(\mathcal{C})$.

Remark 3.3.

(1) If γ is a minimal cycle of $\mathcal{C}$ then it can be seen as a loop in the complement $E_{\mathcal{C}_\gamma^c}$ of $\mathcal{C}_\gamma^c$.
(2) The support of a curve is in general not well defined for a homology class, i.e. two homologuous curves may have different supports.

The following proposition asserts that there always exists a minimal representative within a given (non trivial) free homotopy class.

Proposition 3.4. *For any cycle γ in a curve $\mathcal{C}$, there exists a cycle of $\mathcal{C}$ with the same free homotopy class as γ, and which is contained in the internal support of γ.*

Proof. Let γ_1 and γ_2 be two homotopically equivalent cycles of $\mathcal{C}$. We denote by $\mathcal{C}_{\gamma_1}$ and $\mathcal{C}_{\gamma_2}$ the union of the irreducible components of $\mathcal{C}$ intersecting γ_1 and γ_2, respectively, and we set $\mathcal{C}_{1,2} = \mathcal{C}_{\gamma_1} \cup \mathcal{C}_{\gamma_2}$. Recall that, by the Seifert-Van Kampen Theorem, the fundamental group of an algebraic curve admits a presentation with two kinds of generators, the first kind coming from the genus of the irreducible components, and the second from the 'handles' formed by the intersection of the irreducible components. Note that the second kind of generators can be obtained by lifting a basis of cycles of the incidence graph. Using such presentations, one can easily check that $\big(\pi_1(\mathcal{C}_{\gamma_1})/\pi_1(\mathcal{C}_{\gamma_1} \cap \mathcal{C}_{\gamma_2})\big) \cap \big(\pi_1(\mathcal{C}_{\gamma_2})/\pi_1(\mathcal{C}_{\gamma_1} \cap \mathcal{C}_{\gamma_2})\big)$ is the trivial group. Now, if γ is the class of γ_1 and γ_2 in $\pi_1(\mathcal{C})$, then this implies that γ is an element of $\pi_1(\mathcal{C}_{\gamma_1} \cap \mathcal{C}_{\gamma_2})$, hence has a representative in $\mathcal{C}_{\gamma_1} \cap \mathcal{C}_{\gamma_2}$

This shows that there always exists a cycle supported by the intersection of the supports of two homotopically equivalent cycles, and this is thus also true for the intersection of the supports of all the representatives of the homotopy class of γ. □

Definition 3.5. Two minimal cycles of $\mathcal{C}$ are *combinatorially equivalent* if they have the same projection in the incidence graph $\Sigma_{\mathcal{C}}$ (up to cyclic permutation).

The difference in $\mathrm{H}_1(E_{\mathcal{C}_\gamma^c})$ between two minimal cycles which are homotopically equivalent in $\mathcal{C}$ is combinatorial. More precisely, it is an element of the following subgroup:

Definition 3.6. The *indeterminacy subgroup* $\mathfrak{I}_\gamma(\mathcal{C})$ associated to γ is the subgroup of $\mathrm{H}_1(E_{\mathcal{C}_\gamma^c})$ defined by

$$\mathfrak{I}_\gamma(\mathcal{C}) = \left\langle \sum_{C\in \mathrm{Supp}_P^c(\gamma)} \mathrm{lk}_P(C,D)\cdot m_C \mid \forall D \in \overset{\circ}{\mathrm{Supp}}(\gamma), \forall P\in D\cap \mathrm{Sing}(C) \right\rangle,$$

where

- the summation is over the set $\mathrm{Supp}_P^c(\gamma) = \{C \notin \mathrm{Supp}(\gamma) \mid P \in C\}$. This is the set of all irreducible components which are not in the support of γ containing P, where P is a singular point of $\mathcal{C}$ lying in a component of the internal support,
- $\mathrm{lk}_P(C,D)$ is the local linking number at P of the component C with the component D of $\mathrm{Supp}(\gamma)$ containing P.

Remark 3.7. The linking number $\mathrm{lk}_P(C,D)$ is merely the order of the singular point between C and D at the point P, see e.g. [7, pp.439].

Example 3.8. Let us consider the curve $\mathcal{T}$ of Figure 1, and the case of a cycle γ contained in the cubic C. The support (and the internal support) of γ is only composed of the cubic C. Then we have $\mathcal{T}_\gamma = C$ and $\mathcal{T}_\gamma^c = T_1 \cup T_2$. Remark that $\mathrm{H}_1(E_{\mathcal{T}_\gamma^c}) = \langle x_1, x_2 \mid x_1 + x_2\rangle$ (due to Theorem 2.5). Since the order of the intersection point of C with T_i is 3, then $\mathrm{lk}_{C\cap T_i}(C, T_i) = 3$. Finally, we have

$$\mathrm{H}_1(E_{\mathcal{T}_\gamma^c})/\mathfrak{I}_\gamma(\mathcal{T}) = \langle x_1, x_2 \mid 3x_1, 3x_2, x_1 + x_2\rangle = (\mathbb{Z}_3)^2/\langle x_1 + x_2\rangle \simeq \mathbb{Z}_3.$$

Notation. If there is no ambiguity, $\mathfrak{I}_\gamma(\mathcal{C})$ is also simply denoted by $\mathfrak{I}_\gamma$.

The indeterminacy subgroup of a cycle is determined by the combinatorics and the projection $\Sigma(\gamma)$ of γ in $\Sigma_{\mathcal{C}}$. Thus, if $\mathcal{C}$ and $\mathcal{D}$ are two combinatorially equivalent curves, and γ and μ are combinatorially equivalent cycles in $\mathcal{C}$ and $\mathcal{D}$, respectively, then $\mathfrak{I}_\gamma$ and $\mathfrak{I}_\mu$ are isomorphic. We call *natural isomorphism* the isomorphism

$$\phi : \mathrm{H}_1(E_{\mathcal{C}_\gamma^c})/\mathfrak{I}_\gamma \to \mathrm{H}_1(E_{\mathcal{D}_\mu^c})/\mathfrak{I}_\mu$$

induced by the isomorphism between $\mathrm{H}_1(E_{\mathcal{C}_\gamma^c})$ and $\mathrm{H}_1(E_{\mathcal{D}_\mu^c})$ sending meridian to meridian and respecting both the orientation and the order on $\mathrm{Irr}(\mathcal{C}_\gamma)$ and $\mathrm{Irr}(\mathcal{D}_\mu)$.

