SEMI-ALGEBRAIC GEOMETRY WITH RATIONAL CONTINUOUS FUNCTIONS

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Abstract. Let X be a real algebraic subset of $\mathbb{R}^n$. We investigate on the theory of algebraically constructible functions on X and the description of the semi-algebraic subsets of X when we replace the polynomial functions on X by some rational continuous functions on X.

1. Introduction

The concept of rational continuous maps between smooth real algebraic sets was used the first time by W. Kucharz [10] in order to approximate continuous maps into spheres. In [15], rational continuous functions on smooth real algebraic sets are renamed by "regulous functions" and their systematic study is performed. A theory of vector bundles using these functions is done in [11].

J. Kollár, K. Nowak [9, Prop. 8] and G. Fichou, J. Huisman, F. Mangolte, the author [15, Thm. 4.1] proved independently that the restriction of a regulous function to a real algebraic subset is still rational. It allows us to define the concept of regulous function on a possibly singular affine real algebraic set X by restriction from the ambiant space. On X, we have two classes of functions: rational continuous functions and regulous functions. In cite [9] and [12], they give conditions for a rational continuous function to be regulous. In the second section of the present paper we present some preliminaries and we continue the study of differences between these two classes of functions.

In classical real algebraic geometry, we copy what happens in the complex case, and so we use as sheaf of functions on a real algebraic variety the sheaf of regular functions. Unfortunately and contrary to the complex case, some defects appear: classic Nullstellensatz and theorems A and B of Cartan are no longer valid [4]. In [15] we show that the use of the sheaf of regulous functions instead of the sheaf of regular functions corrects these defects. In this paper, and from the third section, we do the same thing but now in the semi-algebraic framework, we introduce a regulous semi-algebraic geometry i.e a semi-algebraic geometry with regulous functions replacing regular functions (remark that a regulous function is semi-algebraic). The aim of [15] was to study the zero sets of regulous functions, our purpose here is to investigate on their signs.

The third section deals with the theory of algebraically constructible functions, due to C. McCrory and A. Parusiński [17]. This theory has been developed to study singular real algebraic sets. We prove that the theory of algebraically constructible functions can be done using only regulous objects (functions, maps, sets). In particular, we show that the sign of a regulous function is a sum of signs of polynomial functions and we investigate on the number of polynomial functions needed in such representation. This is connected to the work of I. Bonnard in [5] and [6].

In the last sections, we focus on the description of principal semi-algebraic sets when we replace polynomial functions by regulous functions and rational continuous functions.

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2. Regulous functions versus rational continuous functions

2.1. **Regulous functions.** Let $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty\}$, we recall the definition of k-regulous functions on $\mathbb{R}^n$ (see [15]).

Definition 2.1. We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is k-regulous on $\mathbb{R}^n$ if f is C^k on $\mathbb{R}^n$ and f is a rational function on $\mathbb{R}^n$, i.e. there exists a non-empty Zariski open subset $U \subseteq \mathbb{R}^n$ such that $f|_U$ is regular.

A 0-regulous function on $\mathbb{R}^n$ is simply called a regulous function on $\mathbb{R}^n$.

An equivalent definition of a k-regulous function on $\mathbb{R}^n$ is given in [16, Thm. 2.15].

We denote by $\mathcal{R}^k(\mathbb{R}^n)$ the ring of k-regulous functions on $\mathbb{R}^n$. By Theorem 3.3 of [15] we know that $\mathcal{R}^\infty(\mathbb{R}^n)$ coincides with the ring $\mathcal{O}(\mathbb{R}^n)$ of regular functions on $\mathbb{R}^n$.

Denote by $\mathcal{Z}(f)$ the zero set of the real function f. For an integer k, the k-regulous topology of $\mathbb{R}^n$ is defined to be the topology whose closed subsets are generated by the zero sets of regulous functions in $\mathcal{R}^k(\mathbb{R}^n)$. Although the k'-regulous topology is a priori finer than the k-regulous topology when $k' < k$, it has been proved in [15] that in fact they are the same. Hence, it is not necessary to specify the integer k to define the regulous topology on $\mathbb{R}^n$. By [15, Thm. 6.4], the regulous topology on $\mathbb{R}^n$ is the algebraically constructible topology on $\mathbb{R}^n$ (denoted by $\mathcal{C}$-topology). On $\mathbb{R}^n$, the euclidean topology is finer than the $\mathcal{AR}$-topology (the arc-symetrical topology introduced by K. Kurdyka [13]) which is finer than the regulous topology (see [15]) which is the $\mathcal{C}$-topology which is finer than the Zariski topology.

We give now the definition of a regulous function on an affine real algebraic variety [15, Cor. 5.38].

Definition 2.2. Let X be a real algebraic subset of $\mathbb{R}^n$. A k-regulous function on X is the restriction to X of a k-regulous function on $\mathbb{R}^n$. The ring of k-regulous functions on X, denoted by $\mathcal{R}^k(X)$, corresponds to

$$\mathcal{R}^k(X) = \mathcal{R}^k(\mathbb{R}^n)/\mathcal{I}_k(X)$$

where $\mathcal{I}_k(X)$ is the ideal of $\mathcal{R}^k(\mathbb{R}^n)$ of k-regulous functions on $\mathbb{R}^n$ that vanish identically on X.

Remark 2.3. In [15] the previous definition is extended to the case X is a closed regulous subset of $\mathbb{R}^n$.

Let $X \subset \mathbb{R}^n$ be a real algebraic set, we will denote by $\mathcal{O}(X)$ the ring of regular functions on X, by $\mathcal{P}(X)$ the ring of polynomial functions on X and by $\mathcal{K}(X)$ the ring of rational functions on X. By [9, Prop. 8] or [15, Thm. 4.1], a regulous function on X is always rational on X (coincides with a regular function on a dense Zariski open subset of X). Since the regulous topology on X is sometimes strictly finer than the Zariski topology on X, the ring $\mathcal{R}^0(X)$ is not always a subring of $\mathcal{K}(X)$ even if X is Zariski irreducible.

Example 2.4. Let X be the plane cubic with an isolated point $X = \mathcal{Z}(x^2 + y^2 - x^3)$. The curve X is Zariski irreducible but $\mathcal{C}$-reducible. The $\mathcal{C}$-irreducible components of X are F and $\{(0,0)\}$ where $F = \mathcal{Z}(f) \subset \mathbb{R}^2$, with $f = 1 - \dfrac{x^3}{x^2 + y^2}$ extended continuously at the origin, is the smooth branch of X. The ring $\mathcal{R}^0(X)$ is the cartesian product $\mathcal{R}^0(F) \times \mathbb{R}$ and the class of f in $\mathcal{R}^0(X)$ is $(0,1)$. Remark that the ring $\mathcal{R}^0(X)$ is not an integral domain and consequently it is not a subring of $\mathcal{K}(X)$.

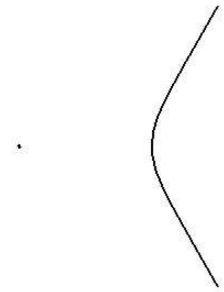

FIGURE 1. Cubic curve with an isolated point.

Let X be a real algebraic subset of $\mathbb{R}^n$. Let $f \in \mathcal{K}(X)$ and let U be a dense Zariski open subset of X, we say that the couple $(U, f|_U)$ or the function $f|_U$ is a regular presentation of f if $f|_U$ is regular. We have a natural ring morphism $\phi^0 : \mathcal{R}^0(X) \to \mathcal{K}(X)$ which send $f \in \mathcal{R}^0(X)$ to the class $(U, f|_U)$ in $\mathcal{K}(X)$, where $(U, f|_U)$ is a regular presentation of f. We have seen that ϕ^0 is not always injective.

Definition 2.5. Let X be a real algebraic subset of $\mathbb{R}^n$. Let $f \in \mathcal{K}(X)$. We say that the rational function f can be extended continuously to X if there exists a regular presentation $f|_U$ of f that can be extended continuously to X.

In the following, we will denote by $\overline{E}^{\tau}$ the closure of the subset E of $\mathbb{R}^n$ for the topology τ on $\mathbb{R}^n$. We prove now that ϕ^0 is injective if and only if $\overline{X_{reg}}^{\mathcal{C}} = X$, X_{reg} denoting the smooth locus of X. The condition $\overline{X_{reg}}^{\mathcal{C}} = X$ means that the Zariski irreducible components of X are also irreducible for the $\mathcal{C}$-topology (see [15]).

Lemma 2.6. *Let X be a real algebraic subset of $\mathbb{R}^n$. Let U be a dense Zariski open subset of X. Then $X_{reg} \subset \overline{U}^{eucl}$.*

Proof. Without loss of generality we can assume X is irreducible. Let Z denote the Zariski closed set $X \setminus U$. Assume $x \in X_{reg} \setminus \overline{U}^{eucl}$. So there exists an open semi-algebraic subset U' of X such that $x \in U'$ and $U' \subset X \setminus \overline{U}^{eucl} \subset Z$. Hence $\dim U' \leq \dim Z < \dim X$, this is impossible by [4, Prop. 7.6.2]. □

Proposition 2.7. *Let X be a real algebraic subset of $\mathbb{R}^n$. The map $\phi^0 : \mathcal{R}^0(X) \to \mathcal{K}(X)$ is injective if and only if $\overline{X_{reg}}^{\mathcal{C}} = X$.*

Proof. Assume $\overline{X_{reg}}^{\mathcal{C}} = X$. Let $f_1, f_2 \in \mathcal{R}^0(X)$ be such that $\phi^0(f_1) = \phi^0(f_2)$. Let $\hat{f}_i \in \mathcal{R}^0(\mathbb{R}^n)$, $i = 1, 2$, be such that $\hat{f}_i|_X = f_i$. Since f_1 and f_2 are two continuous extensions to X of the same rational function on X, they coincide on X_{reg} by Lemma 2.6. Hence $\hat{f}_1 - \hat{f}_2$ vanishes on X since X is the regulous closure of X_{reg}. It implies that $f_1 = f_2$.

Assume $\overline{X_{reg}}^{\mathcal{C}} \neq X$. By [15, Thm. 6.13], we may write $X = \overline{X_{reg}}^{\mathcal{C}} \cup F$ with F a non-empty regulous closed subset of $\mathbb{R}^n$ such that $\dim F < \dim X$. Let $\hat{f} \in \mathcal{R}^0(\mathbb{R}^n)$ be such that $\mathcal{Z}(\hat{f}) = \overline{X_{reg}}^{\mathcal{C}}$ and let f denote the restriction of $\hat{f}$ to X. We have $f \neq 0$ in $\mathcal{R}^0(X)$, $\phi^0(f) = 0$ in $\mathcal{K}(X)$ and thus ϕ^0 is non injective. □

2.2. **Rational continuous functions on central real algebraic sets.** Let n be a positive integer and let $X \subset \mathbb{R}^n$ be a real algebraic set. Let $f \in \mathcal{K}(X)$ be a rational function on X. The domain of f, denoted by $\operatorname{dom}(f)$, is the biggest dense Zariski open subset of X on which f is regular, namely $f = \dfrac{p}{q}$ on $\operatorname{dom}(f)$ where p and q are polynomial functions on $\mathbb{R}^n$ such that $\mathcal{Z}(q) = X \setminus \operatorname{dom}(f)$. The indeterminacy locus or polar locus of f is defined to be the Zariski closed set $\operatorname{indet}(f) = X \setminus \operatorname{dom}(f)$.

Definition 2.8. Let f be a real continuous function on X. We say that f is a rational continuous function on X if f is rational on X i.e there exists a dense Zariski open subset $U \subseteq X$ such that $f|_U$ is regular.

Remark 2.9. We may also define a rational continuous function as a continuous extension of a rational function.

Let $\mathcal{R}_0(X)$ denote the ring of rational continuous functions on X. We have a natural ring morphism $\phi_0 : \mathcal{R}_0(X) \to \mathcal{K}(X)$ which send $f \in \mathcal{R}_0(X)$ to the class $(U, f|_U)$ in $\mathcal{K}(X)$, where $(U, f|_U)$ is a regular presentation of f.

Remark 2.10. We have $\mathcal{R}_0(\mathbb{R}^n) = \mathcal{R}^0(\mathbb{R}^n)$.

Definition 2.11. We say that X is "central" if $\overline{X_{reg}}^{eucl} = X$.

Remark 2.12. The previous definition comes from the introduction of the the central locus of a real algebraic set made in [4, Def. 7.6.3].

The property to be central is the property of an algebraic set that ensures a rational continuous function on it to be the unique possible continuous extension of its associated rational function. The following example illustrates this fact.

Example 2.13. Let $X = \mathcal{Z}(zx^2 - y^2) \subset \mathbb{R}^3$ be the Whitney umbrella. By [15], X is irreducible in

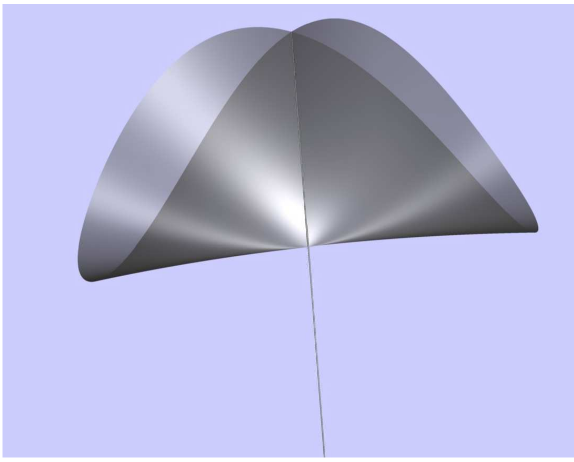

Figure 2. Whitney umbrella.

the $\mathcal{C}$-topology and we have

$$\overline{X_{reg}}^{\mathcal{A}\mathrm{R}} = \overline{X_{reg}}^{\mathcal{C}} = \overline{X_{reg}}^{Zar} = X.$$

The set $X \setminus \overline{X_{reg}}^{eucl}$ is the half of the stick. The function $\frac{y^2}{x^2}|_X$ is regular on X outside of the stick and so it gives rise of a rational function on X. Its class in $\mathcal{K}(X)$ is also the class of the regular function $z|_X$ ($(X \setminus \mathcal{Z}(x^2+y^2), \frac{y^2}{x^2}|_{X\setminus\mathcal{Z}(x^2+y^2)})$ and $(X, z|_X)$ are two regular presentations of the same rational function). This rational function can be extended continuously in different ways to X: we can extend the regular presentation $(X \setminus \mathcal{Z}(x^2+y^2), \frac{y^2}{x^2}|_{X\setminus\mathcal{Z}(x^2+y^2)})$ by z on $X \cap \mathcal{Z}(x^2+y^2)$ (we get the regular function $z|_X$ on X) but we can also extend it by z on $X \cap \mathcal{Z}(x^2+y^2) \cap \{z \geq 0\}$ and by 0 on $X \setminus \overline{X_{reg}}^{eucl} = X \cap \mathcal{Z}(x^2+y^2) \cap \{z < 0\}$. The first extension gives a regulous function on X and the second one only gives a rational continuous function on X. Consequently, the map $\phi_0 : \mathcal{R}_0(X) \to \mathcal{K}(X)$ is not injective.

