



HAL
open science

Combining epistemological and cognitive approaches of geometry with cKc

Mithalal-Le Doze Joris

► **To cite this version:**

Mithalal-Le Doze Joris. Combining epistemological and cognitive approaches of geometry with cKc. CERME 9 - Ninth Congress of the European Society for Research in Mathematics Education, Charles University in Prague, Faculty of Education; ERME, Feb 2015, Prague, Czech Republic. pp.557-563. hal-01287014

HAL Id: hal-01287014

<https://hal.science/hal-01287014>

Submitted on 11 Mar 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Combining epistemological and cognitive approaches of geometry with cKç

Joris Mithalal-Le Doze

Université Paris 4, ESPE de Paris, Laboratoire de Didactique André Revuz, Paris, France, joris.mithalal@espe-paris.fr

In this article, we study the issues in making students use deductive geometry, both from epistemological and cognitive points of view. We show that working on geometrical objects needs at the same time working on diagrams in a specific way (instrumental and dimensional deconstruction), and that this can be trained by specific tasks. In this process, instrumental work is a crucial point of the work and, using the cKç model, we characterize two instrumental deconstruction that are involved in this evolution of the geometrical work.

Keywords: Dynamic 3D geometry, visualization, geometrical paradigm, dimensional deconstruction, cKç model.

INTRODUCTION

The issue of pupils using diagrams in order to solve deductive geometry problems is well known and has been studied a lot in this working group: they can read results so that mathematical proof is useless to them. We presented (Mithalal, 2009) preliminary results about didactical situations, based on Cabri 3D (a 3D Dynamic Geometry Environment), that aimed at making students use natural axiomatic geometry (Houdement & Kuzkiak, 2006). We shown (Mithalal, 2009, 2010) that such conditions led them to work on geometrical objects, and not only on diagrams. At first sight, making them use deductive reasoning could be considered as the major part of this evolution. In this article we will show that, in fact, instrumental activity should be considered as a pivot point that makes the whole process possible. We use Duval's (2005) framework, and show that instrumental deconstruction is the link between iconic and non-iconic visualization. We use the cKç model (Balacheff & Margolinas, 2003) to combine an epistemological and a cognitive understanding of geometrical activity into an operational tool for its precise analysis. We

show that two instrumental deconstructions are to be distinguished. On the one hand, there are only small differences between them, but on the other hand moving from the first to the second one is fundamental in this process. At the end, deductive geometry turns to be meaningful because of new conceptions about construction tasks.

USING DIAGRAMS FOR DEDUCTIVE GEOMETRY

Epistemological and cognitive points of view

Using diagrams is very ambiguous for pupils when learning deductive geometry: they are used to reading results on it, but suddenly this turns to be forbidden. Now, they have to use diagrams for heuristic work but not for proving, which is very confusing as they see what will take much time to be proved. Nevertheless, many authors, such as Parzysz (1988), Chaachoua (1997) or Jahnke (2007), have shown that that geometry involves a mix of practical activities and axiomatic reasoning, and that "inventing hypothesis and testing their consequences is more productive for the understanding of the epistemological proof than forming elaborate chain of deduction" (Jahnke, 2007, p. 79). Working with diagrams is, then, fundamental.

Laborde and Capponi (1994) made a distinction between drawings and figures: a drawing is a graphical object (hand-drawn shape, diagram, digital representation, manipulative...), while a figure is the matching between an ideal object and a set of drawings that represent it correctly. This emphasizes that a drawing can be considered for itself or as a representation of something, and extends Fischbein's (1993) theory of figural concepts. Considering that our question is now "how could we make pupils work on figures instead of drawings?", two kind of issues must be taken into account: epistemological and cognitive ones. We already gave a few words about epistemological issues, and we will use the well known framework of

geometrical paradigms (Houdement & Kuzniak, 2003, 2006) to express it: pupils have to move from natural geometry (GI) to natural axiomatic geometry (GII) which is about figures.

Cognitive point of view

A cognitive point of view is essential, as it explains why using drawings is unavoidable, and why this is so hard to perform. Chaachoua (1997, pp. 32–42) showed that drawings play fundamental roles in the geometrical activity – which not only includes the resolution process –, both for teachers and learners. In order for the resolution to be correctly performed, drawings must satisfy three conditions: (i) they have to display geometrical properties the text doesn't necessarily mention (ii) they may carry out an illustration function, to illustrate either the problem's statements, the resolution steps or the final solution (iii) they may carry out an experimentation function, so that the “geometer” can work on the drawing, which leads to perceive new sub-figures or relations, to make or evaluate conjectures, etc. Therefore, to be able to work with drawings is fundamental, and teaching geometry without using drawings would be a nonsense.