If γ is a minimal cycle in $\mathcal{C}$, then it can be seen as a cycle in the complement of $\mathcal{C}_\gamma^c$. We denote by $\hat{\gamma}$ its value in $\mathrm{H}_1(E_{\mathcal{C}_\gamma^c})$, and by $[\gamma]$ the class of $\hat{\gamma}$ in $\mathrm{H}_1(E_{\mathcal{C}_\gamma^c})/\mathfrak{I}_\gamma$.

We can now define the main object of this paper.

Definition 3.9. Let γ be a minimal cycle of $\mathcal{C}$. The *linking set* of γ, denoted by $\{\gamma\}$, is the subset of $\mathrm{H}_1(E_{\mathcal{C}_\gamma^c})/\mathfrak{I}_\gamma$ formed by the classes of all minimal cycles that are combinatorially equivalent to γ.

Remark 3.10. Adding or removing components of $\mathcal{C}$ lying in the support of γ but not in its internal support does not modify the linking set of γ and the quotient $\mathrm{H}_1(E_{\mathcal{C}_\gamma^c})/\mathfrak{I}_\gamma$.

The next result gives a more explicit description of the linking set, in terms of the homological parametrization $\{\Gamma_C\}_{C\in\mathrm{Irr}(\mathcal{C})}$ fixed at the beginning of this section.

Proposition 3.11. *Let γ be a minimal cycle of $\mathcal{C}$.*

(1) *If the projection $\Sigma(\gamma)$ on the incidence graph $\Sigma_{\mathcal{C}}$ is not contractible, then $\{\gamma\}$ is the set*

$$\langle\gamma\rangle := \left\{ [\gamma] + \sum_{C\in\overset{\circ}{\mathrm{Supp}}(\gamma)} \sum_{g\in\Gamma_C} a_g\cdot[g],\ \text{with } a_g\in\mathbb{Z} \right\}.$$

(2) *If $\overset{\circ}{\mathrm{Supp}}(\gamma)=\{C\}$ is a single component, we have*

$$\{\gamma\} = \left\{ \sum_{g\in\Gamma_C} a_g\cdot[g],\ \text{with } (a_1,\cdots,a_k)\in(\mathbb{Z})^{|\Gamma_C|}\setminus\{(0,\cdots,0)\} \right\}.$$

Remark 3.12. Note in particular that, for any two minimal cycles γ and μ supported in a same single component, we have $\{\gamma\}=\{\mu\}$.

Proof of Proposition 3.11. Suppose that $\Sigma(\gamma)$ is not contractible; in particular we then have that γ is supported by at least two components. Clearly, any cycle of $\mathcal{C}$ which is combinatorially equivalent to γ has the same (internal) support and indeterminacy subgroup. It may however differ from $\hat\gamma$ in $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})$, by a combination of terms of the two following kinds:

(1) an element of $\mathfrak{I}_\gamma$,
(2) an element of the form $\sum_{C\in\overset{\circ}{\mathrm{Supp}}(\gamma)} \sum_{g\in\Gamma_C} a_g\cdot[g]$ with $a_g\in\mathbb{Z}$.

Since terms of the first kind are handled by taking the quotient of $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})$ by $\mathfrak{I}_\gamma$, we are left with the inclusion $\{\gamma\}\subset\langle\gamma\rangle$. To see the other inclusion, note that any element of $\langle\gamma\rangle$ which is not the class of the trivial loop represents a cycle which is combinatorially equivalent to γ, so we only have to prove that $\langle\gamma\rangle$ does not contain the trivial loop. Observe that

(i) adding to γ elements of Γ_C with $C\in\overset{\circ}{\mathrm{Supp}}(\gamma)$ (by a band sum supported in the elements of $\overset{\circ}{\mathrm{Supp}}(\gamma)$) does not change the projection $\Sigma(\gamma)$,
(ii) if the projection of a cycle on $\Gamma_{\mathcal{C}}$ is not contractible, then this cannot be the trivial cycle.

Now, by hypothesis, the projection of γ is not contractible in $\Gamma_{\mathcal{C}}$, which by (i) implies that the projection of any elements of $\langle\gamma\rangle$ also has a non contractible projection. By (ii), this in turns implies that no element of $\langle\gamma\rangle$ is trivial.

Suppose now that γ is supported by a unique irreducible component C of $\mathcal{C}$. Then a cycle is combinatorially equivalent to γ if and only if it is also supported by C, and thus can be expressed as a linear combination in our chosen basis Γ_C of $H_1(C)$; since a cycle is non trivial, these coefficients are not all zero. This proves (2), and concludes the proof. □

In the case where $\Sigma(\gamma)$ is contractible in $\Sigma_{\mathcal{C}}$, we can decompose it as a band sum of subcycles supported by a single component. By Case (2) of Proposition 3.11, we can consider and compute the linking set of each subcycle. Furthermore these linking sets carry more topological information than the linking set of γ himself (each subcycle has smaller indeterminacy subgroup and smaller linking set).

3.2. Main theorem.

We now show that the linking set indeed is an invariant of algebraic curves.

Theorem 3.13. *Let $\mathcal{C}$ and $\mathcal{D}$ be two curves with the same ordered and oriented topology. If γ and μ are two combinatorially equivalent minimal cycles in $\mathcal{C}$ and $\mathcal{D}$ respectively, then we have:*

$$\phi(\{\gamma\})\cap\{\mu\}\neq\emptyset,$$

where ϕ denote the natural isomorphism between $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})/\mathfrak{I}_\gamma$ and $\mathrm{H}_1(E_{\mathcal{D}^c_\mu})/\mathfrak{I}_\mu$.

Proof of Theorem 3.13. Let $\psi : \mathbb{CP}^2 \to \mathbb{CP}^2$ be the homeomorphism sending $\mathcal{C}$ to $\mathcal{D}$, preserving the order and the orientation. Then ψ sends γ to a cycle which is combinatorially equivalent to μ. Since ψ induces a homeomorphism $\psi^* : E_{\mathcal{C}^c_\gamma} \to E_{\mathcal{D}^c_\mu}$, by Definition 3.9 we have that $\{\psi^*(\gamma)\} = \{\mu\}$.

It remains to show that $\phi(\{\gamma\}) \cap \{\psi^*(\gamma)\} \neq \emptyset$. Since ψ preserves both the order and the orientation, the map induced by ψ^* between $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})/\mathfrak{I}_\gamma$ and $\mathrm{H}_1(E_{\mathcal{D}^c_\mu})/\mathfrak{I}_\mu$ is the natural isomorphism ϕ. This implies that the image by ϕ of the class $[\gamma]$ of γ in $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})/\mathfrak{I}_\gamma$ is given by $\phi([\gamma]) = [\psi^*(\gamma)]$. s Thus the intersection of $\phi(\{\gamma\})$ and $\{\psi^*(\gamma)\}$ is not empty. ☐

Remark 3.14. We do not have in general the equality $\phi(\{\gamma\}) = \{\mu\}$. The reason is that we cannot ensure that $\phi([g]) = [\psi^*(g)]$ for *any* $g \in \Gamma_C$ with $C \in \mathring{\mathrm{Supp}}(\gamma)$. See, however, Theorem 4.1 below in the case of rational curves, as well as Theorem 4.6.