Proposition 2.14. *The map $\phi_0 : \mathcal{R}_0(X) \to \mathcal{K}(X)$ is injective if and only if X is central.*

Proof. Under the hypothesis $X = \overline{X_{reg}}^{eucl}$, it follows from Lemma 2.6 that if a rational function of $\mathcal{K}(X)$ has a continuous extension to X then this extension is the unique possible extension.

Assume X is not central. It is always possible to extend the null function on $\overline{X_{reg}}^{eucl}$ to a continuous function f on X such that f is not the null function on X. The function f is rational on X since it has a regular presentation on X_{reg}. Obviously, f is a non-trivial element of the kernel of ϕ_0 and the proof is done. □

In the following, to simplify notation, we sometimes identify a rational continuous function on a central real algebraic set with one of its regular presentations (e.g. $\dfrac{x^3}{x^2+y^2} \in \mathcal{R}^0(\mathbb{R}^2)$). By [9, Prop. 8] or [15, Thm. 4.1], any $f \in \mathcal{R}^0(X)$ can be identified with a unique function in $\mathcal{R}_0(X)$. Hence we get:

Proposition 2.15. *We have the following ring inclusion $\phi_0^0 : \mathcal{R}^0(X) \hookrightarrow \mathcal{R}_0(X)$ and moreover*
$$\phi^0 = \phi_0 \circ \phi_0^0.$$

Remark 2.16. Let X be a real algebraic subset of $\mathbb{R}^n$ such that $\overline{X_{reg}}^{\mathcal{C}} = X$ and X is not central (e.g the Whitney umbrella). By Propositions 2.15 and 2.14, we see that in this case the map ϕ_0^0 is not surjective i.e there is a rational continuous function on X which is not regulous.

In the following example, due to Kollár and Nowak [9, Ex. 2], we will see that, even if X is central, ϕ_0^0 may be not surjective.

Example 2.17. Let $X = \mathcal{Z}(x^3 - (1+z^2)y^3) \subset \mathbb{R}^3$. Then X is a central singular surface with singular locus the z-axis. By [9, Ex. 2], the class of the rational fraction $\dfrac{x}{y}|_X$ in $\mathcal{K}(X)$ can be extended continuously to X (in a unique way) by the function $(1+z^2)^{\frac{1}{3}}$ on the z-axis and gives an element $f \in \mathcal{R}_0(X)$. Moreover, f cannot be extended to an element of $\mathcal{R}_0(\mathbb{R}^3) = \mathcal{R}^0(\mathbb{R}^3)$ (the reason is that the restriction of f to the z-axis $(1+z^2)^{\frac{1}{3}}$ is not rational) and thus f is not in $\mathcal{R}^0(X)$. Here the map $\phi_0^0 : \mathcal{R}^0(X) \hookrightarrow \mathcal{R}_0(X)$ is not surjective and the map $\phi_0 : \mathcal{R}_0(X) \to \mathcal{K}(X)$ is injective.

One of the goal of the paper [9] was to study the surjectivity of the map ϕ_0^0 when X is a central real algebraic set. Notice that "regulous functions" are named "hereditarily rational continuous functions" in [9].

We reformulate with our notation the three principal results of [9] with an improvement of the first one.

The following lemma can be obtained from the arguments used in the proof of [9, Prop. 11].

Lemma 2.18. *(proof of* [9, Prop. 11]*)*
Let $X \subset \mathbb{R}^n$ be a real algebraic set and let $f \in \mathcal{R}_0(X)$. Let $W = \mathrm{indet}(f)$ be the polar locus of f in X. If $f|_W \in \mathcal{R}^0(W)$ has the additional property that $f|_W$ is the restriction to W of $g \in \mathcal{R}^0(\mathbb{R}^n)$ such that g is regular on $\mathbb{R}^n \setminus W$ then
$$f \in \mathcal{R}^0(X).$$
Moreover, f has also the additional property that there exists $\hat{f} \in \mathcal{R}^0(\mathbb{R}^n)$ such that $\hat{f}$ is regular on $\mathbb{R}^n \setminus W$ and $\hat{f}|_X = f$.

We improve Lemma 2.18 by removing the additional property from the hypotheses.

Lemma 2.19. *Let $X \subset \mathbb{R}^n$ be a real algebraic set and let $f \in \mathcal{R}_0(X)$. Let $W = \mathrm{indet}(f)$ be the polar locus of f in X. If $f|_W \in \mathcal{R}^0(W)$ then*
$$f \in \mathcal{R}^0(X).$$

Proof. Assume $f|_W \in \mathcal{R}^0(W)$. By definition, there exists $g \in \mathcal{R}^0(\mathbb{R}^n)$ such that $g|_W = f|_W$. We denote by g_0 the regulous function $g|_W$. We consider the following sequence of regulous functions

$$(g_0, g_1 = (g_0)|_{\mathrm{indet}(g_0)}, g_2 = (g_1)|_{\mathrm{indet}(g_1)}, \ldots)$$

on a sequence of Zariski closed subsets $(W_i = \mathrm{indet}(g_{i-1}))$ of W of dimension strictly decreasing and included one in another. The functions g_i are regulous since they are also a restriction of a regulous function on $\mathbb{R}^n$. We claim that there exists an integer m such that g_m is a regular function on W_m. Indeed, g_m is automatically regular if $\dim W_m = 0$ and we get the claim since $\dim W_{i+1} < \dim W_i$. By [4, Prop. 3.2.3], g_m is the restriction to $W_m = \mathrm{indet}(g_{m-1})$ of regular function $\hat{g}_m$ on $\mathbb{R}^n$. By Lemma 2.18 for $f = g_{m-1}$, $X = W_{m-1}$ and $W = W_m$, we get that g_{m-1} is the restriction to W_{m-1} of a regulous function $\hat{g}_{m-1}$ on $\mathbb{R}^n$ regular on $\mathbb{R}^n \setminus \mathrm{indet}(g_{m-1})$. Repeated application of Lemma 2.18 enables us to see that $g_0 = f|_W$ is the restriction to W of a regulous function $\hat{g}_0$ on $\mathbb{R}^n$ regular on $\mathbb{R}^n \setminus \mathrm{indet}(f|_W)$. Since $\mathrm{indet}(f|_W) \subset \mathrm{indet}(f) = W$, using one last time Lemma 2.18, we get the proof. $\square$

Proposition 2.20. *([9, Prop. 8])*
Let $X \subset \mathbb{R}^n$ be a real algebraic set and let $f \in \mathcal{R}_0(X)$. For any irreducible real algebraic subset $W \subset X$ not contained in the singular locus of X, we have

$$f|_W \in \mathcal{R}_0(W).$$

Theorem 2.21. *([9, Prop. 8, Thm. 10])*
Let $X \subset \mathbb{R}^n$ be a smooth real algebraic set. Then the map $\phi_0^0 : \mathcal{R}^0(X) \hookrightarrow \mathcal{R}_0(X)$ is an isomorphism.

Proof. By [9, Prop. 8], a rational continuous function on a smooth real algebraic set is hereditarily rational. By [9, Thm. 10], a continuous hereditarily rational function on a non necessary smooth real algebraic set $X \subset \mathbb{R}^n$ is the restriction of rational continuous function on $\mathbb{R}^n$ and thus "continuous hereditarily rational" means "regulous". $\square$

We extend the result of Theorem 2.21 to real algebraic sets with isolated singularities using Lemma 2.19.

Theorem 2.22. *Let $X \subset \mathbb{R}^n$ be a real algebraic set with only isolated singularities. Then*

$$\mathcal{R}^0(X) = \mathcal{R}_0(X).$$

Proof. Let $f \in \mathcal{R}_0(X)$. Let $W \subset X$ be a real algebraic subset. If $\dim W = 0$ then $f|_W$ is regular and thus $f|_W \in \mathcal{R}_0(W)$. If W is irreducible and $\dim W \geq 1$ then $f|_W \in \mathcal{R}_0(W)$ by Proposition 2.20. It follows that $f|_W \in \mathcal{R}_0(W)$ without hypothesis on W. We consider the following sequence of continuous rational functions

$$(f_0 = f, f_1 = f|_{\mathrm{indet}(f)}, f_2 = (f_1)|_{\mathrm{indet}(f_1)}, \ldots)$$

on a sequence of real algebraic subsets $(W_i = \mathrm{indet}(f_{i-1}))$ of X of dimension strictly decreasing and included one in another. There exists an integer m such that f_m is regular on W_m. Using several times Lemma 2.19, we get that $f \in \mathcal{R}^0(X)$. $\square$

Corollary 2.23. *Let $X \subset \mathbb{R}^n$ be a real algebraic curve. Then*

$$\mathcal{R}^0(X) = \mathcal{R}_0(X).$$

2.3. **Blow-regular functions on central real algebraic sets.** By [15, thm. 3.11], regulous functions on a smooth real algebraic set $X \subset \mathbb{R}^n$ coincide with blow-regular functions on X, it gives another equivalent definition for regulous functions on X.

Definition 2.24. Let $X \subset \mathbb{R}^n$ be a smooth real algebraic set. Let $f : X \to \mathbb{R}$ be a real function. We say that f is regular after blowings-up on X or f is blow-regular on X if there exists a composition $\pi : M \to X$ of successive blowings-up along smooth centers such that $f \circ \pi$ is regular on M. We denote by $\mathcal{B}(X)$ the ring of blow-regular functions of X.

Theorem 2.25. [15, thm. 3.11]
Let $X \subset \mathbb{R}^n$ be a smooth real algebraic set. We have $\mathcal{R}^0(X) = \mathcal{B}(X)$.

Now we will give a definition of blow-regular function for a non-necessarily smooth real algebraic set.

Definition 2.26. Let $X \subset \mathbb{R}^n$ be a real algebraic set. Let $\mathcal{B}(X)$ denote the ring of real functions f defined on X such that, there exists a resolution of singularities $\pi : \tilde{X} \to X$ such that the composite $f \circ \pi$ is in $\mathcal{B}(\tilde{X}) = \mathcal{R}^0(\tilde{X}) = \mathcal{R}_0(\tilde{X})$. A $f \in \mathcal{B}(X)$ is called a "blow-regular function" on X.

Remark 2.27. According to the definition of blow-regular function on a smooth variety we get: $f \in \mathcal{B}(X)$ if and only if f is a real function defined on X such that there exists a resolution of singularities $\pi : \tilde{X} \to X$ such that $f \circ \pi$ is regular. This justifies the notation "blow-regular".

Remark 2.28. In the definition 2.26 we can change $\exists$ by $\forall$. It is not true in the equivalent definition of the remark 2.27.

We prove in the following that, even in the central case, blow-regular functions and rational continuous functions coincide.

Proposition 2.29. *Let $X \subset \mathbb{R}^n$ be a central real algebraic set. We have*

$$\mathcal{B}(X) = \mathcal{R}_0(X).$$

Proof. Assume $f \in \mathcal{R}_0(X)$ and let $\pi : \tilde{X} \to X$ be a resolution of singularities. Then clearly $f \circ \pi$ is rational on $\tilde{X}$. Since $\pi^{-1}(X) = \tilde{X}$ the we can conclude that $f \circ \pi$ is continuous on $\tilde{X}$ and thus $f \circ \pi \in \mathcal{R}_0(\tilde{X})$.

Assume $f \in \mathcal{B}(X)$ and let $\pi : \tilde{X} \to X$ be a resolution of singularities. Then $f \circ \pi \in \mathcal{R}_0(\tilde{X})$ and thus f is rational on X. The function f is continuous on X since:
• The fibres of π are non-empty i.e π is surjective. Indeed if $\pi^{-1}(x) = \emptyset$ for a $x \in X$ then $\dim X_x < \dim X$ ($\dim X_x$ is the local dimension of X at x [4, Def. 2.8.12]) and thus $x \notin \overline{X_{reg}}^{eucl}$ by [4, Prop. 7.6.2], this contradicts our assumption that X is central. The surjectivity of π can also be deduced from [13, Thm. 2.6, Cor. 2.7].
• The function $f \circ \pi$ is continuous on $\tilde{X}$.
• The function $f \circ \pi$ is constant on the fibers of π.
In fact, the "central" condition forces the strong topology on X to be the quotient topology induced by the strong topology on $\tilde{X}$. □

The next example illustrates the fact that the assumption that X is central cannot be dropped in the previous proposition. In general we only have $\mathcal{R}_0(X) \subset \mathcal{B}(X)$.

Example 2.30. We consider the real algebraic surface introduced in [15, Ex. 6.10]. Let X be the algebraic subset of $\mathbb{R}^4$ defined by $X = \mathcal{Z}((x+2)(x+1)(x-1)(x-2)+y^2) \cap \mathcal{Z}(u^2 - xv^2)$. The set X has two connected components W and Z, W has dimension two and $W = \overline{X_{reg}}^{eucl}$, Z has dimension one and $Z = \mathcal{Z}(((x+2)(x+1)(x-1)(x-2)+y^2)^2 + u^2 + v^2) \cap \{(x,y,u,v) \in \mathbb{R}^4 \mid x < 0\}$. Let f be the real function defined on X by $f(x,y,u,v) = \begin{cases} \dfrac{1}{(x+1)^2+y^2+u^2+v^2} \text{ if } (x,y,u,v) \neq (-1,0,0,0) \\ 0 \text{ if } (x,y,u,v) = (-1,0,0,0) \end{cases}$.
Remark that f is not continuous at the point $(-1,0,0,0)$ and is regular on W. Let $\pi : \tilde{X} \to X$ be a resolution of singularities. Since $\pi^{-1}(Z) = \emptyset$ then $f \circ \pi$ will be regular on $\tilde{X}$ and thus $f \in \mathcal{B}(X) \setminus \mathcal{R}_0(X)$.

3. Algebraically constructible functions

We make reminders on the theory of constructible and algebraically constructible functions due to C. McCrory and A. Parusiński (see [17], [18]). This theory was used to study the topology of singular real algebraic sets. We follow the definitions and the results given in [8].

Let S be a semi-algebraic set. A constructible function on S is a function $f : S \to \mathbb{Z}$ that can be written as a finite sum
$$\varphi = \sum_{i\in I} m_i \,\mathbf{1}_{S_i}$$
where for each $i \in I$, m_i is an integer and $\mathbf{1}_{S_i}$ is the characteristic function of a semi-algebraic subset S_i of S. The set of constructible functions on S provided with the sum and the product form a commutative ring denoted by $\mathrm{F}(S)$. If $\varphi = \sum_{i\in I} m_i \,\mathbf{1}_{S_i}$ is a constructible function then the Euler integral of φ on S is
$$\int_S \varphi \, d\chi = \sum_{i\in I} m_i \chi(S_i)$$
where χ is the Euler characteristic with compact support. Let $f : S \to T$ be a continuous semi-algebraic map and $\varphi \in \mathrm{F}(S)$. The pushforward $f_*\varphi$ of φ along f is the function from T to $\mathbb{Z}$ defined by
$$f_*\varphi(y) = \int_{f^{-1}(y)} \varphi \, d\chi.$$
It is known that $f_*\varphi \in \mathrm{F}(T)$ and that $f_* : \mathrm{F}(S) \to \mathrm{F}(T)$ is a morphism of additive groups.