However, Duval (2005) showed that these operations are very hard to perform, because of visualization issues. He explains that there are two ways in which one sees a drawing: the iconic and the non-iconic visualization. **Iconic Visualization (I.V.)** means that the recognition of an object is due to the similarity of its shape with an already known object: if something looks like a square, it is one of those. A well-known consequence is the obstacle of typical configurations, which leads some pupils to identify a square as a rhombus, but not a square, because its position is the typical position of a rhombus (Figure 1). Another consequence is that you cannot modify or analyse the drawings: adding lines changes the shape, and then the nature of the objects decomposing the object into lines is impossible as nothing but the global shape is

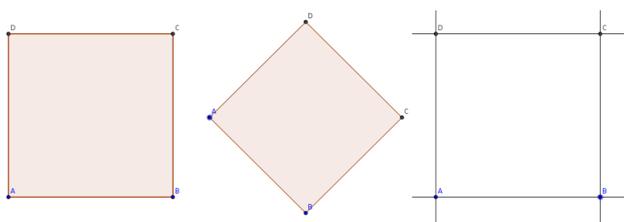


Figure 1: A square, a rhombus and “something from a square” (Iconic visualization)

perceived. This second consequence, more subtle, is fundamental for deductive geometry which involves experimental work and analytic study of the drawings.

With **Non-Iconic Visualization (NI.V.)**, one considers that a drawing is nothing but one of the representations of a geometrical object, and there is no contradiction between modifying the drawing and considering that it remains a representation of the same object. Moreover, the general shape is no longer a fundamental characteristic of the drawing, which results of assembling lower dimension components, the *figural units* (such as points, lines, segments, circles, etc.) Then, visualization is based on three operations: mereological deconstruction, instrumental deconstruction (one answer to the following question “How is it possible to construct this drawing with a given set of tools?”) and dimensional deconstruction (a discursive process in which a figure is seen as a set of figural units linked by geometrical properties, which leads to the geometrical object). Mathematical proof becomes meaningful to the pupils when they use non-iconic visualization and dimensional deconstruction. Nevertheless, Duval (2005, pp. 45–48) claimed that going from iconic to non-iconic visualization is neither easy nor natural. We showed (Mithalal, 2009, 2010) with space geometry, iconic visualization is no longer efficient. Construction tasks with 3D DGEs make 10th grade students use dimensional deconstruction. In this article, we assume this evolution is not a *revolution*, and we will study how continuous this process is. More precisely, we will show that instrumental deconstruction can be related both to iconic and non-iconic visualization, depending of tiny differences, and helps the pupils’ activity evolution.

GOING FROM ICONIC VISUALIZATION TO DIMENSIONAL DECONSTRUCTION: AN EXPERIMENTAL STUDY

The following case study aims at describing such a continuum and the role of instrumental deconstruction. We will use the results of an experiment more precisely exposed in Mithalal (2010), that consists of a knowledge diagnosis simple construction task.

Let $ABCDEHGF_1F_2F_3$ be a truncated cube (Figure 2), the pupils had to find as many ways as possible to construct the missing vertex, and to verify that the cube was still a cube when dragging point A to modify its size: the construction procedure had to be based,

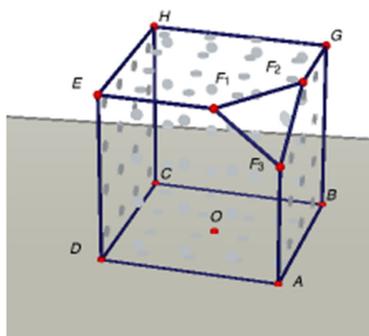


Figure 2: The truncated cube pupils had to complete

explicitly or implicitly, on geometrical properties so that the result was robust (Healy, 1994). They were asked to describe each method in a few words.

Interpreting the question with the various points of view we mentioned led to distinct aims and methods. For instance, with an “iconic” interpretation, the aim was to construct the missing part of the shape without considering geometrical properties, whereas a “non-iconic” interpretation should lead to use the cube properties (such as symmetry) to construct nothing but F. We identified (Mithalal, 2010) four distinct interpretations corresponding to specific resolution strategies: (1) to draw the missing part so that the result looks like a cube (iconic) (2) to draw the missing part so that the result looks like a cube and with the same length as the original (iconic) (3) to draw a point using a construction procedure that fits with a cube (instrumental deconstruction) (4) to construct a point linked to the truncated cube with its geometrical properties (dimensional deconstruction)

Instrumental deconstruction:

Connected both to I.V. and to N.I.V.