Let $\overline{\mathcal{C}}$ and $\overline{\gamma}$ be the images of $\mathcal{C}$ and γ, respectively, under the action of the complex conjugation. We denote by $\mathfrak{c} : \mathrm{H}_1(E_\mathcal{C}) \to \mathrm{H}_1(E_{\overline{\mathcal{C}}})$ the map induced by this action. Since $\mathcal{C}$ and $\overline{\mathcal{C}}$ have the same combinatorics, the quotients $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})/\mathfrak{I}_\gamma$ and $\mathrm{H}_1(E_{\overline{\mathcal{C}^c_\gamma}})/\overline{\mathfrak{I}_\gamma}$ can be identified using the natural isomorphism ϕ.

Proposition 3.15. *Let $\mathcal{C}$ be a curve, and γ be a minimal cycle in $\mathcal{C}$. The linking set of $\overline{\gamma}$ in $\mathrm{H}_1(E_{\overline{\mathcal{C}^c_\gamma}})/\overline{\mathfrak{I}_\gamma}$ is:*

$$\{\overline{\gamma}\} = \{-\phi(g) \mid g \in \{\gamma\}\}\,,$$

where ϕ is the natural isomorphism between $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})/\mathfrak{I}_\gamma$ and $\mathrm{H}_1(E_{\overline{\mathcal{C}^c_\gamma}})/\overline{\mathfrak{I}_\gamma}$.

Proof. Consider the map $\psi : \mathrm{H}_1(E_{\mathcal{C}^c_\gamma})/\mathfrak{I}_\gamma \to \mathrm{H}_1(E_{\overline{\mathcal{C}^c_\gamma}})/\overline{\mathfrak{I}_\gamma}$ such that following diagram is commutative

$$\begin{array}{ccc}
\mathrm{H}_1(E_\mathcal{C}) & \xrightarrow{\ \mathfrak{c}\ } & \mathrm{H}_1(E_{\overline{\mathcal{C}}}) \\
\downarrow & & \downarrow \\
\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})/\mathfrak{I}_\gamma & \xrightarrow{\ \psi\ } & \mathrm{H}_1(E_{\overline{\mathcal{C}^c_\gamma}})/\overline{\mathfrak{I}_\gamma}.
\end{array}$$

Since $\mathfrak{c}$ sends the meridian of each component C_i to the opposite of the meridian of $\overline{C_i}$, the map ψ is the opposite of the natural isomorphism ϕ between $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})/\mathfrak{I}_\gamma$ and $\mathrm{H}_1(E_{\overline{\mathcal{C}^c_\gamma}})/\overline{\mathfrak{I}_\gamma}$. In particular the meridians generating $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})/\mathfrak{I}_{\mathcal{C}_\gamma}$ are sent to their opposite, and the result follows. ☐

4. Some particular cases

We now focus on two particular types of cycles, namely those supported by rational curves, for which the linking set is a singleton, and those supported by a single component, for which Theorem 3.13 can be refined.

4.1. Rational curves, line arrangements and the $\mathcal{I}$-invariant.

Assume that the internal support of the cycle γ is a rational curve, so that any irreducible component has genus 0. Note that such a cycle has a projection $\sigma(\gamma)$ which is necessarily non contractible in the incidence graph. Using Case (1) of Proposition 3.11, the linking set $\{\gamma\}$ is the singleton $\{[\gamma]\}$, and we have the following version of Theorem 3.13.

Theorem 4.1. *Let $\mathcal{C}$ and $\mathcal{D}$ two curves with the same oriented and ordered topology. If γ and μ are combinatorially equivalent cycles in $\mathcal{C}$ and $\mathcal{D}$, respectively, whose internal supports are rational curves, then*

$$\phi(\{\gamma\}) = \{\mu\}\,,$$

where ϕ is the natural isomorphism between $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})/\mathfrak{I}_{\mathcal{C}_\gamma}$ and $\mathrm{H}_1(E_{\mathcal{D}^c_\mu})/\mathfrak{I}_\mu$.

Remark that the previous equality is equivalent to the equality $\phi([\gamma]) = [\mu]$ in $\mathrm{H}_1(E_{\mathcal{C}^c_\mu})/\mathfrak{I}_\mu$.

Remark 4.2. The above theorem is in particular true if the curves $\mathcal{C}$ and $\mathcal{D}$ themselves are rational, which contains the case of line arrangements.

Actually, in the particular case of line arrangements, this invariant is equivalent to the $\mathcal{I}$-invariant introduced in [4]. Let us recall some terminologies of [4] used to define the $\mathcal{I}$-invariant.

Definition 4.3. Let $\mathcal{A}$ be a line arrangement, $\xi : \mathrm{H}_1(E_{\mathcal{A}}) \to \mathbb{C}^*$ be a non-trivial character and σ be a cycle of the incidence graph $\Sigma_{\mathcal{A}}$. The triple $(\mathcal{A}, \xi, \sigma)$ is an inner-cyclic arrangement if

(1) $\xi(x_L) = 1$, for any L such that $v_L \in \sigma$ (where v_L is the vertex associated to L in $\Sigma_{\mathcal{A}}$),
(2) $\xi(x_L) = 1$, for any line L passing through a point P such that $v_P \in \sigma$ (where v_P is the vertex associated to P in $\Sigma_{\mathcal{A}}$),
(3) $\prod_{\ell \ni P} \xi(x_\ell)$, for any singular point P on a line L such that $v_L \in \sigma$ (where the product is taken over all the lines ℓ of $\mathcal{A}$ containing P).