Let $X \subset \mathbb{R}^n$ be a real algebraic set. We say that a constructible function φ on X is algebraically constructible if it can be written as a finite sum
$$\varphi = \sum_{i\in I} m_i f_{i*}(\mathbf{1}_{X_i})$$
where f_i are regular maps from real algebraic sets X_i to X. Algebraically constructible functions on X form a subring, denoted by $\mathrm{A}(X)$, of $\mathrm{F}(X)$. We say that a constructible function φ on X is strongly algebraically constructible if it can be written as a finite sum
$$\varphi = \sum_{i\in I} m_i \,\mathbf{1}_{X_i}$$
where X_i are real algebraic subsets of X. Strongly algebraically constructible functions on X form a subring of $\mathrm{A}(X)$ denoted by $\mathrm{AS}(X)$.

Let A be a ring of semi-algebraic functions on X. For $f \in A$, we define the sign function associated to f as
$$\mathrm{sign}(f) : X \to \{-1, 0, 1\}$$
$$x \mapsto \mathrm{sign}(f)(x) = \begin{cases} -1 \text{ if } f(x) < 0 \\ 0 \text{ if } f(x) = 0 \\ 1 \text{ if } f(x) > 0 \end{cases}$$
Let $f \in A$, we have $\mathrm{sign}(f) \in \mathrm{AS}(X)$ since f is a semi-algebraic function (the inverse image of a semi-algebraic set by a semi-algebraic map is a semi-algebraic set [3, Prop. 2.2.7]). Following [1], we say that two n-tuples $< f_1, \ldots, f_n >$ and $< h_1, \ldots, h_n >$ of elements of A are equivalent, and we write $< f_1, \ldots, f_n > \simeq < h_1, \ldots, h_n >$, if
$$\forall x \in X,\ \mathrm{sign}(f_1(x)) + \cdots + \mathrm{sign}(f_n(x)) = \mathrm{sign}(h_1(x)) + \cdots + \mathrm{sign}(h_n(x)).$$
A (quadratic) form over A is an equivalence class of a n-tuple for this relation. If ρ is the class of the n-tuple $< f_1, \ldots, f_n >$, we simply write $\rho = < f_1, \ldots, f_n >$ and n is called the dimension of ρ and denoted by $\dim(\rho)$. For two forms $< f_1, \ldots, f_n >$ and $< g_1, \ldots, g_m >$ over A, we define the sum (denoted by $\perp$) and the product (denoted by $\otimes$):
$$< f_1, \ldots, f_n > \perp < g_1, \ldots, g_m > = < f_1, \ldots, f_n, g_1, \ldots, g_m >,$$
$$< f_1, \ldots, f_n > \otimes < g_1, \ldots, g_m > = < f_1g_1, \ldots, f_ng_1, f_1g_2, \ldots, f_ng_2, \ldots, f_ng_m > .$$

We call two forms $< f_1,\dots,f_n >$ and $< g_1,\dots,g_m >$ over A similar, and write

$$< f_1,\dots,f_n > \sim < g_1,\dots,g_m >,$$

if

$$\forall x\in X,\ \operatorname{sign}(f_1(x))+\cdots+\operatorname{sign}(f_n(x))=\operatorname{sign}(g_1(x))+\cdots+\operatorname{sign}(g_m(x)).$$

With the operations $\perp$ and $\otimes$, the set of similarity classes of forms is a ring called the reduced Witt ring of degenerate forms over A, we will denote it by $\mathrm{W}(A)$. The form ρ is called isotropic if there is a form τ with $\rho\sim\tau$ and $\dim(\rho)>\dim(\tau)$. Otherwise, ρ is called anisotropic. The form $< 0 >$ is considered isotropic.

Since A is a ring of semi-algebraic functions on X, we have a signature map

$$\Lambda : \mathrm{W}(A)\to F(X)$$
$$< f_1,\dots,f_n > \mapsto \operatorname{sign}(f_1)+\cdots+\operatorname{sign}(f_n)$$

which is a ring morphism. The signature map is clearly injective by definition of similarity for forms.

Parusiński and Szafraniec haved proved that algebraically constructible functions correspond to sums of signs of polynomial functions.

Theorem 3.1. [19, Thm. 6.1]
Let $X\subset\mathbb{R}^n$ be a real algebraic set. Then

$$\mathrm{A}(X)=\Lambda(\mathrm{W}(\mathcal{P}(X)))=\Lambda(\mathrm{W}(\mathcal{O}(X))).$$

We prove now that algebraically constructible functions correspond to sums of signs of regulous functions. It is a very natural result since the topology generated by zero sets of regulous functions is the algebraically constructible topology.

Theorem 3.2. *Let $X\subset\mathbb{R}^n$ be a real algebraic set. Then*

$$\mathrm{A}(X)=\Lambda(\mathrm{W}(\mathcal{R}^0(X))).$$

Proof. We proceed by induction on the dimension of X. If $\dim(X)=0$ then regulous means regular and the result follows from Theorem 3.1.

Assume $\dim(X)>0$ and let $f\in\mathcal{R}^0(X)$. Let W denote $\operatorname{indet}(f)$. There exist $p,q\in\mathcal{P}(X)$ such that $f=\dfrac{p}{q}$ on $\operatorname{dom}(f)$ and $\mathcal{Z}(q)=W$. Notice that $\Lambda(< f >)=\Lambda(< pq >)$ on $X\setminus W$. We have $f|_W\in\mathcal{R}^0(W)$ and by induction there exists $h_1,\dots,h_k\in\mathcal{P}(W)$ such that $\Lambda(< f|_W >)=\Lambda(< h_1,\dots,h_k >)$. The polynomial functions h_i are restrictions of polynomial functions on X still denoted by h_i [4, prop. 3.2.3]. The proof is done since

$$\Lambda(< f >)=\Lambda(< pq > \perp < 1,-q^2 > \otimes < h_1,\dots,h_k >)$$

on X. □

In the next section, we will count the number of signs of polynomial functions we need in the sum to be the sign of a regulous function.

We prove now that strongly algebraically constructible functions are exactly finite sums of characteristic functions of regulous closed sets.

Proposition 3.3. *Let $X\subset\mathbb{R}^n$ be a real algebraic set. Then*

$$\mathrm{AS}(X)=\{\sum_{i\in I} m_i\,\mathbf{1}_{W_i}\,,\ I\text{ finite},\ m_i\in\mathbb{Z},\ W_i\subset X\text{ regulous closed}\}.$$

Proof. Let W be a closed regulous subset of X. Let $f\in\mathcal{R}^0(X)$ be such that $\mathcal{Z}(f)=W$. By [15, Thm. 4.1], there exist a finite stratification $X=\coprod_{i\in I}W_i$ with W_i Zariski locally closed subsets of X such that $f|_{W_i}$ is regular $\forall i\in I$. Given $i\in I$, there exists $p_i,q_i\in\mathcal{P}(X)$ such that $\dfrac{p_i}{q_i}|_{W_i}=f|_{W_i}$ and

$\mathcal{Z}(q_i) \cap W_i = \emptyset$. Hence $S_i = W \cap W_i = \mathcal{Z}(p_i) \cap W_i$ is also Zariski locally closed. So there is a finite stratification $W = \coprod_{i\in I} S_i$ with S_i Zariski locally closed subsets of X. It means that $S_i = Z_i \cap (X \setminus Z_i')$ where Z_i and Z_i' are real algebraic subsets of X. Then

$$\mathbf{1}_W = \sum_{i\in I} \mathbf{1}_{S_i} = \sum_{i\in I} (\mathbf{1}_{Z_i}(\mathbf{1}_X - \mathbf{1}_{Z_i'})) = \sum_{i\in I} (\mathbf{1}_{Z_i} - \mathbf{1}_{Z_i \cap Z_i'}) \in \mathrm{AS}(X).$$

□

We characterize algebraically constructible functions using regulous closed sets and regulous maps.

Theorem 3.4. *Let $X \subset \mathbb{R}^n$ be a real algebraic set. Then*

$$\mathrm{A}(X) = \{ \sum_{i\in I} m_i f_{i*}(\mathbf{1}_{W_i}) \,,\, I \,\text{finite},\, m_i \in \mathbb{Z},\, W_i \,\text{regulous closed},\, f_i : W_i \to X \,\text{regulous map}\}.$$

Proof. By Proposition 3.3 and since f_* is additive, it is sufficient to prove that $f_*(\mathbf{1}_Y) \in \mathrm{A}(X)$ when $f : Y \to X$ is a regulous map between two real algebraic sets. We proceed by induction on the dimension of Y. If $\dim(Y) = 0$ then f is regular and there is nothing to prove. Assume $\dim(Y) > 0$. We may also assume that Y is irreducible. By [15, Thm. 3.11], there exists a regular birational map $\pi : \tilde{Y} \to Y$ such that $f \circ \pi$ is a regular map (solve the singularities of Y and then use [15, Thm. 3.11]). The birational map π is biregular from $\tilde{Y} \setminus \pi^{-1}(Z)$ to $Y \setminus Z$ with Z a real algebraic subset of Y of positive codimension. Then

$$f_*(\mathbf{1}_Y) = (f \circ \pi)_*(\mathbf{1}_{\tilde{Y}}) - (f \circ \pi)_*(\mathbf{1}_{\pi^{-1}(Z)}) + f_*(\mathbf{1}_Z)$$

and $f_*(\mathbf{1}_Z) \in \mathrm{A}(X)$ by the induction hypothesis. □

Now we look at sum of signs of rational continuous functions.

The proof of the next result is due to the author and G. Fichou.

Proposition 3.5. *Let $X \subset \mathbb{R}^n$ be a central real algebraic set. Let $f \in \mathcal{R}_0(X)$ be a rational continuous function on X. Then $\mathcal{Z}(f)$ is a closed regulous subset of X.*

Proof. Since X is central then the function f is semi-algebraic (its graph is the euclidean closure of the graph of any of its regular presentation). It follows that $\mathcal{Z}(f)$ is a semi-algebraic set. Denote by A (resp. V) the $\mathcal{C}$-closure (resp. Zariski closure) of $\mathcal{Z}(f)$ in X. We want to show that $A = \mathcal{Z}(f)$.

By [4, Prop. 2.8.2], we have $\dim \mathcal{Z}(f) = \dim A = \dim V = d$. By Proposition 2.29 there exists $\sigma : \tilde{X} \to X$ a resolution of the singularities of X such that $f \circ \sigma$ is regular on $\tilde{X}$. We can assume moreover, performing more blowings-up if necessary, that σ restricts to a resolution of the singularities of V. This is possible because V is included in X and σ is surjective on X (X is central). Since V is the Zariski closure of $\mathcal{Z}(f)$ then we have $\dim E \cap \mathcal{Z}(f) = \dim E$ for each Zariski irreducible component E of V (otherwise we replace E by $\overline{E \cap \mathcal{Z}(f)}^{Zar}$). According to the above remark and since $f \circ \sigma$ is regular on $\tilde{V}$ ($\tilde{V}$ is the strict transform of V by σ) then $f \circ \sigma$ vanishes identically on $\tilde{V}$. It follows that $V_{reg} \subset \mathcal{Z}(f)$ and thus $\overline{\mathrm{Reg}\, A}^{eucl} \subset \mathcal{Z}(f)$ where $\mathrm{Reg}\, A = \{x \in A : \dim_x A = d \,\text{and}\, A \text{ is smooth at } x\}$.

So the difference between $\mathcal{Z}(f)$ and its $\mathcal{C}$-closure A is of dimension strictly less than the dimension d of $\mathcal{Z}(f)$. Denote by C the $\mathcal{C}$-closure of that difference, so that $A = \mathcal{Z}(f) \cup C$, where C is a $\mathcal{C}$-set of dimension strictly less than d. Now, let D be one of the irreducible $\mathcal{C}$-components of C of maximal dimension (i.e. of dimension $\dim C$).

Note that $\dim D \cap \mathcal{Z}(f) = \dim D$. Otherwise one may replace in C the component D by the $\mathcal{C}$-closure of $D \cap \mathcal{Z}(f)$ which would be strictly smaller (notably in dimension by [14, Prop. 4.3]), in contradiction with the fact that $\dim D$ is maximal. Let us play the same game as previously: since $D \subset X$, there exists a resolution of the singularities $\sigma : \tilde{X} \to X$ of X such that $f \circ \sigma$ is regular on $\tilde{X}$, and σ restricts to a resolution of the singularities of the Zariski closure of D in X. By [13, Thm. 2.6, Cor. 2.7], there exists a smooth irreducible real algebraic subset $\tilde{D}$ in $\tilde{X}$ such that $\sigma(\tilde{D}) = \overline{\operatorname{Reg} D}^{eucl}$ ($\overline{\operatorname{Reg} D}^{eucl}$ is thus the union of the $\mathcal{A}$R-irreducible components of maximal dimension of D, and we have $D = \overline{\operatorname{Reg}(\overline{D}^{Zar})}^{\mathcal{C}}$ [15, Thm. 6.13]).

For the same reason as before, $f\circ\sigma$ vanishes on $\tilde{D}$ so f vanishes on $\overline{\operatorname{Reg} D}^{eucl}$, in contradiction with the fact that D is an irreducible $\mathcal{C}$-component of maximal dimension not included in $\mathcal{Z}(f)$. Therefore $C = \emptyset$ and the proof is achieved. □

Theorem 3.6. *Let $X \subset \mathbb{R}^n$ be a central real algebraic set. Then*

$$\mathrm{A}(X) = \Lambda(\mathrm{W}(\mathcal{R}_0(X))).$$

Proof. Let $f \in \mathcal{R}_0(X)$, we have to prove $\Lambda(< f >) \in \mathrm{A}(X)$. Let $Y = \{(x,t) \in X \times \mathbb{R} \,|\, f(x) = t^2\}$. By Proposition 3.5, $Y \in \mathcal{C}$ i.e Y is a closed regulous subset of the central real algebraic set $X \times \mathbb{R}$. By Proposition 3.3, $\mathbf{1}_Y \in \mathrm{AS}(X \times \mathbb{R})$. We get

$$\Lambda(< f >) = \pi_*(\mathbf{1}_Y) - \mathbf{1}_X \in \mathrm{A}(X)$$

where $\pi : X \times \mathbb{R} \to X$ is the projection. □

Example 3.7. We go back to Example 2.17. We have $f = \dfrac{x}{y} \in \mathcal{R}_0(X) \setminus \mathcal{R}^0(X)$ but $f^3 \in \mathcal{R}^0(X)$. We have $\Lambda(< f >) = \Lambda(< f^3 >) \in \mathrm{A}(X)$.

Remark 3.8. To conclude this section, we remark that it follows from above results that, if $X \subset \mathbb{R}^n$ is a real algebraic set, the following rings $\mathrm{W}(\mathcal{P}(X))$, $\mathrm{W}(\mathcal{O}(X))$, $\mathrm{W}(\mathcal{R}^0(X))$, $\mathrm{A}(X)$ are all isomorphic. If in addition X is central then we may add $\mathrm{W}(\mathcal{R}_0(X))$ to the previous list.

4. Signs of regulous functions (part 1)

Throughout this section X will denote a real algebraic subset of dimension d of $\mathbb{R}^n$. By Theorem 3.4, the sign of a regulous function on X can be written as a sum of signs of polynomial functions on X. The goal of this section is to bound in terms of d the number of polynomial functions needed in such representation. This is connected to the work of I. Bonnard in [5] and [6] that concerns the representation of general algebraically constructible functions as sums of signs of poynomial functions. However, the author cautions the reader that the results of this text concern specifically algebraically constructible functions that are signs of regulous functions and depend strongly of the nice properties verified by the regulous functions. It seems unlikely to be able to generalize this work to general algebraically constructible functions.