We propose now to illustrate the evolutions of two groups’ geometrical work, representative of usual evolutions: on the one hand Pierre and Ludovic, moving from iconic visualization to instrumental deconstruction, on the other hand Paul, Julie and Marie, whose interpretation of the task went from instrumental (3rd strategy) to dimensional deconstruction (4th strategy). Pierre and Ludovic’s aim was to construct the shape: “Why do you absolutely want to use ‘triangle?’” – “Because it is a triangle!” (5’43) They spent about 25 minutes trying to put a point at the right place, but this is almost impossible. They concluded so after multiple trials and viewpoint changes, and decided that they needed new methods:

16’06 Ludovic: Anyway, we can’t see anything.

21’57 Pierre: It looked good, but when we move it, it looks like nothing...

26’22 Pierre: We have to find another solution.

Their aim still was that “it looks good”, but they realised that the means for it had to be improved. Then, they tried various methods, including intersecting the three truncated edges (two would have been enough), constructing parallels, or trying to use vectors (they failed): they started using instrumental deconstruction. Very often, pupils interpreted robustness properties as mechanical ones, but did not connect it to geometrical properties. This is why Duval (2005) considered there was no link between the two deconstructions. However Cabri 3D’s mechanical tools are based on geometry, so even “mechanical” problems make dimensional deconstruction very useful for anticipating the strategies and their validities, and for convincing each other. This happened with Marie, Paul and Julie. They first decided that “reconstructing the cube” was not sufficient and that they had to fill it in: it was a shape problem, and they used similar methods to the former group.

13’19 Paul: Maybe we should try with a point, just a point here, at the right place.

13’24 Marie: But in this case you leave it to chance!

13’27 Paul: No! I mean, yes! Well, you make it...

Their problem was to be sure that the construction was good, and they eventually decided that it was “approximately good”, which was enough for them. This is why they decided to use more sophisticated methods based on dimensional deconstructions of a cube, such as edges intersections or symmetry, but during half an hour the correctness was determined by the shape (iconic) and the robustness (instrumental). But the fact that a new method was *really new* turned to be a dimensional deconstruction problem. They previously constructed F by intersecting 3 edges (Figure 3, left). Then, they tried another solution, constructing the symmetrical of the truncated cube, and using it to draw lines (Figure 3, 2nd & 3rd pictures).

25’46 Julie: Look, you construct the same next to it. You’ll see, the shape will do

schlack, schlack, you see it will be a rhombus. Then, you use from this one to that one. To its symmetrical. Then, you only have to do this... Then you create the line, and you're done.

31'50 Marie: Wait, wait, because I just thought about it: the [second] cube is absolutely useless...

32'14 Paul: Yes, in fact you just did the same!

32'15 Marie: Exactly. That's it, you just added a cube on the two sides.

The result was visually very different, and so was the construction process, but they used the same lines: this statement made them decide this was not a new method, no matter they constructed it differently, because they used *the same dimensional deconstruction of the cube*. This shows how the two deconstructions can be strongly linked: the dimensional one is a good way for designing instrumental deconstruction and for controlling its validity, and reciprocally the instrumental problems make dimensional deconstruction (and deductive geometry) play a greater and greater role.

The fundamental role of Instrumental Deconstruction

The example we mentioned here show the two main reasons for the pupils to move from iconic to non-iconic visualization. First, they needed to act more easily and to make vision more efficient. Then, they needed to better control and anticipate their actions, so that dimensional deconstruction was required to control instrumental processes. This points out that, in fact there are two kinds of instrumental deconstruction. The first one (we call it $I.D._{iv}$) is linked to iconic visualization problems, and a second one ($I.D._{niv}$) is a way for dimensional deconstruction to be operative. More than these two processes, moving from $I.D._{iv}$ to $I.D._{niv}$ is fundamental because it strongly modifies the way

problems are interpreted, becoming more theoretical, which makes deductive geometry meaningful. Our aim is now to confirm this statement. External signs of these two instrumental deconstructions are usually very similar, which make it hard to analyse. Same processes might be used with different purposes, and a similar aim may generate different strategies, depending on the pupil's knowledge, cognitive abilities, and interpretation. To get a more precise description of the deconstructions, we used the cKç model that perfectly fits the duality between acting and controlling the actions.