Proposition 4.4. *Let $\mathcal{A}$ be a line arrangement. The following assertions are equivalent:*

(1) $\mathrm{H}_1(E_{\mathcal{A}^c_\gamma})/\mathfrak{I}_\gamma$ *is not trivial.*
(2) *There is a non trivial character ξ for which the triple $(\mathcal{A}, \xi, \Sigma(\gamma))$ is an inner-cyclic arrangement.*

Proof. Let the triple $(\mathcal{A}, \xi, \Sigma(\gamma))$ be an inner-cyclic arrangement. Due to the conditions (1) and (2) in Definition 4.3, the character ξ sends the meridian of the line of $\mathcal{A}_\gamma$ to 1. This means that we do not 'loose information' by considering the restriction of ξ to $\mathrm{H}_1(E_{\mathcal{A}^c_\gamma})$. Condition (3) in Definition 4.3 and the definition of the indeterminacy subgroup (Definition 3.6) imply that ξ induces a non trivial character ξ_* on the quotient $\mathrm{H}_1(E_{\mathcal{A}^c_\gamma})/\mathfrak{I}_\gamma$. Finally, the fact that there exists a non trivial character (namely ξ_*) on the quotient is equivalent to the fact that this quotient is not trivial. □

In [4], the $\mathcal{I}$-invariant is then defined as

$$\mathcal{I}(\mathcal{A}, \xi, \mu) = \xi \circ i_*(\tilde{\mu}),$$

where i_* is the map induced by the inclusion i of $B_{\mathcal{A}}$ (the boundary of a tubular neighbourhood $\mathrm{Tub}(\mathcal{A})$ of $\mathcal{A}$) in $E_{\mathcal{A}}$, and where $\tilde{\mu}$ is a suitably chosen lift of the cycle μ in $B_{\mathcal{A}}$. More precisely, this lift is a 'nearby cycle', in the terminology of [4], which roughly means that this cycle is contained in $B_{\mathcal{A}_\gamma} \setminus \mathrm{Tub}(\mathcal{A}^c_\gamma)$ (see [4, Def. 2.11] for a precise definition).

We can now state the equivalence of our linking invariant with the $\mathcal{I}$-invariant in the case of line arrangements.

Proposition 4.5. *Let $\mathcal{A}$ be a line arrangement and γ be a cycle of $\mathcal{A}$. If ξ_* is a character on $\mathrm{H}_1(E_{\mathcal{A}^c_\gamma})/\mathfrak{I}_\gamma$, then ξ_* induces a character (denoted by ξ) on $\mathrm{H}_1(E_{\mathcal{A}})$ such that $(\mathcal{A}, \xi, \Sigma(\gamma))$ is an inner-cyclic arrangement and we have*

$$\mathcal{I}(\mathcal{A}, \xi, \Sigma(\gamma)) = \xi_*(\{\gamma\}).$$

Proof. To construct ξ from ξ_* we use the reverse process of the proof of Proposition 4.4. Let $\tilde{\gamma}$ be a lift of $\Sigma(\gamma)$ in $B_{\mathcal{A}}$ which is a nearby cycle (see above). By construction, γ and $\tilde{\gamma}$ can be regarded as two cycles in $E_{\mathcal{A}^c_\gamma}$. The definition of an inner-cyclic arrangement implies that $\xi \circ i_*(\tilde{\gamma}) = \xi_{|\mathrm{H}_1(E_{\mathcal{A}^c_\gamma})}(\tilde{\gamma})$. But in $E_{\mathcal{A}^c_\gamma}$, the cycles $\tilde{\gamma}$ and γ are homotopically equivalent up to an element of $\mathfrak{I}_\gamma$. We conclude the proof using the fact that, by definition, ξ sends any element of $\mathfrak{I}_\gamma$ to 1. □

4.2. **Cycles supported by a single component.** We now assume that γ is a cycle in $\mathcal{C}$ such that $\overset{\circ}{\text{Supp}}(\gamma) = \{C\}$ is a single component. By Case (2) of Proposition 3.11, we have

$$\{\gamma\} = \left\{ \sum_{g\in\Gamma_C} a_g \cdot [g], \text{ with } (a_1, \cdots, a_k) \in (\mathbb{Z})^{|\Gamma_C|} \setminus \{(0,\cdots,0)\} \right\}. \tag{1}$$

We then have the following.

Theorem 4.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be two curves with the same ordered topology. If γ and μ are two combinatorially equivalent cycles of $\mathcal{C}$ and $\mathcal{D}$ respectively, then*

$$\phi(\{\gamma\}) = \{\mu\},$$

where ϕ is the natural isomorphism between $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})/\mathfrak{I}_\gamma$ and $\mathrm{H}_1(E_{\mathcal{D}^c_\mu})/\mathfrak{I}_\mu$.

This result is stronger than Theorem 3.13, not only because we have an equality for the linking sets, but also because we no longer assume that the two curves have same oriented topology. This latter simplification relies on the following lemma.

Lemma 4.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be two curves with different oriented and ordered topologies. If there is no order and orientation preserving homeomorphism between $(\mathbb{CP}^2, \mathcal{C})$ and $(\mathbb{CP}^2, \overline{\mathcal{D}})$, where $\overline{\mathcal{D}}$ is the complex conjugate curve of $\mathcal{D}$, then there is no order preserving homeomorphism between $(\mathbb{CP}^2, \mathcal{C})$ and $(\mathbb{CP}^2, \mathcal{D})$.*

Proof. This proof is based on a proof of Artal-Carmona-Cogolludo-Marco in [2], and follows from the following three facts.

Fact 1: There is no homeomorphism between $(\mathbb{CP}^2, \mathcal{C})$ and $(\mathbb{CP}^2, \mathcal{D})$ preserving the order and the orientation of $\mathbb{CP}^2$, and reversing the orientations of all the components.

Indeed, if such a homeomorphism exists, then we can compose it with the complex conjugation, and we then obtain a homeomorphism preserving both the order and the orientation between $(\mathbb{CP}^2, \mathcal{C})$ and $(\mathbb{CP}^2, \overline{\mathcal{D}})$. This is in contradiction with the hypotheses.

Fact 2: There is no homeomorphism between $(\mathbb{CP}^2, \mathcal{C})$ and $(\mathbb{CP}^2, \mathcal{D})$ preserving the order, the orientation of $\mathbb{CP}^2$ and the orientation of at least one component.

If $\mathcal{C}$ consists of a unique irreducible component, then the hypothesis of different oriented and ordered topologies between $\mathcal{C}$ and $\mathcal{D}$ is equivalent to Fact 2. Thus, we assume that $\mathcal{C}$ contains at least two irreducible components.

Suppose that such a homeomorphism ϕ exists. By Fact 1, there should be at least one component C_1 of $\mathcal{C}$ whose orientation is preserved by ϕ and by hypothesis, at least one component C_2 of $\mathcal{C}$ whose orientation is reversed. Since two components of a curve always intersects at least one point, we can pick one such point P and compute their local intersection number $(C_1 \cdot C_2)_P = k \in \mathbb{N}^*$ at P. Now, by hypothesis, we have that $(\phi(C_1) \cdot \phi(C_2))_{\phi(P)} = -k$, since only one of the components has its orientation reversed by ϕ. But since ϕ preserves the orientation of the plane, we also have $((\phi(C_1) \cdot \phi(C_2))_{\phi(P)} = k$, hence a contradiction. This proves that the orientation of C_2 cannot be reversed. The result then holds by connectivity of $\mathcal{C}$.