Notice that the zero-dimensional case is trivial since regulous means here regular.

Definition 4.1. Given $f \in \mathcal{R}^0(X)$, the number $\ell(f)$, called the length of the sign of f, will denote the smallest integer l such that $\operatorname{sign}(f)$ can be written as a sum of l signs of polynomial functions on X. So there is a form ρ over $\mathcal{P}(X)$ such that $\Lambda(\rho) = \Lambda(< f >)$ on X and $\dim(\rho) = \ell(f)$. It is clear that ρ is anisotropic and then it is unique. We denote by $\rho(f)$ the corresponding anisotropic form of dimension $\ell(f)$. The number $\ell(X)$ will denote the maximum of the $\ell(f)$ for $f \in \mathcal{R}^0(X)$.

Let $f_1, \dots, f_m$ be continuous semi-algebraic functions on X. In the sequel, we will use the following notations:

$$\mathcal{S}(f_1, \dots, f_m) = \{x \in X |\, f_1(x) > 0, \dots, f_m(x) > 0\}$$
$$\bar{\mathcal{S}}(f_1, \dots, f_m) = \{x \in X |\, f_1(x) \geq 0, \dots, f_m(x) \geq 0\}.$$

If all the functions f_i lie in a ring A of continuous semi-algebraic functions, the set $\mathcal{S}(f_1, \dots, f_m)$ (resp. $\bar{\mathcal{S}}(f_1, \dots, f_m)$) is called A-basic open (resp. A-basic closed). If $m = 1$, we replace "basic" by "principal". If $A = \mathcal{P}(X)$ then we omet A. If $A = \mathcal{R}^0(X)$, we will sometimes write "regulous basic" (resp. "regulous principal") instead of "$\mathcal{R}^0(X)$-basic" (resp. "$\mathcal{R}^0(X)$-principal").

In the following example, we prove that for curves the sign of a regulous function is not always the sign of a polynomial function.

Example 4.2. (Example 2.4)
Let $X=\mathcal{Z}(y^2-x^2(x-1))$ and let f be the restriction to X of the plane regulous function $1-\dfrac{x^3}{x^2+y^2}$. The function f is zero on the one-dimensional connected component of X and has value 1 on the isolated point of X. If a polynomial function g has the sign of f on the one-dimensional connected component of X then g vanishes on whole X since X is Zariski irreducible. However the sign of f is the sum of signs of two polynomial functions on X, more precisely we have $\rho(f)=<1,-(x^2+y^2)>$ and therefore $\ell(f)=2$.

The defect in the previous example is the fact that the curve is Zariski irreducible but $\mathcal{C}$-reducible i.e the fact that X is not central since $\dim(X)=1$ (see [15]). We will prove now that, under the hypothesis that the curve is central, the sign of a regulous function coincides with the sign of a polynomial function.

We will use several times the following lemma which is a consequence of Łojasiewicz inequality.

Lemma 4.3. [3, Lem. 7.7.10]
Let S be a closed semi-algebraic subset of X. Let $f,g\in\mathcal{P}(X)$. There exist $p,q\in\mathcal{P}(X)$ such that $p>0$ on X, $q\geq 0$ on X, $\Lambda(<pf+qg>)=\Lambda(<f>)$ on S and $\mathcal{Z}(q)=\overline{\mathcal{Z}(f)\cap S}^{Zar}$.

Proposition 4.4. *Assume* $\dim(X)=1$ *and X is central. Let $f\in\mathcal{R}^0(X)$. There exists $h\in\mathcal{P}(X)$ such that $\Lambda(<h>)=\Lambda(<f>)$ on X i.e*
$$\ell(f)=\ell(X)=1.$$

Proof. By [7], any open semi-algebraic subset of X is principal and thus there exists $p_1\in\mathcal{P}(X)$ such that $\mathcal{S}(f)=\mathcal{S}(p_1)$. Since X is central then the Zariski irreducible components of X correspond exactly to the $\mathcal{C}$-irreducible components of X and thus $\mathcal{Z}(f)$ is Zariski closed (see [15], a $\mathcal{C}$-irreducible component of dimension 1 of $\mathcal{Z}(f)$ is necessarily Zariski closed and also Zariski irreducible). So we can multiply p_1 by the square of a polynomial equation of $\mathcal{Z}(f)$ and we get:
$$\mathcal{S}(f)=\mathcal{S}(p_1)\text{ and }\mathcal{Z}(f)\subset\mathcal{Z}(p_1).$$
Similarly there exists $p_2\in\mathcal{P}(X)$ such that $\mathcal{S}(-f)=\mathcal{S}(-p_2)$. Let S denote the closed semi-algebraic set $\bar{\mathcal{S}}(f)$. By Lemma 4.3, there exist $p,q\in\mathcal{P}(X)$ such that $p>0$ on X, $q\geq 0$ on X, $\Lambda(<pp_1+qp_2>)=\Lambda(<p_1>)$ on S and $\mathcal{Z}(q)=\overline{\mathcal{Z}(p_1)\cap S}^{Zar}$. Let h denote the polynomial function pp_1+qp_2. We want to prove that $\Lambda(<h>)=\Lambda(<f>)$ on X. We have $\Lambda(<h>)=\Lambda(<p_1>)=\Lambda(<f>)$ on S since $\mathcal{S}(f)=\mathcal{S}(p_1)$ and since $\mathcal{Z}(f)\subset\mathcal{Z}(p_1)$. Assume now $x\notin S$. Notice that it is equivalent to suppose that $f(x)<0$. So $p_2(x)<0$ (since $\mathcal{S}(-f)=\mathcal{S}(-p_2)$), $p_1(x)\leq 0$ (since $\bar{\mathcal{S}}(-f)=\bar{\mathcal{S}}(-p_1)$). The proof is done if we prove that $q(x)>0$ since in that case we would have $h(x)<0$. We have $S\cap\mathcal{Z}(p_1)=\bar{\mathcal{S}}(f)\cap\mathcal{Z}(p_1)\subset\mathcal{Z}(f)\cap\mathcal{Z}(p_1)=\mathcal{Z}(f)$ since $\mathcal{S}(f)=\mathcal{S}(p_1)$ (you can not have simultaneously $f(y)>0$ and $p_1(y)=0$). We have already noticed that $\mathcal{Z}(f)$ is Zariski closed, hence
$$\mathcal{Z}(q)=\overline{\mathcal{Z}(p_1)\cap S}^{Zar}\subset\overline{\mathcal{Z}(f)}^{Zar}=\mathcal{Z}(f)$$
and it follows that $x\notin\mathcal{Z}(q)$. $\square$

Remark 4.5. Proposition 4.4 still holds if we replace "X is central" by "$\mathcal{Z}(f)$ is Zariski closed" in the assumptions. Look at Theorem 6.1 for a generalization of Proposition 4.4 in any dimension and in the case $\mathcal{Z}(f)$ is Zariski closed.

Example 4.6. Let $X=\mathcal{Z}(x^2-y^3)\subset\mathbb{R}^2$ be the cuspidal curve and let $f=\dfrac{y^2}{x}|_X$. We have $f\in\mathcal{R}^0(X)\setminus\mathcal{P}(X)$ but $\Lambda(<f>)=\Lambda(<x>)$ on X.

Proposition 4.7. *Assume* $\dim(X) = 1$ *and let* $f \in \mathcal{R}^0(X)$. *There exist* $h_1, h_2 \in \mathcal{P}(X)$ *such that* $\Lambda(< h_1, h_2 >) = \Lambda(< f >)$ *on* X *i.e*
$$\ell(f) \leq \ell(X) \leq 2.$$

Proof. By the previous results we may assume that $\mathcal{Z}(f)$ is not Zariski closed (in particular, X cannot be central [15]). We also assume that X is irreducible to simplify the proof. By [15], $X = F \coprod \{x_1, \ldots, x_m\}$ where $F = \overline{X_{reg}}^{eucl}$ is the one-dimensional irreducible regulous component of X and $x_1, \ldots, x_m$ are the isolated points of X. Since $\mathcal{Z}(f)$ is not Zariski closed, we must have $\dim \mathcal{Z}(f) = 1$ and thus $F \subset \mathcal{Z}(f)$ (see [15]). For each x_i let $p_i \in \mathcal{P}(X)$ such that $p_i \geq 0$ on X and $\mathcal{Z}(p_i) = \{x_i\}$. We set h_1 to be the product of the p_i such that $f(x_i) \leq 0$ and h_2 to be the $(-1)\times$ the product of the p_i such that $f(x_i) \geq 0$. For this choice of h_1 and h_2, we get the proof. □

Definition 4.8. Let $f \in \mathcal{R}^0(X)$.
We set $f_0 = f$, $X_0 = X$ and $X_1 = \operatorname{indet}(f_0)$.
If $X_1 \neq \emptyset$ i.e if f_0 is not regular on X_0 then we set $f_1 = f_0|_{X_1} \in \mathcal{R}^0(X_1)$ and $X_2 = \operatorname{indet}(f_1)$.
By repeating the same process, it stops after at most d steps since $\dim(X_{i+1}) < \dim(X_i)$ and $X_{i+1} = \emptyset$ if $\dim X_i = 0$.
At the step of index i we associate to the regulous function f_i on X_i a rational representation $(p_i, q_i) \in \mathcal{P}(X) \times \mathcal{P}(X)$ such that $f_i = \dfrac{p_i}{q_i}$ on $X_i \setminus X_{i+1}$ and $\mathcal{Z}(q_i) \cap X_i = X_{i+1}$.
The following sequence
$$((f_0, X_0, p_0, q_0), \ldots, (f_k, X_k, p_k, q_k))$$
is called a "polar sequence" associated to f. We have $X_i \neq \emptyset$ for $i = 1, \ldots, k$ and $X_{k+1} = \emptyset$ i.e f_k is regular on X_k.
The number k of the previous sequence is called the "polar depth" of f and we denote it by pol-depth(f).

Remark 4.9. If $f \in \mathcal{R}^0(X)$ then obviously pol-depth$(f) \leq d$.

Proposition 4.10. *If* $f \in \mathcal{R}^0(X)$ *then* $\operatorname{codim}(\operatorname{indet}(f) \setminus \operatorname{Sing}(X)) \geq 2$.

Proof. We may assume X is irreducible and suppose $\dim((\operatorname{indet}(f) \setminus \operatorname{Sing}(X)) = d - 1$. Under this assumption there exists a resolution of singularities $\pi : \tilde{X} \to X$ of X and also of $\operatorname{indet}(f)$ such that $\tilde{f} = f \circ \pi \in \mathcal{R}^0(\tilde{X})$, $\operatorname{indet}(\tilde{f}) = Z$ where Z is the strict transform of $\operatorname{indet}(f)$ and $\dim Z = d-1$. Let W be an irreducible component of Z of dimension $d-1$. Since the local ring $\mathcal{O}_{\tilde{X},W}$ is a discrete valuation ring, we may write the rational function $\tilde{f} = t^m u$ with t an uniformizing parameter of $\mathcal{O}_{\tilde{X},W}$, $m < 0$ and u a unit of $\mathcal{O}_{\tilde{X},W}$. There exists a non-empty Zariski open subset U of W where u doesn't vanish and thus it is impossible to extend continuously the rational function $t^m u$ to W, a contradiction. □

Proposition 4.11. *Let* $X \subset \mathbb{R}^n$ *be a real algebraic set of dimension* d. *Let* $f \in \mathcal{R}^0(X)$, $k =$ pol-depth(f) *and* $((f_0, X_0, p_0, q_0), \ldots, (f_k, X_k, p_k, q_k))$ *a "polar sequence" associated to* f. *Then*
$$\Lambda(< f >) = \Lambda(< p_0 q_0 > \perp_{i=1}^{k} (< 1, -\prod_{j=0}^{i-1} q_j^2 > \otimes < p_i q_i >))$$
on X. *Therefore,*
$$\ell(f) \leq 1 + 2\,\text{pol-depth}(f).$$

Proof. The proof is straightforward since we have $\Lambda(< f >) = \Lambda(< p_0 q_0 >)$ on $X \setminus X_1$ and
$$\Lambda(< f >) = \Lambda(< p_0 q_0 > \perp_{i=1}^{m} (< 1, -\prod_{j=0}^{i-1} q_j^2 > \otimes < p_i q_i >))$$
on $X \setminus X_{m+1}$ for $m = 1, \ldots, k$ and $X_{k+1} = \emptyset$. □

It follows from Propositions 4.11 and 4.7:

Theorem 4.12. *Let $X \subset \mathbb{R}^n$ be a real algebraic set of dimension d. Then*

$$\ell(X) = 1 \text{ if } d = 0,$$
$$\ell(X) \leq 2 \text{ if } d = 1,$$
$$\ell(X) \leq 2d + 1 \text{ else.}$$

It follows from Propositions 4.11, 4.4 and 4.10:

Theorem 4.13. *Let $X \subset \mathbb{R}^n$ be a real algebraic set of dimension d such that* $\operatorname{codim}(\operatorname{Sing}(X)) > 1$. *Then*

$$\ell(X) = 1 \text{ if } d = 0 \,\text{or}\, 1,$$
$$\ell(X) \leq 2d - 1 \text{ else.}$$

We will improve the results of Theorems 4.13 and 4.12 in the sixth section.

Example 4.14. We prove the optimality of the bound given in Theorem 4.13 for $X = \mathbb{R}^2$. Consider the regulous function $f = -1 + \dfrac{x^3}{x^2 + y^2}$. Notice that we have a partition of $\mathbb{R}^2$ given by $\mathbb{R}^2 = \mathcal{S}(-f) \coprod \mathcal{Z}(f) \coprod \mathcal{S}(f)$. We can not write $\Lambda(< f >) = \Lambda(< h >)$ with $h \in \mathbb{R}[x, y]$ since $\mathcal{Z}(f)$ is not Zariski closed.
We can not write $\Lambda(< f >) = \Lambda(< h_1, h_2 >)$ with $h_1, h_2 \in \mathbb{R}[x, y]$ since it would imply that $h_1 h_2$ vanishes on $\mathcal{S}(-f) \cup \mathcal{S}(f)$ and thus vanishes on whole $\mathbb{R}^2$.
By Proposition 4.11, we get

$$\rho(f) =< -x^2 - y^2 + x^3, -1, x^2 + y^2 > .$$

5. Regulous principal semi-algebraic sets

5.1. **Regulous principal semi-algebraic sets versus polynomial principal semi-algebraic sets.** Let $X \subset \mathbb{R}^n$ be a real algebraic set of dimension d.

In this section we raise and study the following questions:
Given a regulous principal open (resp. closed) semi-algebraic subset of X, is it a principal open (resp. closed) semi-algebraic subset of X?
By taking the complementary set, we only have to look at the question concerning open sets. If $d = 0$ the answer is trivially "yes". For $d = 1$ the answer is also "yes" by [7] since in this case any open (resp. closed) semi-algebraic subset of X is principal.

For $d = 2$ the answer can be negative:

Example 5.1. As usual consider $X = \mathbb{R}^2$ and $f = 1 - \dfrac{x^3}{x^2 + y^2}$. Let $S = \mathcal{S}(f)$. Since $S \cap \overline{\operatorname{Bd}(S)}^{Zar} = \{(0, 0)\} \neq \emptyset$ then S cannot be basic [7, Prop. 2.2] ($\operatorname{Bd}(S) = \overline{S}^{eucl} \setminus \mathring{S}$).