ANALYSING INSTRUMENTAL DECONSTRUCTION WITH CKç

The cKç model

cKç is a knowledge model (Balacheff, 2011), linked to the Theory of Conceptual Fields (Vergnaud, 1990) and to the Theory of Didactical Situations (Brousseau, 1997). Mathematical knowledge is characterized by the problem it solves. It is both determined by a subject and the milieu that generates the problem, so this is a [subject<>milieu] system balance, with an action/feedback loop: the subject acts, and the feedback from the milieu has to be good. This is very local, deeply linked to a specific context, and Balacheff (2011) calls it *conception*. Then, *knowledge* is a set of one subject's conceptions, and *concept* is a more general set of "knowledges". The social, more general, textual construct is called *savoir*.

Conceptions are what pupils work with during problem solving, and the cKç model describes it as a collection of 4 sets. **P** is a set of problems (p). A problem is basically a disturbance of the system balance, and the conception may solve these problems. **R** is a set of operators (r_i), that turn a problem into another problem belonging to **P**. It causes action, and this is the most directly visible part of the conception. **L** is a representation system for **P** and **R** expression. Σ is a control structure that ensures the conception is coherent and judges whether an operator has to be used

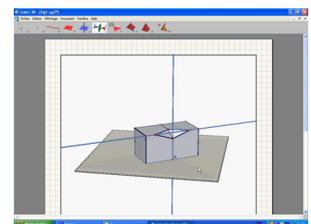
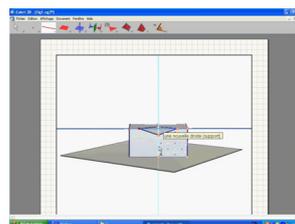
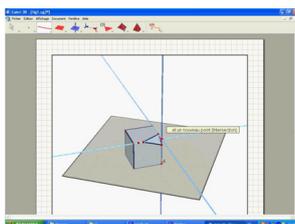


Figure 3: Construction with three lines or with the symmetrical of the cube

or not, whether a problem is solved or not... Σ is a set of controls (σ). As we can see, this is a very formal description, and it is essential to be conscious that it does not give any theoretical explanation for what happens, but it methodologically helps describing observations in a way that is supposed to be precise enough. In our case, L is always the same – drawings, oral language – and the possibilities of initial problems of a conception are well identified. The main evolution during the experiment is about R, Σ , and the relation between it.

Some nuances in very similar examples

Let us go back to the former examples. Pierre and Ludovic, Marie, Paul and Julie, used exactly the same construction process: construct the lines from the truncated edges, make the intersection, fill in the cube so that “it looks like a cube”. From a mathematical point of view, the third line was useless, so that we consider that all of them tried to draw the original shape. Pierre and Ludovic “[had] to find another solution” (26’22) and reproduced what another group was doing (*We’ll do the same as they do!*, 27’25), but the shape was so important that they also constructed the three “missing” triangles. Marie, Paul and Julie tried this strategy at the beginning:

- 3’11 Paul: (Marie created a line on a truncated edge.) Tell me what you’re doing!
- 3’20 Marie: You perfectly see what I’m doing.
- 3’22 Paul: Yes, you put a line, but explain us what you want to do.
- 3’28 Paul: (Marie: created the two other lines.) Ok, and then?
- 3’30 Marie: And then, you think I’m stupid! Don’t you see the cube is done?

Marie knew what she was doing and that the solution was right, because she used the properties of the cube to control, a priori, the construction process. This

helps understand why she didn’t need visual control to validate the construction, and at the end little differences are very meaningful.

In other words, the two procedures and the results were very similar, but our interpretations of it are different: Marie, Paul & Julie considered that all was about the shape, but they could use geometrical knowledge to control their actions. Pierre & Ludovic failed when geometrical controls were unavoidable. The great difference was about controlling, which leads to the following conclusion: the first group used a I.D._{iv} interpretation, whereas the second one interpreted the task with I.D._{niv}, so that geometrical knowledge and mechanical properties were connected.

Operators and controls

Analysing these examples, we didn’t fully describe the students’ conceptions. Would it be possible, this paper is too small. We used cKç to point out the main aspects of their conceptions, but this is more accurate when studying operators and controls.