Fact 3: There is no homeomorphism between $(\mathbb{CP}^2, \mathcal{C})$ and $(\mathbb{CP}^2, \mathcal{D})$ preserving the order and reversing the orientation of the plane.

If such a homeomorphism exists, then it is an orientation reversing homeomorphism from $\mathbb{CP}^2$ to $\mathbb{CP}^2$. This is impossible, since the signature of the intersection form is non null. ☐

Proof of Theorem 4.6. By Proposition 3.15 and Case (2) of Proposition 3.11, we know that in the case of an internal support consisting of a single component, we have $\phi\{\gamma\} = \{\psi^*(\gamma)\}$, where

ψ^* is the map between $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})/\mathfrak{I}_\gamma$ and $\mathrm{H}_1(E_{\overline{\mathcal{C}^c_\gamma}})/\overline{\mathfrak{I}_\gamma}$ induced by complex conjugation. Applying Lemma 4.7 then yields the result. □

5. Computation of the linking set

In this section, we describe a practical method for the computation of our invariant, based on a modification of the braid monodromy and using the usual linking number of links in the 3-sphere.

Definition 5.1. A cycle γ in $\mathcal{C}$ is *admissible* if there is a generic projection $\pi : \mathbb{CP}^2 \setminus \{*\} \to \mathbb{CP}^1$ such that $\pi(\gamma)$ has no self-intersection, and $\pi^{-1}(\pi(\gamma)) \cap \mathrm{Sing}(\mathcal{C}^c_\gamma) = \emptyset$.

Note that the latter condition can always be fulfilled, up to a small modification of π.

Let γ be an admissible cycle in $\mathcal{C}$. For any point p of $\pi(\gamma)$, we consider the fiber F_p over p, and the intersection points of F_p with $\mathcal{C}$. Since γ may contain singular points in $\mathcal{C}_\gamma$, the number of intersection points may be lower than d above some points. We define L_γ as the (possibly singular) link formed by the intersection of $\cup_{p\in\gamma} F_p$ with $\mathcal{C}$. In other words

$$L_\gamma = \pi^{-1}(\pi(\gamma)) \cap \mathcal{C} \subset \mathbb{CP}^2.$$

Note that L_γ contains the initial cycle γ as a component. Note also that this link is defined as the closure of a (possibly singular) d-component braid, which is defined, as with the usual braid monodromy, by considering the configuration of points in the fibers above the loop $\pi(\gamma)$.

We also define L^c_γ as the (non singular) sublink of L_γ given by the components which are contained in $\mathcal{C}^c_\gamma$ (i.e the components which are not in the support of γ) and the cycle γ:

$$L^c_\gamma = \left(\pi^{-1}(\pi(\gamma)) \cap \mathcal{C}^c_\gamma\right) \cup \gamma \subset L_\gamma.$$

Now, this link L_γ naturally sits in a copy of S^3, as follows. Let $\mathcal{D}_\gamma$ be the disc bounded by $\pi(\gamma)$ in $\mathbb{CP}^1$. Pick a polydisc $\mathcal{P}$ of $\mathbb{CP}^2$ such that $\pi(\mathcal{P}) = D_\gamma$ and $\pi^{-1}(D_\gamma) \cap \mathcal{C} \subset \mathcal{P}$. By construction, the link L_γ lies in the boundary of $\mathcal{P}$, which is homeomorphic to S^3.

Recall that $\hat{\gamma}$ denote the class of γ in $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})$, and let ρ be the map from $\mathrm{H}_1(S^3 \setminus L^c_\gamma)$ to $\mathrm{H}_1(E_{\mathcal{C}^c_\gamma})$ sending the meridian of each component of L_{γ^c} to the meridian of its associated component in $\mathcal{C}^c_\gamma$. We denote by $\dot{\gamma}$ the class of γ in $\mathrm{H}_1(S^3 \setminus L^c_\gamma)$. As a consequence of the definitions, we have the following.

Fact 5.2. Let γ be an admissible cycle of $\mathcal{C}$ for the projection π, then

$$\hat{\gamma} = \rho(\dot{\gamma}),$$

This Fact provides a practical tool to compute $\hat{\gamma}$ in terms of the usual linking numbers of $\dot{\gamma}$ in the link L^c_γ.

Remark 5.3. The meridian of $\dot{\gamma}$ is the only one sent to 0 by ρ.

Example 5.4. Let us recall from Example 3.8 that, in the case of the curve $\mathcal{T}$ defined as the cubic $C : x^3 - xz^2 - y^2z = 0$ with its two real tangent lines T_1 and T_2, pictured in Figure 1, we have

$$\mathrm{H}_1(E_{\mathcal{T}^c_\gamma})/\mathfrak{I}_\gamma(\mathcal{T}) = \langle x_1, x_2 \mid 3x_1, 3x_2, x_1 + x_2\rangle$$

for a cycle γ contained in the cubic C. Figure 1 gives the curve $\mathcal{T}$ for $x \in (-1, 3)$. The computation of the linking set decomposes in three main steps.

Step 1 Pick a basis Γ_C for each $C \in \overset{\circ}{\mathrm{Supp}}(\gamma)$.

Let g_1 and g_2 be the two cycles of C in Figure 1, where g_1 is the real one. The orientation of the

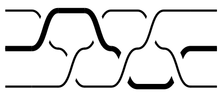

FIGURE 2. Braid defining $L^c_{g_2}$. The thick strand corresponds to the cycle g_2.

cycles is indicated in Figure 1. We have $g_1 = C \cap \{[x : y : z] \mid z \neq 0, x/z \in (-1, 0), y/z \in i\mathbb{R}\}$ and $g_2 = C \cap \{[x : y : z] \mid z \neq 0, x/z \in (0, 1), y/z \in \mathbb{R}\}$. Note that $\{g_1, g_2\}$ forms a basis of $\mathrm{H}_1(C)$.

STEP 2 Compute the image of Γ_C in $\mathrm{H}_1(E_{\mathcal{T}^c_\gamma})$.