In the following we will prove that under the topological condition "$S \cap \overline{\operatorname{Bd}(S)}^{Zar} = \emptyset$", the answer to the previous question, for the regulous principal open semi-algebraic set S, is "yes".

Remark 5.2. Let $f \in \mathcal{R}^0(X)$. Set $S = \mathcal{S}(f)$ and assume $f = \dfrac{p}{q}$ on $\operatorname{dom}(f)$ with $p, q \in \mathcal{P}(X)$ and $\mathcal{Z}(q) = \operatorname{indet}(f)$. If we assume in addition that $S \cap \overline{\operatorname{Bd}(S)}^{Zar} = \emptyset$, we will prove in the following that there exists $r \in \mathcal{P}(X)$ such that $S = \mathcal{S}(r)$ but it may happen that we can not choose r to be equal to pq. Consider $X = \mathbb{R}^2$, $f = \dfrac{y^2 + x^2(1 - x)^2}{x^2 + y^2} = \dfrac{p}{q}$. Since $f = 1 + \dfrac{x^4 - 2x^3}{x^2 + y^2}$ then we see that $f \in \mathcal{R}^0(\mathbb{R}^2)$. We have $S = \mathcal{S}(f) = \mathbb{R}^2 \setminus \{(1, 0)\}$, $\overline{\operatorname{Bd}(S)}^{Zar} = \{(1, 0)\}$ and $\mathcal{S}(pq) = \mathbb{R}^2 \setminus \{(1, 0), (0, 0)\}$.

We can answer affirmatively to the previous question if the set S does not meet the polar locus.

Proposition 5.3. *Let $f \in \mathcal{R}^0(X)$ and $S = \mathcal{S}(f)$. Assume $S \cap \operatorname{indet}(f) = \emptyset$. The set S is then a principal open semi-algebraic set and more precisely we have $\mathcal{S}(f) = \mathcal{S}(pq)$ where $p, q \in \mathcal{P}(X)$ satisfy $f = \dfrac{p}{q}$ on $\operatorname{dom}(f)$ and $\mathcal{Z}(q) = \operatorname{indet}(f)$.*

Proof. Assume $f = \dfrac{p}{q}$ on $\operatorname{dom}(f)$ with $p, q \in \mathcal{P}(X)$ and $\mathcal{Z}(q) = \operatorname{indet}(f)$. We clearly have $\mathcal{S}(f) \setminus \operatorname{indet}(f) = \mathcal{S}(pq) \setminus \operatorname{indet}(f) = \mathcal{S}(pq)$. By assumption $\mathcal{S}(f) \setminus \operatorname{indet}(f) = \mathcal{S}(f)$ and thus $\mathcal{S}(f) = \mathcal{S}(pq)$. □

Remark 5.4. Let $f \in \mathcal{R}^0(X)$. Set $S = \mathcal{S}(f)$ and assume $((f_0, X_0, p_0, q_0), \dots, (f_k, X_k, p_k, q_k))$ is a polar sequence associated to f. We have

$$S = \coprod_{i=0}^{k} \mathcal{S}(p_i q_i) \cap X_i.$$

We will use several times the following other consequence of Hörmander-Łojasiewicz inequality.

Lemma 5.5. [1, Prop. 1.16, Chap. 2]
Let C be a closed semi-algebraic subset of X and let $f, g \in \mathcal{P}(X)$ such that $\mathcal{Z}(f) \cap C \subset \mathcal{Z}(g)$. There exist $h \in \mathcal{P}(X)$ and $l \in \mathbb{N}$ odd such that

$$\Lambda(< (1+h^2)f + g^l >) = \Lambda(< f >)$$

on C.

Theorem 5.6. *Let $f \in \mathcal{R}^0(X)$ and $S = S(f)$. There exists $r \in \mathcal{P}(X)$ such that*

$$\mathcal{S}(r) \subset S \text{ and } S \setminus \mathcal{S}(r) \subset \overline{\operatorname{Bd}(S)}^{Zar} \cap \operatorname{indet}(f).$$

More precisely, if $((f_0, X_0, p_0, q_0), \dots, (f_k, X_k, p_k, q_k))$ is a polar sequence associated to f then, for $i = 0, \dots, k$, there exists $r_i \in \mathcal{P}(X)$ such that

$$\mathcal{S}(r_i) \cap X_i \subset S \cap X_i \text{ and } (S \setminus \mathcal{S}(r_i)) \cap X_i \subset \overline{\operatorname{Bd}(S)}^{Zar} \cap X_{i+1}.$$

Proof. We set $S_i = S \cap X_i$ for $i = 0, \dots, k$. We proceed by decreasing induction on $i = k, \dots, 0$.
- For $i = k$ there is nothing to do since f_k is regular on X_k.
- Assume $i \in \{0, \dots, k-1\}$ and there exists $r_{i+1} \in \mathcal{P}(X)$ such that

$$\mathcal{S}(r_{i+1}) \cap X_{i+1} \subset S \cap X_{i+1} \text{ and } (S \setminus \mathcal{S}(r_{i+1})) \cap X_{i+1} \subset \overline{\operatorname{Bd}(S)}^{Zar} \cap X_{i+2}.$$

Let F denote the closed semi-algebraic subset of X_i defined by $F = \overline{\mathcal{S}(r_{i+1}) \cap X_i}^{eucl} \cap (X_i \setminus S_i)$.

We have

$$X_{i+1} \cap F \subset \mathcal{Z}(r_{i+1}) \cap X_i. \tag{1}$$

If $x \in X_{i+1} \cap F$ then $x \in X_{i+1}$ and $x \notin S_i \cap X_{i+1} = S_{i+1}$. By induction hypothesis we have $\mathcal{S}(r_{i+1}) \cap X_{i+1} \subset S_{i+1}$ and thus $r_{i+1}(x) \leq 0$. Since $x \in \overline{\mathcal{S}(r_{i+1}) \cap X_i}^{eucl}$ then $x \in \overline{\mathcal{S}(r_{i+1}) \cap X_i}^{eucl} \setminus (\mathcal{S}(r_{i+1}) \cap X_i) = \operatorname{Bd}(\mathcal{S}(r_{i+1}) \cap X_i)$ i.e $r_{i+1}(x) = 0$ and it proves (1).

By (1) and since $X_{i+1} = \mathcal{Z}(-q_i^2) \cap X_i$ then Lemma 5.5 provides us $h' \in \mathcal{P}(X)$, l' an odd positive integer such that $r'_{i+1} = (1 + h'^2)(-q_i^2) + r_{i+1}^{l'}$ verifies $\Lambda(< r'_{i+1} >) = \Lambda(< -q_i^2 >)$ on F. Since $\Lambda(< r'_{i+1} >) = \Lambda(< r_{i+1} >)$ on X_{i+1} then r'_{i+1} satisfies the same induction hypotheses than r_{i+1} namely

$$\mathcal{S}(r'_{i+1}) \cap X_{i+1} \subset S_{i+1} \tag{2}$$

and

$$(S_{i+1} \setminus \mathcal{S}(r'_{i+1})) \subset \overline{\operatorname{Bd}(S)}^{Zar} \cap X_{i+2}. \tag{3}$$

We claim that r'_{i+1} satisfies the third property

$$\mathcal{S}(r'_{i+1})\cap X_i\subset S_i. \tag{4}$$

If $x\in\mathcal{S}(r'_{i+1})\cap X_i$ then $r_{i+1}(x)$ must be >0 and if $x\notin S_i$ then $x\in F$ and the sign of $r'_{i+1}(x)$ is the sign of $-q_i^2(x)$, which is impossible. We have proved (4).

Set $C=\overline{S_i}^{eucl}\setminus(\mathcal{S}(r'_{i+1})\cap X_i)$. Let $t\in\mathcal{P}(X)$ such that $\mathcal{Z}(t)=\overline{\mathcal{Z}(p_i)\cap C}^{Zar}$. Since $f_i\in\mathcal{R}^0(X_i)$ then $\mathcal{Z}(q_i)\cap X_i\subset\mathcal{Z}(p_i)\cap X_i$ [15, Prop. 3.5] and thus we get $\mathcal{Z}(p_iq_i)\cap C\subset\mathcal{Z}(t)\subset\mathcal{Z}(t^2r'_{i+1})$. By Lemma 5.5, there exist $h\in\mathcal{P}(X)$ and l an odd positive integer such that $r_i=(1+h^2)p_iq_i+t^{2l}r'^{l}_{i+1}$ verifies $\Lambda(<r_i>)=\Lambda(<p_iq_i>)$ on C. We prove now that r_i is the function we are looking for.

Assume $x\in X_i\setminus S_i$. If $x\in X_{i+1}$ then $p_i(x)q_i(x)=0$, else $x\in X_i\setminus(S_i\cup X_{i+1})$ and the sign of $p_i(x)q_i(x)$ is the sign of $f_i(x)$; thus $p_i(x)q_i(x)\le 0$. By (4) we get $r'_{i+1}(x)\le 0$ and thus $r_i(x)\le 0$. We have proved that

$$\mathcal{S}(r_i)\cap X_i\subset S_i. \tag{5}$$

It remains to prove

$$S_i\setminus(\mathcal{S}(r_i)\cap X_i)\subset\overline{\mathrm{Bd}(S)}^{Zar}\cap X_{i+1}. \tag{6}$$

Assume $x\in S_i\setminus X_{i+1}$. We have $f_i(x)=\dfrac{p_i(x)}{q_i(x)}$ and thus $p_i(x)q_i(x)>0$. If $r'_{i+1}(x)\ge 0$ then $r_i(x)>0$. If $r'_{i+1}(x)<0$ then $x\in C$ and we know that the sign of $r_i(x)$ is the sign of $p_i(x)q_i(x)$. We have proved that $S_i\setminus X_{i+1}\subset\mathcal{S}(r_i)\cap(X_i\setminus X_{i+1})$ and by (5) then $S_i\setminus(\mathcal{S}(r_i)\cap X_i)\subset X_{i+1}$. So in order to get (6) we are left to prove

$$S_{i+1}\setminus(\mathcal{S}(r_i)\cap X_{i+1})\subset\overline{\mathrm{Bd}(S)}^{Zar}. \tag{7}$$

We have $(\mathcal{Z}(p_i)\cap C)\setminus X_{i+1}\subset\mathrm{Bd}(S_i)$ since $C\subset\overline{S_i}^{eucl}$ and $S_i\setminus X_{i+1}=(\mathcal{S}(p_iq_i)\cap X_i)\setminus X_{i+1}$. By (3) and since $\mathcal{Z}(q_i)\cap X_i=X_{i+1}\subseteq\mathcal{Z}(p_i)\cap X_i$ we get $\mathcal{Z}(p_i)\cap C\cap X_{i+1}=((\overline{S_i}^{eucl}\setminus S_i)\cup(S_i\setminus\mathcal{S}(r'_{i+1})))\cap X_{i+1}\subset(\mathrm{Bd}(S_i)\cap X_{i+1})\cup(\overline{\mathrm{Bd}(S)}^{Zar}\cap X_{i+2})\subset\overline{\mathrm{Bd}(S)}^{Zar}$. From the above it follows that

$$\mathcal{Z}(t)\subset\overline{\mathrm{Bd}(S)}^{Zar}. \tag{8}$$

Since $r_i=t^{2l}r'^{l}_{i+1}$ on X_{i+1} then $S_{i+1}\setminus(\mathcal{S}(r_i)\cap X_{i+1})=(S_{i+1}\setminus\mathcal{S}(r'_{i+1}))\cup(\mathcal{Z}(t)\cap S\cap X_{i+1})$. Combining (3) and (8) we get (7), and the proof is complete.

□

Remark that Theorem 5.6 implies Proposition 5.3. Let us mention consequences of Theorem 5.6.

Theorem 5.7. *Let $f\in\mathcal{R}^0(X)$ and $S=\mathcal{S}(f)$. Then S is a principal open semi-algebraic set if and only if $S\cap\overline{\mathrm{Bd}(S)}^{Zar}=\emptyset$.*

Theorem 5.8. *Let $f\in\mathcal{R}^0(X)$. Then $\bar{\mathcal{S}}(f)$ is a principal closed semi-algebraic set if and only if $\mathcal{S}(-f)\cap\overline{\mathrm{Bd}(\mathcal{S}(-f))}^{Zar}=\emptyset$.*

Proof. It is easily seen that an open (resp. closed) semi-algebraic subset S of X is principal open (resp. closed) if and only if $X\setminus S$ is principal closed (resp. open). According to the above remark, the proof follows from Theorem 5.7. □

Corollary 5.9. *Let $f\in\mathcal{R}^0(X)$ such that $\mathcal{Z}(f)$ is Zariski closed. Then the sets $\mathcal{S}(f)$, $\mathcal{S}(-f)$, $\bar{\mathcal{S}}(f)$ and $\bar{\mathcal{S}}(-f)$ are principal semi-algebraic sets.*

Proof. Assume $\mathcal{Z}(f)$ is Zariski closed. Since $\mathrm{Bd}(\mathcal{S}(f))\subset\mathcal{Z}(f)$, we get $\mathcal{S}(f)\cap\overline{\mathrm{Bd}(\mathcal{S}(f))}^{Zar}\subset\mathcal{S}(f)\cap\overline{\mathcal{Z}(f)}^{Zar}=\mathcal{S}(f)\cap\mathcal{Z}(f)=\emptyset$. By Theorems 5.7 and 5.8 the proof is complete. □

5.2. **Characterization of regulous principal semi-algebraic sets.** Let $X \subset \mathbb{R}^n$ be a real algebraic set of dimension d.

In this section, we will give an answer to the following question: Under which conditions an open semi-algebraic set is regulous principal?

Definition 5.10. A semi-algebraic subset S of X is said to be generically principal on X if S coincides with a principal open semi-algebraic subset of X outside a real algebraic subset of X of positive codimension.

The next result is a regulous version of Lemma 5.5.

Lemma 5.11. *Let C be a closed semi-algebraic subset of X and let $f, g \in \mathcal{R}^0(X)$ such that $\mathcal{Z}(f) \cap C \subset \mathcal{Z}(g)$. There exist $h \in \mathcal{P}(X)$ and $l \in \mathbb{N}$ odd such that $h > 0$ on X and*

$$\Lambda(< hf + g^l >) = \Lambda(< f >)$$

on C.

Proof. We can see C as a closed semi-algebraic subset of $\mathbb{R}^n$ and $f, g \in \mathcal{R}^0(\mathbb{R}^n)$ by definition of regulous functions on X. By [4, Thm. 2.6.6], for a sufficiently big positive odd integer l the function $\frac{g^l}{f}$ is semi-algebraic and continuous on C. By [4, Thm. 2.6.2], $|\frac{g^l}{f}|$ is bounded on C by a polynomial function h which is positive definite on $\mathbb{R}^n$. The proof is done by restricting these functions to X. $\square$

Proposition 5.12. *Let S be a semi-algebraic subset of X. The set S is regulous principal open if and only if we have:*
1) $S \cap \overline{\mathrm{Bd}(S)}^{\mathcal{C}} = \emptyset$,
and there exists an algebraic subset W of X of positive codimension such that:
2) there exists $p \in \mathcal{P}(X)$ such that $S \setminus W = \mathcal{S}(p) \setminus W$,
3) there exists $g \in \mathcal{R}^0(X)$ such that $S \cap W = \mathcal{S}(g) \cap W$.