An operator is an association between a purpose (I want to do this) and a mean for it ([aim \Rightarrow action]). For instance, an operator could be “I want to construct a parallel to (d), so I have to use the “parallel tool” of Cabri 3D to select (d) and a point”. Nor the aim, neither the action, reflect the kind of geometry used, the interpretation has to be based on their association (\Rightarrow). Let us consider another frequently used operator, “I want to construct a parallel to (d), so I have to use the line tool, select two points to create a line (d’), and move one of the points so that (d) and (d’) are parallel”. The purpose is the same, but the action shows that neither robustness nor geometrical properties are taken into account, and that “are parallel” mean something very perceptual, so that this operator is associated more to iconic than to non-iconic visualization. A control isn’t directly linked to an action, it is a judgment on it, and then it is a [Statement1 \Rightarrow Statement2] system. The first one is a fact, the second one is a conclusion, and here again the association (\Rightarrow) mean something about the kind of geometry used.

	Pierre and Ludovic	Marie, Paul and Julie
Interpretation, final control	Shape (iconical)	
Operators	Three lines, 3 triangles	Three lines
Control of actions	Visual-instrumental	geometrical

Table 1: Two interpretations of the problems

One of the greatest issues with defining operators and controls is the not so clear frontier between them. Balacheff (2003) mention that, for instance, “the symmetrical of a segment is another segment” could be either an operator – that indicates how to draw it – or a control – to judge the correctness of a construction. Our formal description helps understanding this: a single property may be linked to an operator or a control, because it mainly justifies the (\Rightarrow) link in both cases, but it is a tool in one case – linked to an action – and a fact, an object, in the other one. Operators are easier to catch, and also give information about their visualization. For instance, constructing three lines was useless, so this combination of operators (we call it procedure) indicated an iconic interpretation of the problem.

The two main ways to catch controls are dialog analysis and inferences from the observed procedures: an observation is associated to a procedure, linked to a set of controls and a type of conception. It is all the more important that the differences we shown between the two former instrumental deconstructions were mainly based on some specific controls, and on their role, but not on operators. Indeed, controlling the final validity with some dimensional deconstruction-based control mean something very different than using this deconstruction only as a tool to control the action. This is why we need to take into account the moment when the controls are used.

1. Before acting, controls determine the interpretation of the problems, a set of possibly adequate operators, i.e. a selection function (Vadcard, 2000), and a priori reasons for the validity of a strategy.

2. During the action, controls ensure that operators are adequate and help interpreting the feedback.
3. After acting, controls provide a posteriori judgment about the validity of a solution.

(2) is close to a “tool” role for using operators, and then for the kind of geometry they are based on. (1) and (3) are more about how geometry problems are interpreted and about the pupils’ knowledge. Of course, using deductive geometry requires that (1) and (3) mainly involve controls based on dimensional deconstruction, but identifying (2) controls is very important. It gives information about the coherence of the conception, about knowledge that are used only for practical reasons, and most of all about the evolution mechanism of the conception.

Characterization of two instrumental deconstructions

With this description, it is now possible to make the difference more clear. The two instrumental deconstructions both consist in considering the objects as the result of a construction process that involves figural units and some specifications about “how it works”. But the kind of objects involved, their properties, the tools pupils use and the controls that are needed, can be different. Of course, these are prototypes.

Pierre and Ludovic gave us a very good example of I.D._{iv} we already analysed. Leelah and Catherine’s deconstruction was very different. At the very beginning they tried approximate shape constructions, but they gave up immediately and repeated many times that the shape didn’t matter (6’06, Catherine:

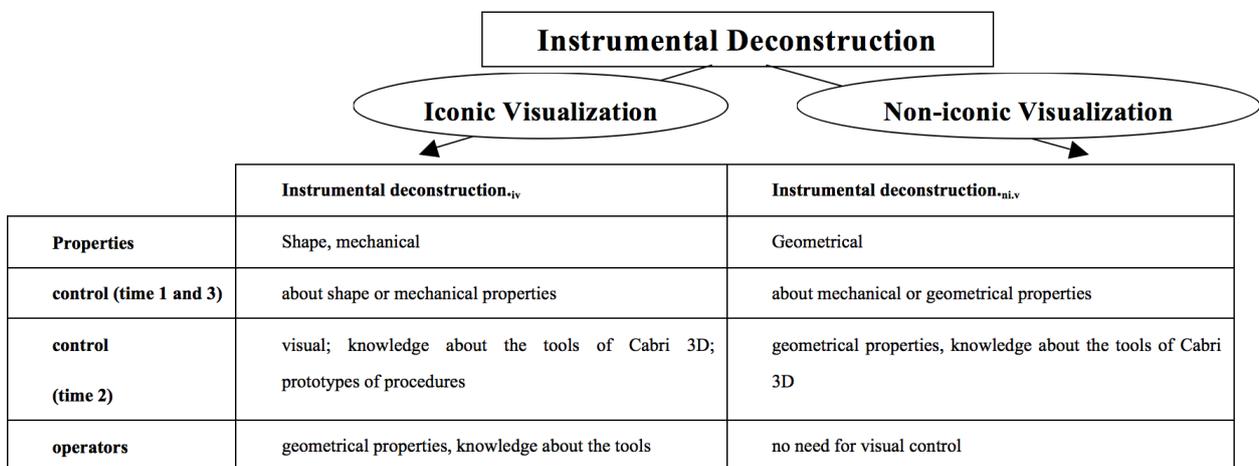


Figure 4: The two instrumental deconstructions

Yes, but no, the point can be anywhere.; 17'44, C.: But you don't need to create the tetrahedrons to get the points, the intersection of the [planes] already made it.; 32'44, C.: No, but the aim is not the numerical stuff, for us...). Contrary to the other groups we mentioned, they anticipated and validated every strategy before the action:

6'06 Leelah: I'd like to put a plane like this, a plane like this, a plane like this. [...] By extending the truncated sides...

22'51 Catherine: Later I thought of a much easier solution! Symmetry with respect to this point: hop, done!

Then, a priori controls based on dimensional deconstruction and deductive geometry allowed these pupils to validate a strategy before action, and to design precisely their procedure by transforming the geometrical information into an operational process. One great consequence is that visual control is no longer required, as Catherine expressed it very clearly: "Middle of this segment... We can't see it, but it's done!" (21'59) Eventually, we can describe the differences this way:

CONCLUSION

Many studies about teaching geometry underline how difficult it is to make pupils use axiomatic geometry and proof, and consider that there is no continuity of the cognitive process (Duval, 2005). We showed here that in particular cases, such as construction tasks with 3D DGEs, a continuous evolution is possible. This is linked to a more general statement: there are in fact two instrumental deconstructions, which are at the same time very different cognitive process, but very close ways of acting. Eventually, characterising the differences needs a very precise tool combining epistemological and cognitive points of view, which was provided by cKç. We only could underline the similarities between $I.D._{iv}$ and $I.D._{niv}$, and the evolution process between it remains to be studied.

REFERENCES

Balacheff, N. (2011). *cKç, un modèle pour relier connaissance et preuve en didactique des mathématiques*.

- Chaachoua, H. (1997). *Fonctions du dessin dans l'enseignement de la géométrie dans l'espace*. PhD thesis, Université Joseph Fourier, Grenoble.
- Duval, R. (1999). Representation, vision and visualization: cognitive functions in mathematical thinking. Basic issues for learning. In *Proceedings of the Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, number 21*. Cuernavaca, Morelos, Mexico.
- Duval, R. (2005). Les conditions cognitives de l'apprentissage de la géométrie. *Annales de Didactique et de Sciences Cognitives*, 10, 5–53.
- Healy, L., Hoelzl, R., Hoyles, C., & Noss, R. (1994). Messing up. *Micromath*, 10(1), 14–17.
- Houdement, C., & Kuzniak, A. (2003). Elementary geometry split into different geometrical paradigms. In A. M. Marriotii (Ed.), *Proceedings of CERME3*. Bellaria, Italy: ERME. See: http://www.mathematik.uni-dortmund.de/~erme/CERME3/Groups/TG7/TG7_list.php.
- Houdement, C., & Kuzniak, A. (2006). Paradigmes géométriques et enseignement de la géométrie. *Annales de Didactique et de Sciences Cognitives*, 11, 175–193.
- Laborde, C. (2000). Dynamic geometry as a source of rich learning contexts for the complex activity of proving. *Educational Studies in Mathematics*, 44, 151–161.
- Mithalal, J. (2009). 3D geometry and learning of mathematical reasoning. In V. Durand-Guerrier, S. Soury-Lavergne, & F. Arzarello (Eds.), *Proceedings of CERME6* (pp. 796–805). Lyon, France: INRP.
- Mithalal, J. (2010). *Déconstruction instrumentale et déconstruction dimensionnelle dans le contexte de la géométrie dynamique tridimensionnelle*. PhD thesis, Université de Grenoble.
- Parzys, B. (1988). "Knowing" vs "seeing", problems of the plane representation of space geometry figures. *Educational Studies in Mathematics*, 19, 79–92.