For the projection $pr_1 : [x : y : z] \mapsto [x + y/2 : z]$, resp. $pr_2 : [x : y : z] \mapsto [x + iy/2 : z]$, the cycle g_1, resp. g_2, is admissible. Using the procedure described above, we have that the link $L^c_{g_1}$ is the closure of the trivial braid of $\mathbb{B}_3$, while the link $L^c_{g_2}$ is the closure of the braid $\sigma_1\sigma_2\sigma_1^{-1}\sigma_2\sigma_1\sigma_2^{-1} \in \mathbb{B}_3$ represented in Figure 2. The orientation on L_{g_i} is induced by that of the defining braids (oriented from left to right in the figures). This implies that

$$\dot{g_1} = 0 \in \mathrm{H}_1(S^3 \setminus L^c_{g_1}) \quad \text{and} \quad \dot{g_2} = -\alpha_1 + \alpha_3 \in \mathrm{H}_1(S^3 \setminus L^c_{g_2}),$$

where α_i denotes the meridian of the component s_i in $\mathrm{H}_1(S^3 \setminus L^c_\gamma)$ (see Figure 2). The map $\rho_2 : \mathrm{H}_1(S^3 \setminus L^c_{g_2}) \to \mathrm{H}_1(E_{\mathcal{T}^c_{g_2}})$ is defined by $(\alpha_1, \alpha_2, \alpha_3) \mapsto (x_2, 0, x_1)$. Since $\mathcal{T}^c_\gamma = \mathcal{T}^c_{g_i}$, we have that $\widehat{g_1} = 0$ and $\widehat{g_2} = x_1 - x_2$ in $\mathrm{H}_1(E_{\mathcal{T}^c_\gamma})$.

STEP 3 Determine the linking set of γ.

Since γ is any cycle in C we can consider that $\gamma = g_1$. To compute the linking set of γ, we use the Case (2) of Proposition 3.11, and we obtain:

$$\{\gamma\} = \{[0], [x_1 - x_2], [-x_1 + x_2]\}.$$

6. APPLICATION

In order to illustrate the strength of the linking invariant, we use it to distinguish a family of new Zariski pairs of curves.

6.1. **Definition of the curves.** Let C be the cubic of $\mathbb{CP}^2$ defined by

$$x^3 - xz^2 - y^2z = 0.$$

Since C is a smooth cubic, it admits 9 inflexion points. They are given by:

$$[x : y : z] \in \{[0 : 0 : 1], [\alpha : \beta : 1]\},$$

for α a root of $3X^4 - 6X^2 - 1 = 0$ and $\beta = \pm\sqrt{\alpha^3 - \alpha}$, and their approximate values are

$$\begin{array}{cc}
P_0 = [0 : 0 : 1] & P_1 \simeq [-1.468 : 1.302i : 1] \\
P_2 \simeq [-1.468 : -1.302i : 1] & P_3 \simeq [1.468 : 1.302 : 1] \\
P_4 \simeq [1.468 : -1.302 : 1] & P_5 \simeq [-0.393i : 0.477 + 0.477i : 1] \\
P_6 \simeq [0.393i : 0.477 - 0.477i : 1] & P_7 \simeq [-0.393i : -0.477 - 0.477i : 1] \\
P_8 \simeq [0.393i : -0.477 + 0.477i : 1] &
\end{array}$$

In the following, we will only consider the seven inflexion points $P_0, \cdots, P_6$.

Let L_i be the tangent to C at P_i. We set

$$C_7 = C \cup \Big(\bigcup_{k=0}^{6} L_k \Big).$$

The portion of the curve C_7 over the segment $x = (-1; 3)$ is pictured in Figure 3.

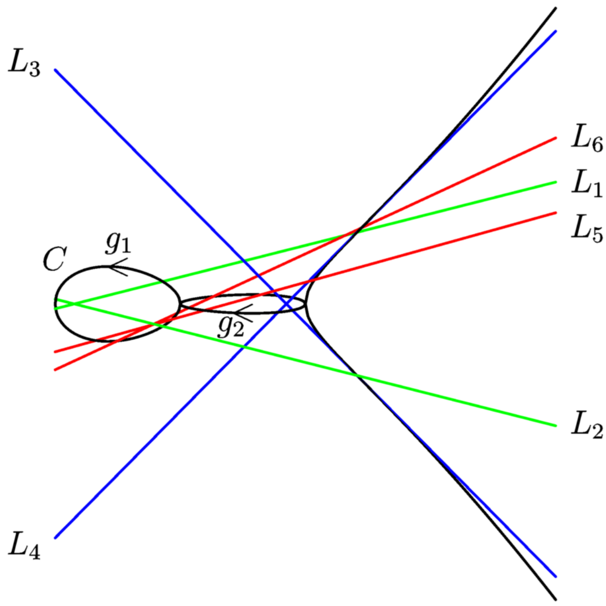

FIGURE 3. Representation of the curve C_7 over the segment $x = (-1; 3)$.

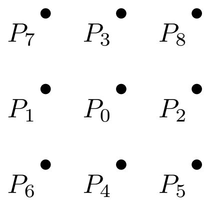

FIGURE 4. Representation of the nine inflexion points of C in $(\mathbb{F}_3)^2$.

Remark 6.1. The lines L_1 and L_2 are only passing through the real oval, the lines L_3 and L_4 are passing only through the complex oval (i.e. are two real tangent lines), and the lines L_5 and L_6 are passing through both the real and the complex ovals.

Notation. We denote by $\overline{\{i,j\}}$ the set $\{0,\cdots,6\} \setminus \{i,j\}$.

Definition 6.2. Let $C^{[i,j]}$ be the curve defined by

$$C^{[i,j]} = C \cup \Big(\bigcup_{k \in \overline{\{i,j\}}} L_k \Big) = C_7 \setminus \{L_i, L_j\} \,.$$

In the following, we will focus our interest on a particular couple of curve $C^{[i,j]}$, which is formed by the curves $C^{[5,6]}$ and $C^{[1,3]}$. This choice of pair is not arbitrary: indeed, these two curves have a property similar to the sextics of the historical example of Zariski [18], as we now explain.

It is well known that any triple of inflexion points of a smooth cubic are aligned (see [12]). This configuration of points can be represented as the points of $(\mathbb{F}_3)^2$: in the notation introduced at the beginning of this section, we have the configuration of Figure 4.

Let $\mathcal{Z}^{[i,j]}$ be the conic defined by the 5 singular points of $\mathcal{C}^{[i,j]}$. Geometrically, the curves $C^{[5,6]}$ and $C^{[1,3]}$ can be distinguished by the following property.