Proof. Assume $S = \mathcal{S}(f)$ with $f \in \mathcal{R}^0(X)$ such that $f = \frac{p}{q}$ on $\mathrm{dom}(f)$ with $p, q \in \mathcal{P}(X)$ and $\mathcal{Z}(q) = \mathrm{indet}(f)$. We have $S \cap \overline{\mathrm{Bd}(S)}^{\mathcal{C}} = \emptyset$ since $\overline{\mathrm{Bd}(S)}^{\mathcal{C}} \subset \mathcal{Z}(f)$. Moreover $S \setminus \mathrm{indet}(f) = \mathcal{S}(pq)$ and $f|_{\mathrm{indet}(f)} \in \mathcal{R}^0(\mathrm{indet}(f))$. We have proved one implication.

Assume now S satisfies the the three conditions of the Proposition. We may assume $W \subset \mathcal{Z}(p)$ changing p by $q^2 p$ where $q \in \mathcal{P}(X)$ satisfies $W = \mathcal{Z}(q)$.

Set $F = \overline{\mathcal{S}(g)}^{eucl} \setminus S$. Assume $x \in W \cap F$ then $x \in W \setminus (S \cap W)$ and thus $g(x) \leq 0$. Then $x \in \mathrm{Bd}(\mathcal{S}(g)) \subset \mathcal{Z}(g)$. We have proved that $\mathcal{Z}(-q^2) \cap F \subset \mathcal{Z}(g)$. By Lemma 5.11 there exist $h \in \mathcal{P}(X)$, $l \in \mathbb{N}$ odd and $g' \in \mathcal{R}^0(X)$ such that $h > 0$ on X, $g' = -hq^2 + g^l$ and $\Lambda(< g' >) = \Lambda(< -q^2 >)$ on F. Clearly, the function g' satisfies again the property 3) of the proposition, namely

$$S \cap W = \mathcal{S}(g') \cap W. \tag{9}$$

The function g' satisfies in addition the following property

$$\mathcal{S}(g') \subset S. \tag{10}$$

Assume $g'(x) > 0$ then $g(x) > 0$ and moreover if $x \notin S$ then $x \in F$ and we get a contradiction since then the sign of $g'(x)$ would be the sign of $-q^2(x)$. We have proved (10).

Set $C = \overline{S}^{eucl} \setminus \mathcal{S}(g')$. Let $t \in \mathcal{R}^0(X)$ be such that $\mathcal{Z}(t) = \overline{\mathcal{Z}(p) \cap C}^{\mathcal{C}}$. We clearly have $\mathcal{Z}(p) \cap C \subset \mathcal{Z}(t^2 g')$. By Lemma 5.11, there exist $p' \in \mathcal{P}(X)$ positive definite on X and a positive odd integer l' such that $f = p'p + t^{2l'} g'^{l'}$ is regulous on X and satisfies $\Lambda(< f >) = \Lambda(< p >)$ on C.

Assume $x \notin S$. We have $p(x) \leq 0$ since $W \subset \mathcal{Z}(p)$. We have $g'(x) \leq 0$ by (10). Hence $f(x) \leq 0$ and we have proved that

$$S(f) \subset S. \tag{11}$$

Assume $x \in S \setminus W$. We have $p(x) > 0$. If $g'(x) > 0$ then clearly $f(x) > 0$. If $g'(x) \leq 0$ then $x \in C$ and $f(x) > 0$ since $\Lambda(< f >) = \Lambda(< p >)$ on C. We have proved that

$$S \setminus W \subset \mathcal{S}(f) \setminus W. \tag{12}$$

Since $W \subset \mathcal{Z}(p)$ and using (9) it follows that

$$(S \cap W) \setminus (\mathcal{S}(f) \cap W) \subset \mathcal{Z}(t) = \overline{\mathcal{Z}(p) \cap C}^{\mathcal{C}}. \tag{13}$$

We prove now that

$$\mathcal{Z}(p) \cap C \subset \mathrm{Bd}(S). \tag{14}$$

Assume $y \in \mathcal{Z}(p) \cap C \cap W = W \cap C$. We have $p(y) = 0$, $y \in \overline{S}^{eucl} \cap W$ and $g'(y) \leq 0$. We have $y \notin S \cap W$ by (9). Hence $y \in \mathrm{Bd}(S) \cap W$. Assume $y \in \mathcal{Z}(p) \cap C$ and $y \notin W$. Since $p(y) = 0$ and $y \notin W$ then $y \notin S$. We get $y \in \overline{S}^{eucl}$ since $y \in C$ and it proves (14).

From (11), (12), (13) and (14) it follows that

$$S \setminus \mathcal{S}(f) \subset \overline{\mathcal{Z}(p) \cap C}^{\mathcal{C}} \cap W \subset \overline{\mathrm{Bd}(S)}^{\mathcal{C}} \cap W.$$

Since $S \cap \overline{\mathrm{Bd}(S)}^{\mathcal{C}} = \emptyset$ we finally get

$$S = \mathcal{S}(f).$$

□

Theorem 5.13. *Let S be a semi-algebraic subset of X. The set S is regulous principal open if and only if we have:*
1) for any real algebraic subset V of X then $S \cap V$ is generically principal,
and
2) $S \cap \overline{\mathrm{Bd}(S)}^{\mathcal{C}} = \emptyset$.

Proof. If $S = \mathcal{S}(f)$ with $f \in \mathcal{R}^0(X)$ then we have already seen that $S \cap \overline{\mathrm{Bd}(S)}^{\mathcal{C}} = \emptyset$. Moreover, $S \cap V$ is generically principal for any real algebraic subset V of X since $f|_V \in \mathcal{R}^0(V)$ and thus $\mathcal{S}(f|_V)$ coincides with a principal open semi-algebraic subset of V on $V \setminus \mathrm{indet}(f|_V)$.

Assume now the set S satisfies the conditions 1) and 2) of the theorem. We denote the set X by X_0 and S by S_0. Since S_0 is generically principal there exist $p_0 \in \mathcal{P}(X_0)$ and an algebraic subset X_1 of X_0 of positive codimension such that $S_0 \setminus X_1 = \mathcal{S}(p_0) \setminus X_1$. If $X_1 = \emptyset$ then we are done since S is even principal. If $X_1 \neq \emptyset$ then we denote by S_1 the set $S_0 \cap X_1$. Remark that S_1 satisfies the conditions 1) and 2) of the theorem as an open semi-algebraic subset of X_1 and we can repeat the process used for S_0 but here for the set S_1. So we build a finite sequence

$$((X_0, S_0, p_0), \ldots, (X_k, S_k, p_k))$$

such that for $i = 0, \ldots, k-1$, X_{i+1} is an algebraic subset of X_i of positive codimension, $S_i = S \cap X_i$ satisfies the conditions 1) and 2), $p_i \in \mathcal{P}(X)$, $S_i \setminus X_{i+1} = (\mathcal{S}(p_i) \cap X_i) \setminus X_{i+1}$ and $S_k = S \cap X_k = \mathcal{S}(p_k) \cap X_k$ with $p_k \in \mathcal{P}(X)$. By Proposition 5.12, there exists $g_{k-1} \in \mathcal{R}^0(X)$ such that $S_{k-1} = \mathcal{S}(g_{k-1}) \cap X_{k-1}$. By successive application of Proposition 5.12, there exists $g_i \in \mathcal{R}^0(X)$ such that $S_i = \mathcal{S}(g_i) \cap X_i$ for $i = k-2, \ldots, 0$, which establishes in particular that S is regulous principal open. □

6. Signs of regulous functions (part 2)

6.1. **Upper bounds for the lengths of signs of regulous functions.** We can use Corollary 5.9 to improve some of the results of section 4 concerning the lengths of signs of regulous functions.

We extend the result of Proposition 4.4 (see Remark 4.5), which concerns curves, to any real algebraic set of any dimension.

Theorem 6.1. *Let $f \in \mathcal{R}^0(X)$. Then $\mathcal{Z}(f)$ is Zariski closed if and only if $\ell(f) = 1$.*

Proof. The proof of the "if" is trivial.

Assume $\mathcal{Z}(f)$ is Zariski closed. By Corollary 5.9, there exist p_1, p_2 in $\mathcal{P}(X)$ such that $\mathcal{S}(f) = \mathcal{S}(p_1)$ and $\mathcal{S}(-f) = \mathcal{S}(p_2)$. We conclude by proceeding analogously to the end of the proof of Proposition 4.4. □

Corollary 6.2. *Let $f \in \mathcal{R}^0(X)$, $k = \text{pol-depth}(f)$ and $((f_0, X_0, p_0, q_0), \ldots, (f_k, X_k, p_k, q_k))$ a "polar sequence" associated to f. Let*

$$t = \min\{i \in \{0, \ldots, k\} |\ \mathcal{Z}(f) \cap X_i \,\text{is Zariski closed}\}.$$

Therefore,

$$\ell(f) \leq 1 + 2t.$$

Proof. The proof is straightforward using Proposition 4.11 and Theorem 6.1. □

In the following proposition, we improve the result of Theorem 4.12 in the two dimensional case and the result of Theorem 4.13 in the three dimensional case.

Proposition 6.3. *Assume $d = \dim(X) \leq 3$ and moreover that $\operatorname{codim}(\operatorname{Sing}(X)) > 1$ in the case $d = 3$. Then*

$$\ell(X) \leq 3.$$

More precisely, if $f \in \mathcal{R}^0(X)$ and $f = \dfrac{p}{q}$ on $\operatorname{dom}(f)$, $p, q \in \mathcal{P}(X)$, $\mathcal{Z}(q) = \operatorname{indet}(f)$ then there exists $h \in \mathcal{P}(X)$ such that $\Lambda(< f >) = \Lambda(< pq > \perp < 1, -q^2 > \otimes < h >)$ on X.

Proof. By Theorems 4.12 and 4.13, we may assume that $d = 2$ or $d = 3$ and also that $\operatorname{codim}(\operatorname{Sing}(X)) > 1$ in the case $d = 3$. Let $f \in \mathcal{R}^0(X)$. By Proposition 4.10, we have $\dim(\operatorname{indet}(f)) \leq 1$. We get the proof, using Corollary 6.2, if $\dim(\operatorname{indet}(f)) < 1$ or if $\mathcal{Z}(f) \cap \operatorname{indet}(f)$ is Zariski closed. So we assume $\dim(\operatorname{indet}(f)) = 1$ and $\mathcal{Z}(f) \cap \operatorname{indet}(f)$ is not Zariski closed. We write $f = \dfrac{p}{q}$ on $\operatorname{dom}(f)$ with $p, q \in \mathcal{P}(X)$ and $\mathcal{Z}(q) = \operatorname{indet}(f)$. We decompose $\mathcal{Z}(q) = \operatorname{indet}(f)$ as a union $C_1 \cup \cdots \cup C_t \cup W$ where the C_i are irreducible real algebraic curves and $\dim(W) = 0$. For each curve C_i, we denote by F_i the regulous closed set $\overline{(C_i)_{reg}}^{\mathcal{C}} = \overline{(C_i)_{reg}}^{eucl}$. By [15, Thm. 6.7], the sets F_i are $\mathcal{C}$-irreducible and $C_i \setminus F_i$ is empty or a finite set of points. Since $\mathcal{Z}(f) \cap \operatorname{indet}(f)$ is not Zariski closed, we have $\dim(\mathcal{Z}(f) \cap \operatorname{indet}(f)) = 1$. Since the F_i are $\mathcal{C}$-irreducible, we get that $F_i \subset \mathcal{Z}(f)$ if and only if $\dim(\mathcal{Z}(f) \cap C_i) = 1$. Remark that there exists at least one F_i such that $F_i \subset \mathcal{Z}(f)$ but $C_i \not\subset \mathcal{Z}(f)$ since $\mathcal{Z}(f) \cap \operatorname{indet}(f)$ is not Zariski closed. If $F_i \subset \mathcal{Z}(f)$ then $\Lambda(< f >) = \Lambda(< pq >)$ on C_i outside a finite number of points. If $F_i \not\subset \mathcal{Z}(f)$ then $\mathcal{Z}(f) \cap C_i$ is Zariski closed. It follows that there exists a real algebraic subset Y of $\operatorname{indet}(f)$ such that $\mathcal{Z}(f) \cap Y$ is Zariski closed and such that $\Lambda(< f >) = \Lambda(< pq >)$ on $X \setminus Y$. By Theorem 6.1, there exists $h \in \mathcal{P}(X)$ such that $\Lambda(< f >) = \Lambda(< h >)$ on Y. Let $r \in \mathcal{P}(X)$ be such that $\mathcal{Z}(r) = Y$. The proof is done since

$$\Lambda(< f >) = \Lambda(< pq > \perp < 1, -r^2 > \otimes < h >) \,\text{on}\, X.$$

□

Example 6.4. Consider $f = z - \dfrac{x^3}{x^2 + y^2} \in \mathcal{R}^0(\mathbb{R}^3)$. So $\mathcal{Z}(z - \dfrac{x^3}{x^2 + y^2}) \subset \mathbb{R}^3$ is the "canopy" of the Cartan umbrella $V = \mathcal{Z}(z(x^2 + y^2) - x^3) \subset \mathbb{R}^3$. Moreover, $\operatorname{indet}(f)$ is the stick of the umbrella and $\mathcal{Z}(f) \cap \operatorname{indet}(f) = \{(0, 0, 0)\}$. According to Corollary 6.2 we get:

$$\Lambda(< f >) = \Lambda(< (x^2 + y^2)f > \perp < 1, -x^2 - y^2 > \otimes < z >)$$

on $\mathbb{R}^3$. Remark that since $\mathcal{Z}(f)$ is not Zariski closed then $\ell(f) > 1$ (Theorem 6.1). If $\Lambda(< f >) = \Lambda(< p_1, p_2 >)$ on $\mathbb{R}^3$ with $p_1, p_2 \in \mathcal{P}(\mathbb{R}^3)$ then it is easy to see that the product $p_1 p_2$ vanishes identically on $\mathbb{R}^3$. It follows that the form $< p_1, p_2 >$ is isotropic, a contradiction because $\ell(f) > 1$. Hence $\ell(f) = 3$ and $\rho(f) = < (x^2 + y^2)f > \perp < 1, -x^2 - y^2 > \otimes < z >$.

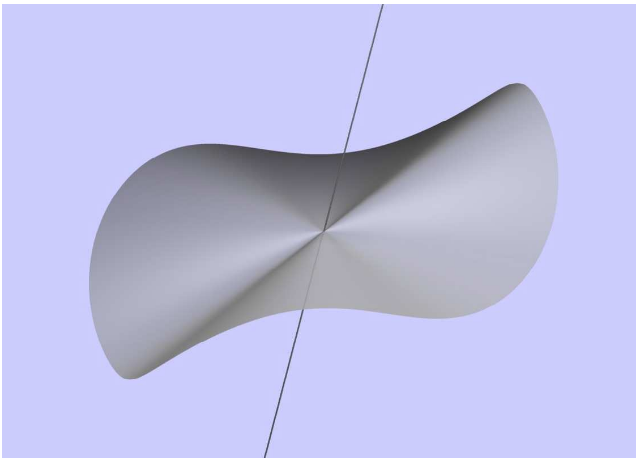

FIGURE 3. Cartan umbrella.

According to the proof of Proposition 6.3, we get:

Corollary 6.5. *Let $f \in \mathcal{R}^0(X)$. If* $\dim(\mathrm{indet}(f)) \leq 1$ *then* $\ell(f|_{\mathrm{indet}(f)}) = 1$.

Corollary 6.6. *Let $f \in \mathcal{R}^0(X)$. If* $\dim(\mathrm{indet}(f)) \leq 1$ *then* $\ell(f) \leq 3$.

Using Proposition 6.3, we improve the upper bounds on ℓ given in Theorems 4.12 and 4.13.

Theorem 6.7. *Let $X \subset \mathbb{R}^n$ be a real algebraic set of dimension d. Then*

$$\ell(X) = 1 \text{ if } d = 0,$$

$$\ell(X) \leq 2 \text{ if } d = 1,$$

$$\ell(X) \leq 2d - 1 \text{ else.}$$

Proof. By Theorem 4.12 and Proposition 6.3, we are left to prove the theorem for $d > 2$. Let $f \in \mathcal{R}^0(X)$. By Proposition 4.11, we can assume that $1 + 2\,\text{pol-depth}(f) > 2d - 1$ i.e pol-depth$(f) = d$. Let $((f_0, X_0, p_0, q_0), \ldots, (f_d, X_d, p_d, q_d))$ be a polar sequence associated to f. For $i = 0, \ldots, d$, we have $\dim X_i = d - i$. In particular $\dim X_{d-2} = 2$ and by Proposition 6.3 there exists $h \in \mathcal{P}(X)$ such that $\Lambda(< f_{d-2} >) = \Lambda(< p_{d-2}q_{d-2} > \perp < 1, -q_{d-2}^2 > \otimes < h >)$ on X_{d-2}. Then

$$\Lambda(< f >) = \Lambda(< p_0q_0 > \perp_{i=1}^{d-2} (< 1, -\prod_{j=0}^{i-1} q_j^2 > \otimes < p_iq_i >) \perp < 1, -\prod_{j=0}^{d-2} q_j^2 > \otimes < h >)$$

on X and the proof is done. $\square$

Theorem 6.8. *Let $X \subset \mathbb{R}^n$ be a real algebraic set of dimension d such that* $\mathrm{codim}(\mathrm{Sing}(X)) > 1$. *Then*

$$\ell(X) = 1 \text{ if } d = 0 \,\text{or}\, 1,$$

$$\ell(X) \leq 3 \text{ if } d = 2$$

$$\ell(X) \leq 2d - 3 \text{ else.}$$

Proof. For $d \leq 3$ the theorem follows from Theorem 4.13 and Proposition 6.3. For $d \geq 4$, copy the proof of Theorem 6.7 and use Proposition 4.10. $\square$

6.2. **Characterization of regulous functions with length of sign equal to one.** We give a general result which concerns semi-algebraic functions.

Proposition 6.9. *Let $X\subset\mathbb{R}^n$ be a real algebraic set. Let f be a continuous semi-algebraic function on X. There exists $p\in\mathcal{P}(X)$ such that $\Lambda(<f>)=\Lambda(<p>)$ if and only if the three following conditions are satisfied:*
1) $\mathcal{Z}(f)$ is Zariski closed.
2) $\mathcal{S}(f)$ is principal.
3) $\mathcal{S}(-f)$ is principal.

Proof. One implication is trivial. For the other one, this follows by the same arguments as in the end of the proof of Proposition 4.4. □

Remark 6.10. For a regulous function, condition 1) of Proposition 6.9 implies conditions 2) and 3) (Corollary 5.9).

We give some several equivalent characterizations of regulous functions with length of sign equal to one for central real algebraic sets.

Proposition 6.11. *Let $X\subset\mathbb{R}^n$ be a central real algebraic set. Let $f\in\mathcal{R}^0(X)$. The following properties are equivalent:*
a) $\ell(f)=1$.
b) $\mathcal{Z}(f)$ is Zariski closed.
c) $\mathcal{S}(f^2)=\mathcal{S}(f)\cup\mathcal{S}(-f)=X\setminus\mathcal{Z}(f)$ is principal.
d) $\mathcal{S}(f^2)\cap\overline{\mathrm{Bd}(\mathcal{S}(f^2))}^{Zar}=\emptyset$.

Proof. Equivalence between a) and b) (resp. c) and d)) is Theorem 6.1 (resp. Theorem 5.7) and the assumption that X is central is not required. It is clear that b) implies c). We are reduced to proving c) implies b). Assume $\mathcal{S}(f^2)=X\setminus\mathcal{Z}(f)$ is principal, namely $\mathcal{S}(f^2)=\mathcal{S}(p)$ for $p\in\mathcal{P}(X)$. There is no loss of generality in assuming X is irreducible. We assume $\mathcal{Z}(f)$ is a proper subset of X since otherwise $\mathcal{Z}(f)$ is already Zariski closed. Since $\overline{X_{reg}}^{eucl}=X$ (X is central) and X is irreducible then it follows from [15, Prop. 6.6] that $\dim\mathcal{Z}(f)<\dim X$. Notice that $\mathcal{S}(-p)\subset\mathcal{Z}(f)$. If $\mathcal{S}(-p)\neq\emptyset$ then we claim that $\dim\mathcal{S}(-p)=\dim X$: Let $\tilde{\mathcal{S}}(-p)$ be the constructible subset of $\operatorname{Spec}_r\mathcal{P}(X)$ associated to $\mathcal{S}(-p)$ (see [4, Ch. 7]). We have $\dim\mathcal{S}(-p)=\dim\tilde{\mathcal{S}}(-p)$ [4, Prop. 7.5.6]. Since X is central and $\mathcal{S}(-p)$ is non-empty and open then $\mathcal{S}(-p)\cap X_{reg}\neq\emptyset$. By [4, Prop. 7.6.2], $\tilde{\mathcal{S}}(-p)\cap\operatorname{Spec}_r\mathcal{K}(X)\neq\emptyset$ and we get $\dim\tilde{\mathcal{S}}(-p)=\dim X$ [4, Prop. 7.5.8] which gives the claim. It follows from the claim and above remarks that $\mathcal{S}(-p)=\emptyset$ and thus $\mathcal{Z}(f)=\mathcal{Z}(p)$ is Zariski closed. □

Corollary 6.12. *Let $X\subset\mathbb{R}^n$ be a central real algebraic set. Let $f\in\mathcal{R}^0(X)$ such that $\mathcal{S}(f)$ is principal and f is nonnegative on X. Then $\ell(f)=1$.*

Example 6.13. The assumption that X is central in Proposition 6.11 and Corollary 6.12 is a necessary assumption. Consider the regulous function $f=1-\dfrac{x^3}{x^2+y^2}$ restricted to $X=\mathcal{Z}(y^2-x^3+x^2)$ of Example 2.4, f is non-negative on X, $\mathcal{S}(f)\cap X$ is principal ($\mathcal{S}(f)\cap X=\mathcal{S}(1-x)\cap X$) but $\mathcal{Z}(f)\cap X$ is not Zariski closed.

Example 6.14. We have already seen that if f is a regulous function on a real algebraic set X then the property that $\mathcal{Z}(f)$ is Zariski closed (condition 1) of Proposition 6.9) implies that $\mathcal{S}(f)$ and $\mathcal{S}(-f)$ are both principal (conditions 2) and 3) of Proposition 6.9). We prove now that the converse is not always true even if X is central. Consider the following regulous functions on the plane: $h=(1-\dfrac{x^3}{x^2+y^2})^2$, $g=-(y^2+(x+\dfrac{1}{2})(x-\dfrac{1}{2})(x-4)(x-5))$, $f=hg$. We have $\mathrm{Bd}(\mathcal{S}(f))=\mathcal{Z}(g)=\overline{\mathrm{Bd}(\mathcal{S}(f))}^{Zar}$, hence

$\mathcal{S}(f)$ is principal (Theorem 5.7) and more precisely $\mathcal{S}(f)=\mathcal{S}(g)$. We have $\mathrm{Bd}(\mathcal{S}(-f))=\mathcal{Z}(g)\cup\mathcal{Z}(h)$, hence $\overline{\mathrm{Bd}(\mathcal{S}(-f))}^{Zar}=\mathcal{Z}(g)\cup\mathcal{Z}((x^2+y^2)^2h)=\mathcal{Z}(g)\cup\mathcal{Z}(h)\cup\{(0,0)\}$. Since g and f are both positive at the origin then $\overline{\mathrm{Bd}(\mathcal{S}(-f))}^{Zar}\cap\mathcal{S}(-f)=\emptyset$ and thus $\mathcal{S}(-f)$ is principal; more precisely $\mathcal{S}(-f)=\mathcal{S}(-g(x^2+y^2)^2h)$. We also have $\mathcal{S}(f^2)\cap\overline{\mathrm{Bd}(\mathcal{S}(f^2))}^{Zar}=\{(0,0)\}$ and thus $\mathcal{Z}(f)$ is not Zariski closed (Proposition 6.11).

In the previous example, the problems arise in part because of the $\mathcal{C}$-reducibility of the zero set of the regulous function f.

Proposition 6.15. *Let $X\subset\mathbb{R}^n$ be a central real algebraic set of dimension d. Let $f\in\mathcal{R}^0(X)$ be such that $\mathcal{S}(f)$ is principal, $\mathcal{S}(-f)$ is principal, $\mathcal{Z}(f)$ is $\mathcal{C}$-irreducible and $\mathrm{Bd}(\mathcal{S}(f))\cap\mathrm{Bd}(\mathcal{S}(-f))\neq\emptyset$. Then $\mathcal{Z}(f)$ is Zariski closed.*

Proof. The sets $\mathcal{S}(-f)$ and $\mathcal{S}(f)$ are both non-empty since $\mathrm{Bd}(\mathcal{S}(f))\cap\mathrm{Bd}(\mathcal{S}(-f))\neq\emptyset$. We may assume X is irreducible. As we have already explained in the proof of Proposition 6.11 and since X is central, we have $\dim\mathcal{S}(f)=\dim\mathcal{S}(-f)=d$. We claim that $\dim\mathrm{Bd}(\mathcal{S}(f))=d-1$. There exist $x\in X_{reg}$ and a semi-algebraic neighbourhood U of x in X satisfying the following three properties:
- There exists a semi-algebraic homeomorphism from U onto a semi-algebraic U' of the origin in $\mathbb{R}^d$ (mapping x to the origin).
- $\mathcal{S}(f)\cap U\neq\emptyset$.
- $(X\setminus\overline{\mathcal{S}(f)}^{eucl})\cap U\neq\emptyset$.

The first property follows from [4, Prop. 3.3.11]. The second and the third properties are consequences of the assumption $\mathrm{Bd}(\mathcal{S}(f))\cap\mathrm{Bd}(\mathcal{S}(-f))\neq\emptyset$ and also because X is central and irreducible. Since $\mathrm{Bd}(\mathcal{S}(f))\cap U=U\setminus((\mathcal{S}(f)\cap U)\cup((X\setminus\overline{\mathcal{S}(f)}^{eucl})\cap U))$, we get $\dim\mathrm{Bd}(\mathcal{S}(f))\geq d-1$ applying [4, lem. 4.5.2]. Since X is irreducible and central then $\dim\mathcal{Z}(f)\leq d-1$ ([15, Prop. 6.6]). Since $\mathrm{Bd}(\mathcal{S}(f))\subset\mathcal{Z}(f)$, we get the claim and moreover we see that $\dim\mathcal{Z}(f)=d-1$.

By the same arguments we get $\dim\mathrm{Bd}(\mathcal{S}(-f))=d-1$. Since X is irreducible and central and since $\dim\mathcal{Z}(f)\leq d-1$ then $X=\overline{X\setminus\mathcal{Z}(f)}^{eucl}$ and thus $\mathcal{Z}(f)=\mathrm{Bd}(\mathcal{S}(f))\cup\mathrm{Bd}(\mathcal{S}(-f))$. Since $\mathcal{Z}(f)=\mathrm{Bd}(\mathcal{S}(f))\cup\mathrm{Bd}(\mathcal{S}(-f))$, $\dim\mathcal{Z}(f)=\dim\mathrm{Bd}(\mathcal{S}(f))=\dim\mathrm{Bd}(\mathcal{S}(-f))=d-1$ and since by assumption $\mathcal{Z}(f)$ is $\mathcal{C}$-irreducible the we get

$$\mathcal{Z}(f)=\overline{\mathrm{Bd}(\mathcal{S}(f))}^{\mathcal{C}}=\overline{\mathrm{Bd}(\mathcal{S}(-f))}^{\mathcal{C}}.$$

Hence $\overline{\mathcal{Z}(f)}^{Zar}=\overline{\mathrm{Bd}(\mathcal{S}(f))}^{Zar}=\overline{\mathrm{Bd}(\mathcal{S}(-f))}^{Zar}$ and thus $\overline{\mathrm{Bd}(\mathcal{S}(f^2))}^{Zar}=\overline{\mathcal{Z}(f)}^{Zar}=\overline{\mathrm{Bd}(\mathcal{S}(f))}^{Zar}=\overline{\mathrm{Bd}(\mathcal{S}(-f))}^{Zar}$. Since $\mathcal{S}(f)$ is principal then $\overline{\mathrm{Bd}(\mathcal{S}(f^2))}^{Zar}\cap\mathcal{S}(f)=\emptyset$. Since $\mathcal{S}(-f)$ is principal then $\overline{\mathrm{Bd}(\mathcal{S}(f^2))}^{Zar}\cap\mathcal{S}(-f)=\emptyset$. Hence $\overline{\mathrm{Bd}(\mathcal{S}(f^2))}^{Zar}\cap\mathcal{S}(f^2)=\emptyset$ and the proof is done (Proposition 6.11). ☐

6.3. **Complexity of regulous principal semi-algebraic sets.**

Theorem 6.16. [1, Prop. and Def. 3.7 Ch. 1], [17, Thm. 2.8]
Let $X\subset\mathbb{R}^n$ be a real algebraic set of dimension d. The cokernel of the inclusion map $\mathrm{A}(X)\subset\mathrm{F}(X)$ has two primary torsion and moreover

$$2^d\,\mathrm{F}(X)\subset\mathrm{A}(X).$$

From the previous theorem, we can introduce some invariants of semi-algebraic sets (see [1, Prop. and Def. 3.7 Ch. 1] for the original definitions).

Definition 6.17. Let $X\subset\mathbb{R}^n$ be a real algebraic set. Let C be a non-empty semi-algebraic subset of X.
- The minimal number $k>0$ such that $k\,\mathbf{1}_C\in\mathrm{A}(X)$ is a power of two, say $k=2^{\mathrm{w}(C)}$.
- There exists a form ρ over $\mathcal{P}(X)$ such that $\Lambda(\rho)=2^{\mathrm{w}(C)}\,\mathbf{1}_C$. The form ρ can always be chosen

anisotropic and then it is unique. We denote by $\rho(C)$ the corresponding anisotropic form and by $\ell(C)$ the dimension of $\rho(C)$.
• The number $\mathrm{w}(C)$ is called the width of C, the number $\ell(C)$ is called the length of C and the form $\rho(C)$ is called the defining form of C.

Corollary 6.18. [17, Thm. 2.8]
Let $X \subset \mathbb{R}^n$ be a real algebraic set of dimension d. Let C be a non-empty semi-algebraic subset of X. Then

$$\mathrm{w}(C) \leq d.$$

The following proposition characterizes the widths of regulous closed sets and regulous principal sets.