Proposition 6.3 (Zariski-like)**.** *The cardinality of the intersection $\mathcal{Z}^{[i,j]} \cap C$ has value 5 for $(i,j) = (5,6)$, and value 6 for $(i,j) = (1,3)$.*

Proof. We consider the cubic defined by the two lines (P_1, P_2) and (P_3, P_4). By the alignment property of inflexion points of smooth cubics, this cubic must also contains the point P_0 (see Figure 4), and thus is $\mathcal{Z}^{[5,6]}$. By a similar argument, the cubic $\mathcal{Z}^{[1,3]}$ is given by the lines (P_0, P_2) and (P_6, P_5). By Figure 4, it is clear that $P_1 \in \mathcal{Z}^{[1,3]} \cap C$, so that $\#\mathcal{Z}^{[1,3]} \cap C = 6$. In the case of $\mathcal{Z}^{[5,6]}$, we have that P_0 is an intersection point of multiplicity 2 between $\mathcal{Z}^{[5,6]}$ and C, so that $\#\mathcal{Z}^{[5,6]} \cap C = 5$. □

This geometric distinction between $C^{[5,6]}$ and $C^{[1,3]}$ is also true for the pairs $(C^{[5,6]}, C^{[1,4]})$, $(C^{[5,6]}, C^{[2,3]})$ or $(C^{[5,6]}, C^{[2,4]})$.

6.2. **Computation of the invariant.** Let γ be a (non trivial) cycle of C. Let us compute the linking set $\{\gamma\}$ in both $C^{[5,6]}$ and $C^{[1,3]}$, using the procedure illustrated in the examples of the previous sections.

Step 0: Compute the indeterminacy subgroup.
Since the local linking number of a tangent at an inflexion point of a cubic is 3, we have

$$\mathfrak{I}_\gamma(C^{[i,j]}) = \mathfrak{I}_\gamma^{[i,j]} = \langle 3x_k \mid k \in \overline{\{i,j\}} \rangle.$$

Furthermore, we have that $(C^{[i,j]})_\gamma^c$ is the line arrangement $\bigcup_{k \in \overline{\{i,j\}}} L_k$. We then have that $\mathrm{H}_1(E_{(C^{[i,j]})_\gamma^c})/\mathfrak{I}_\gamma^{[i,j]} \simeq (\mathbb{Z}_3)^5 / \langle \sum_{k \in \overline{\{i,j\}}} x_k \rangle$, and is generated by x_k for $k \in \overline{\{i,j\}}$.

Step 1: Pick a basis Γ_C.
Let g_1 and g_2 be the two cycles of C indicated in Figure 3, and which form a basis for $\mathrm{H}_1(C)$ (as in Example 5.4). By definition, we have that $(C^{[i,j]})_\gamma^c = (C^{[i,j]})_{g_k}^c$. (Note that the same applies to the curve C_7).

Step 2: Compute the image of Γ_C in $\mathrm{H}_1(E_{(C^{[i,j]})_\gamma^c})$.
In order to reduce the amount of computations, we first work in $\mathrm{H}_1(E_{(C_7)_\gamma^c})$, and then restrict to $\mathrm{H}_1(E_{(C^{[i,j]})_\gamma^c})$. We consider the projections pr_1 and pr_2 as in Example 5.4 such that g_1 and g_2 are admissible and we compute their associated links in C_7. The link $L_{g_1}^c$ is the closure of the braid Σ_1 pictured on the left of Figure 5, and defined by

$$\Sigma_1 = \sigma_1^{-1}\sigma_2\sigma_4\sigma_3\sigma_4^{-1}\sigma_4^{-1}\sigma_3\sigma_4\sigma_5\sigma_5\sigma_1\sigma_1\sigma_2\sigma_1^{-1} \in \mathbb{B}_8,$$

and $\rho_1 : \mathrm{H}_1(S^3 \setminus L_{g_1}^c) \to \mathrm{H}_1(E_{(C_7)_\gamma^c})$ is defined by $(\alpha_1, \cdots, \alpha_8) \mapsto (0, x_2, x_1, x_5, x_6, x_4, x_3, x_0)$, where α_i denotes the meridian of the component s_i in $\mathrm{H}_1(S^3 \setminus L_{g_1}^c)$ (see Figure 5). Similarly, $L_{g_2}^c$ is the closure of the braid Σ_2 pictured on the right of Figure 5 and defined by

$$\Sigma_2 = \sigma_1^{-1}\sigma_2\sigma_5\sigma_5\sigma_6^{-1}\sigma_3\sigma_1\sigma_3\sigma_4\sigma_3\sigma_3\sigma_5\sigma_5\sigma_6\sigma_4^{-1}\sigma_2^{-1}\sigma_1\sigma_2 \in \mathbb{B}_8,$$

and $\rho_2 : \mathrm{H}_1(S^3 \setminus L_{g_2}^c) \to \mathrm{H}_1(E_{(C_7)_\gamma^c})$ is defined by $(\alpha_1, \cdots, \alpha_8) \mapsto (0, x_3, x_4, x_5, x_6, x_2, x_1, x_0)$.

Computing the linking number of $\dot{\gamma}$ in the links $L_{g_i}^c$ and using the maps ρ_i, we obtain that the classes of g_1 and g_2 in $\mathrm{H}_1(E_{(C_7)_\gamma^c})$ are:

$$\widehat{g_1} = x_1 - x_2 - x_5 + x_6 \quad \text{and} \quad \widehat{g_2} = -x_3 + x_4 + x_5 + x_6.$$

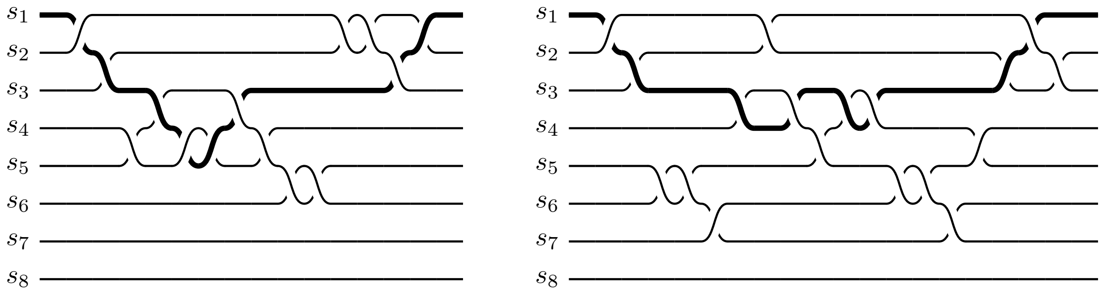

FIGURE 5. The braids Σ_1 and Σ_2. The thick strands correspond to the cycles g_1 and g_2, respectively.

Restricting the computation done in $\mathrm{H}_1(E_{(C_7)^c_\gamma})$ to $\mathrm{H}_1(E_{(C^{[5,6]})^c_\gamma})$ and $\mathrm{H}_1(E_{(C^{[1,3]})^c_\gamma})$, we obtain that:

$$\begin{array}{rclcrcl} \widehat{g_1}^{[5,6]} &=& x_1 - x_2 & \text{and} & \widehat{g_2}^{[5,6]} &=& -x_3 + x_4, \\ \widehat{g_1}^{[1,3]} &=& -x_2 - x_5 + x_6 & \text{and} & \widehat{g_2}^{[1,3]} &=& x_4 + x_5 + x_6, \end{array}$$

where the notation $\widehat{g_k}^{[i,j]}$ means that g_k is regarded in $\mathrm{H}_1(E_{(C^{[i,j]})^c_\gamma})$.

STEP 3: Determine the linking set of γ.
Since γ is any cycle in the cubic C, we can take $\gamma = g_1$. Then, using Case (2) of Proposition 3.11, we deduce that:

$$\left\{\gamma^{[5,6]}\right\} = \begin{Bmatrix} [x_1 - x_2], & [-x_1 + x_2], & [x_1 - x_2 - x_3 + x_4], & [-x_1 + x_2 + x_3 - x_4] \\ [x_3 - x_4], & [-x_3 + x_4], & [x_1 - x_2 + x_3 - x_4], & [-x_1 + x_2 - x_3 + x_4] \end{Bmatrix},$$

$$\left\{\gamma^{[1,3]}\right\} = \begin{Bmatrix} [x_4 + x_5 + x_6], & [-x_4 - x_5 - x_6], & [-x_2 - x_5 + x_6], & [-x_2 + x_4 - x_6] \\ [-x_2 - x_4 + x_5], & [x_2 + x_5 - x_6], & [x_2 + x_4 - x_5], & [x_2 - x_4 + x_6] \end{Bmatrix}.$$

Hence we have the following, as a direct consequence of Theorem 4.6.

Theorem 6.4. *There is no order preserving homeomorphism between the pair* $(\mathbb{CP}^2, C^{[5,6]})$ *and* $(\mathbb{CP}^2, C^{[1,3]})$*. In other words, the curves* $C^{[5,6]}$ *and* $C^{[1,3]}$ *form an ordered Zariski pair.*

Proof. The order on $C^{[5,6]}$ and $C^{[1,3]}$ is given by the lexicographic order on the indices of the lines. This implies that the natural isomorphism ϕ is given by $(x_0, x_1, x_2, x_3, x_4) \mapsto (x_0, x_2, x_4, x_5, x_6)$. It is now easy to verify that $\phi\left(\left\{\gamma^{[5,6]}\right\}\right) \cap \left\{\gamma^{[1,3]}\right\} = \emptyset$. $\square$

Remark 6.5. Recall that the elements of a linking set are only representatives of an equivalence class in a quotient, so proving the above disjointness also uses the computation of the indeterminacy subgroup in Step 0.

By a strictly similar argument, one can check that the couples $(C^{[5,6]}, C^{[1,4]})$, $(C^{[5,6]}, C^{[2,3]})$ and $(C^{[5,6]}, C^{[2,4]})$ are ordered Zariski pairs.

Remark 6.6. If we consider the following order on $C^{[1,3]}$: $L_4 < L_2 < L_0 < L_5 < L_6$, and the lexicographic one on $C^{[5,6]}$, then the natural isomorphism ϕ is defined by:

$$(x_0, x_1, x_2, x_3, x_4) \mapsto (x_4, x_2, x_0, x_5, x_6).$$

In this case, we have that $\phi([x_1 - x_2 - x_3 + x_4]) = [x_4 - x_2 - x_6]$ since $[x_1 - x_2 - x_3 + x_4] = [x_0 - x_1 - x_4]$ in $\mathrm{H}_1(E_{(C^{[5,6]})^c_\gamma})/\mathfrak{I}_\gamma$. Then $\phi\left(\left\{\gamma^{[5,6]}\right\}\right) \cap \left\{\gamma^{[1,3]}\right\} \neq \emptyset$.

6.3. **Deletion of the ordered condition.** Unfortunately, we can not delete the order hypothesis as we did for the orientation in Lemma 4.7. In order to derive from the above a genuine Zariski pair, we need to add an additional generic line D to the curves $C^{[5,6]}$ and $C^{[1,3]}$ passing through the inflexion point $P_0 = [0 : 0 : 1]$. We denote by $\widetilde{C}^{[i,j]}$ the curve $C^{[i,j]} \cup D$ obtained in this way. This additional line will allow us to identify L_0 combinatorially.

Proposition 6.7. *The group of automorphisms of the combinatorics of $\widetilde{C}^{[i,j]}$ is S_4.*

Proof. By construction, C is the only component of degree 3. It is thus fixed by the automorphisms of the combinatorics. By a similar combinatorial argument, we can ensure that L_0 and D are fixed. Thus only the lines L_k with $k \in \overline{\{i,j\}} \setminus \{0\}$ can be modified be an automorphism of the combinatorics. By construction, these 4 lines play similar roles in the combinatorics: they are all generic with the other lines and have only one intersection point with the cubic. Hence an automorphism of the combinatorics permutes these 4 lines, and the result holds. □

Theorem 6.8. *There is no homeomorphism between $(\mathbb{CP}^2, \widetilde{C}^{[5,6]})$ and $(\mathbb{CP}^2, \widetilde{C}^{[1,3]})$. In other words, $\widetilde{C}^{[5,6]}$ and $\widetilde{C}^{[1,3]}$ form a Zariski pair.*

Proof. By Theorem 6.4, there is no order preserving homeomorphism between $(\mathbb{CP}^2, \widetilde{C}^{[5,6]})$ and $(\mathbb{CP}^2, \widetilde{C}^{[1,3]})$. Assume that there exists one which does not preserve the order, and denote by $\sigma \in S_4$ the induced automorphism of the combinatorics. Let $\rho : \mathrm{H}_1(E_{(C^{[i,j]})^c_\gamma})/\mathfrak{I}_\gamma \to (\mathbb{Z}_3)^4$ defined by sending the class of an element to its representative having 0 as coordinate for x_0. Note that this application is well defined since L_0 is fixed by the automorphisms of the combinatorics. It is clear that $\sigma \cdot \rho(\{\gamma^{[5,6]}\}) \cap \rho(\{\gamma^{[1,3]}\}) = \emptyset$ (where σ acts by permutation on the coordinates of $(\mathbb{Z}_3)^4$), which is in contradiction with the hypothesis. □

Remark 6.9. Strictly similar methods allow us to define three Zariski pairs from the ordered pairs $(C^{[5,6]}, C^{[1,4]})$, $(C^{[5,6]}, C^{[2,3]})$ and $(C^{[5,6]}, C^{[2,4]})$.

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