Proposition 6.19. *Let $X \subset \mathbb{R}^n$ be a real algebraic set. Let $f \in \mathcal{R}^0(X)$. In case the considered set is non-empty, we get: $\mathrm{w}(\mathcal{Z}(f)) = 0$, $\mathrm{w}(X \setminus \mathcal{Z}(f)) = 0$, $\mathrm{w}(\mathcal{S}(f)) \leq 1$ and $\mathrm{w}(\bar{\mathcal{S}}(f)) \leq 1$. If f is non-negative on X then $\mathrm{w}(\mathcal{S}(f)) = \mathrm{w}(\bar{\mathcal{S}}(f)) = 0$. Moreover $\mathrm{w}(\mathcal{S}(f)) = \mathrm{w}(\mathcal{S}(-f))$ in case $\mathcal{S}(f)$ and $\mathcal{S}(-f)$ are both non-empty.*

Proof. We have $\Lambda(< 1 >\perp \rho(-f^2)) = \mathbf{1}_{\mathcal{Z}(f)}$, $\Lambda(\rho(f^2)) = \mathbf{1}_{X\setminus\mathcal{Z}(f)}$, $\Lambda(\rho(f) \perp \rho(f^2)) = 2\,\mathbf{1}_{\mathcal{S}(f)}$ and $\Lambda(< 1 >\perp \rho(f) \perp< 1 >\perp \rho(-f^2)) = 2\,\mathbf{1}_{\bar{\mathcal{S}}(f)}$. If f is non-negative on X then $\Lambda(\rho(f)) = \mathbf{1}_{\mathcal{S}(f)}$ and $\Lambda(< 1 >) = \mathbf{1}_{\bar{\mathcal{S}}(f)}$.

Assume $\mathcal{S}(f)$ and $\mathcal{S}(-f)$ are both non-empty. If $\mathrm{w}(\mathcal{S}(-f)) = 0$ then

$$\Lambda(< -1 > \otimes\rho(\mathcal{S}(-f)) \perp< 1 >\perp< -1 > \otimes\rho(\mathcal{Z}(f))) = \mathbf{1}_{\mathcal{S}(f)}$$

if $\mathcal{Z}(f) \neq \emptyset$ and

$$\Lambda(< -1 > \otimes\rho(\mathcal{S}(-f)) \perp< 1 >) = \mathbf{1}_{\mathcal{S}(f)}$$

if $\mathcal{Z}(f) = \emptyset$. It follows that $\mathrm{w}(\mathcal{S}(f)) = 0$ and the proof is done. □

We compare the lengths of regulous closed sets and regulous principal sets and the lengths of the signs of regulous functions.

Proposition 6.20. *Let $X \subset \mathbb{R}^n$ be a real algebraic set. Let $f \in \mathcal{R}^0(X)$. In case the considered set is non-empty, we get:*
• *$\ell(\mathcal{Z}(f)) \leq 1 + \ell(f^2) \leq 1 + \ell(f)^2$ and $\rho(\mathcal{Z}(f))$ is the anisotropic form similar to $< 1 >\perp \rho(-f^2)$.*
• *$\ell(X \setminus \mathcal{Z}(f)) = \ell(f^2)$ and $\rho(X \setminus \mathcal{Z}(f)) = \rho(f^2)$.*
• *If f is non-negative on X then $\ell(\mathcal{S}(f)) = \ell(f)$ and $\rho(\mathcal{S}(f)) = \rho(f)$.*
• *If $\mathrm{w}(\mathcal{S}(f)) = 1$ then $\ell(\mathcal{S}(f)) \leq \ell(f) + \ell(f^2) \leq \ell(f)(1 + \ell(f))$ and $\rho(\mathcal{S}(f))$ is the anisotropic form similar to $\rho(f) \perp \rho(f^2)$.*
• *If f is non-negative on X then $\ell(\bar{\mathcal{S}}(f)) = 1$ and $\rho(\mathcal{S}(f)) =< 1 >$.*
• *If $\mathrm{w}(\bar{\mathcal{S}}(f)) = 1$ then $\ell(\bar{\mathcal{S}}(f)) \leq 2 + \ell(f) + \ell(f^2) \leq 2 + \ell(f)(1 + \ell(f))$ and $\rho(\bar{\mathcal{S}}(f))$ is the anisotropic form similar to $< 1, 1 >\perp \rho(f) \perp \rho(-f^2)$.*
• *If $\mathcal{S}(f)$ and $\mathcal{S}(-f)$ are both non-empty and if $\mathrm{w}(\mathcal{S}(f)) = 0$ then $\ell(f) \leq \ell(\mathcal{S}(f)) + \ell(\mathcal{S}(-f))$ and $\rho(f)$ is the anisotropic form similar to $\rho(\mathcal{S}(f)) \perp< -1 > \otimes\rho(\mathcal{S}(-f))$.*
• *If $\mathcal{S}(f)$ and $\mathcal{S}(-f)$ are both non-empty and if $\mathrm{w}(\mathcal{S}(f)) = 1$ and $\mathcal{Z}(f) \neq \emptyset$ then $\ell(f) \leq \inf\{\ell(\mathcal{S}(f)), \ell(\mathcal{S}(-f))\} + \ell(\mathcal{Z}(f)) + 1$ and $\rho(f)$ is the anisotropic form similar to $\rho(\mathcal{S}(f)) \perp< -1 >\perp \rho(\mathcal{Z}(f))$ and $< -1 > \otimes\rho(\mathcal{S}(-f)) \perp< 1 >\perp< -1 > \otimes\rho(\mathcal{Z}(f))$.*
• *If $\mathcal{S}(f)$ and $\mathcal{S}(-f)$ are both non-empty and if $\mathrm{w}(\mathcal{S}(f)) = 1$ and $\mathcal{Z}(f) = \emptyset$ then $\ell(f) \leq \inf\{\ell(\mathcal{S}(f)), \ell(\mathcal{S}(-f))\} + 1$ and $\rho(f)$ is the anisotropic form similar to $\rho(\mathcal{S}(f)) \perp< -1 >$ and $< -1 > \otimes\rho(\mathcal{S}(-f)) \perp< 1 >$.*

Proof. Note that trivially $\ell(f) = \ell(-f)$ and $\ell(f^2) \leq \ell(f)^2$ since $\Lambda(\rho(f) \otimes \rho(f)) = \Lambda(\rho(f^2))$ on X. We give the proof of the last three statements. Assume $\mathcal{S}(f)$ and $\mathcal{S}(-f)$ are both non-empty. By Proposition 6.19 we know that $\mathrm{w}(\mathcal{S}(f)) = \mathrm{w}(\mathcal{S}(-f))$. If $\mathrm{w}(\mathcal{S}(f)) = 0$ then verify that $\Lambda(\rho(\mathcal{S}(f)) \perp<$

$-1> \otimes\rho(\mathcal{S}(-f))) = \Lambda(<f>)$ on X. If $\mathrm{w}(\mathcal{S}(f))=1$ and $\mathcal{Z}(f)\neq\emptyset$ then verify that $\Lambda(\rho(\mathcal{S}(f))\perp< -1>\perp \rho(\mathcal{Z}(f))) = \Lambda(<-1>\otimes\rho(\mathcal{S}(-f))\perp<1>\perp<-1>\otimes\rho(\mathcal{Z}(f)))) = \Lambda(<f>)$ on X. If $\mathrm{w}(\mathcal{S}(f))=1$ and $\mathcal{Z}(f)=\emptyset$ then we can remove the form $\rho(\mathcal{Z}(f))$ from the above formulas. The rest of the proof follows essentially from the arguments given in the proof of Proposition 6.19. $\square$

Remark 6.21. The reader may find more general upper bounds for the length of semi-algebraic sets in [1, Thm. 2.5, Ch. 5]. Note that the result given in [1, Rem. 2.6, Ch. 5] seems to be incorrect. Consider the set $X=\{(0,0)\}\sqcup F$ of Example 2.4 and let $f=x$ restricted to X. We have $\mathcal{Z}(f)=\{(0,0)\}$. We get $\mathrm{w}(\mathcal{Z}(f))=0$ and $\ell(\mathcal{Z}(f))\leq 2$ since $\Lambda(<1,-x^2>)=\mathbf{1}_{\{0,0\}}$ (or use Proposition 6.19). Since $\mathrm{w}(\mathcal{Z}(f))=0$, in [1, Rem. 2.6, Ch. 5] they predict that $\ell(\mathcal{Z}(f))=1$ i.e there exists a polynomial function that does not vanish at the origin and vanishing identically on F, impossible. In this example, $\ell(\mathcal{Z}(f))=2=1+\ell(f^2)$ (the bound given in the first statement of Proposition 6.20 is the best possible in this case).

We may improve the result of Proposition 6.20 if we assume that the regulous function changes of signs sufficiently.

Proposition 6.22. *Let $X\subset\mathbb{R}^n$ be an irreducible real algebraic set. Let $f\in\mathcal{R}^0(X)$ be such that $\dim\mathcal{S}(f)=\dim\mathcal{S}(-f)=\dim X$. Then $\mathrm{w}(\mathcal{S}(f))=\mathrm{w}(\mathcal{S}(-f))=1$, $\ell(\mathcal{Z}(f))\geq 2$, $\ell(\mathcal{S}(f))\geq 2$ and $\ell(\mathcal{S}(-f))\geq 2$.*

Proof. Assume $\mathrm{w}(\mathcal{S}(f))=0$ and $\rho(\mathcal{S}(f))$ is the similarity class of the anisotropic form $<p_1,\ldots,p_k>$, $p_1,\ldots,p_k\in\mathcal{P}(X)$. We claim there exists $x\in\mathcal{S}(f)$ such that $p_i(x)\neq 0$ for $i=1,\ldots,k$. Otherwise $\prod_{i=1}^k p_i$ vanishes identically on $\mathcal{S}(f)$ and thus also on X since by assumption $\overline{\mathcal{S}(f)}^{Zar}=X$. Since $\mathcal{P}(X)$ is an integral domain then it follows that $<p_1,\ldots,p_k>$ is isotropic, a contradiction. Since $\sum_{i=1}^k \mathrm{sign}(p_i)(x)=1$, it follows that k is odd. By the above arguments, there exists $y\in\mathcal{S}(-f)$ such that $p_i(y)\neq 0$ for $i=1,\ldots,k$ and it follows that k is even. Using Proposition 6.19 we conclude that $\mathrm{w}(\mathcal{S}(f))=1$. Hence we get $\ell(\mathcal{S}(f))\geq 2$. Changing f by $-f$ in the above arguments or using Proposition 6.19 we get $\mathrm{w}(\mathcal{S}(-f))=1$ and $\ell(\mathcal{S}(-f))\geq 2$. Assume now that $\ell(\mathcal{Z}(f))=1$. There exists a non-zero $p\in\mathcal{P}(X)$ such that $\Lambda(<p>)=1$ on $\mathcal{Z}(f)$ and $\Lambda(<p>)=0$ on $\mathcal{S}(f)\cup\mathcal{S}(-f)$. It impossible because X is irreducible. $\square$

Proposition 6.23. *Let $X\subset\mathbb{R}^n$ be a real algebraic set. Let $f\in\mathcal{R}^0(X)$. The following properties are equivalent:*
a) $\ell(f)=1$.
b) $\mathcal{Z}(f)$ is Zariski closed.
c) $\ell(X\setminus\mathcal{Z}(f))=1$.

Proof. Equivalence between a) and b) is Theorem 6.1. Assume $\ell(f)=1$. There exists $p\in\mathcal{P}(X)$ such that $\Lambda(<p>)=\Lambda(<f>)$ on X. Thus $\Lambda(<p^2>)=\mathbf{1}_{X\setminus\mathcal{Z}(f)}$ and so $\ell(X\setminus\mathcal{Z}(f))=1$. Assume $\ell(X\setminus\mathcal{Z}(f))=1$. Then clearly $\mathrm{w}(X\setminus\mathcal{Z}(f))=0$ and thus there exists $p\in\mathcal{P}(X)$ such that $\Lambda(<p>)=\mathbf{1}_{X\setminus\mathcal{Z}(f)}$. Hence $\mathcal{Z}(f)=\mathcal{Z}(p)$ i.e $\mathcal{Z}(f)$ is Zariski closed. $\square$

Proposition 6.24. *Let $X\subset\mathbb{R}^n$ be a real algebraic set. Let $f\in\mathcal{R}^0(X)$. Then $\mathcal{S}(f)$ is principal if $\ell(\mathcal{S}(f))\leq 2$.*

Proof. We assume $\mathcal{S}(f)$ is non-empty and $\ell(\mathcal{S}(f))\leq 2$. By Proposition 6.19 we have $\mathrm{w}(\mathcal{S}(f))\leq 1$. We study all the possible couples $(\ell(\mathcal{S}(f)),\mathrm{w}(\mathcal{S}(f)))$.

- Assume $\ell(\mathcal{S}(f))=2$ and $\mathrm{w}(\mathcal{S}(f))=1$. There exist $p,q\in\mathcal{P}(X)$ such that $\Lambda(<p,q>)=2\,\mathbf{1}_{\mathcal{S}(f)}$ and $<p,q>$ is anisotropic. We clearly have $\mathcal{S}(f)\subset\mathcal{S}(p)$ and $\mathcal{S}(f)\subset\mathcal{S}(q)$. We claim that $\mathrm{Bd}(\mathcal{S}(f))\subset\mathcal{Z}(pq)$: Otherwise we may assume there exists $x\in\mathrm{Bd}(\mathcal{S}(f))$ such that $p(x)<0$ and $q(x)>0$. Thus there exists $y\in\mathcal{S}(f)$ such that $p(y)<0$, impossible. Hence $\overline{\mathrm{Bd}(\mathcal{S}(f))}^{Zar}\subset\mathcal{Z}(pq)$. Since $\mathcal{S}(f)\subset\mathcal{S}(p,q)$ then it follows that $\mathcal{S}(f)\cap\overline{\mathrm{Bd}(\mathcal{S}(f))}^{Zar}=\emptyset$. By Theorem 5.7, we conclude that $\mathcal{S}(f)$

is principal.
• The case $\ell(\mathcal{S}(f)) = 1$ and $\mathrm{w}(\mathcal{S}(f)) = 1$ is clearly impossible.
• Assume $\ell(\mathcal{S}(f)) = 1$ and $\mathrm{w}(\mathcal{S}(f)) = 0$. There exists $p \in \mathcal{P}(X)$ such that $\Lambda(< p >) = \mathbf{1}_{\mathcal{S}(f)}$ and thus $\mathcal{S}(f) = \mathcal{S}(p)$.
• Assume $\ell(\mathcal{S}(f)) = 2$ and $\mathrm{w}(\mathcal{S}(f)) = 0$. There exist $p, q \in \mathcal{P}(X)$ such that $\Lambda(< p, q >) = \mathbf{1}_{\mathcal{S}(f)}$ and $< p, q >$ is anisotropic. We clearly have $\mathcal{S}(f) \subset \bar{\mathcal{S}}(p)$ and $\mathcal{S}(f) \subset \bar{\mathcal{S}}(q)$. Thus $\overline{\mathcal{S}(f)}^{eucl} \subset \bar{\mathcal{S}}(p, q)$ and it follows that $\mathrm{Bd}(\mathcal{S}(f)) \subset \bar{\mathcal{S}}(p, q)$. Since $\Lambda(< p, q >) = 0$ on $\mathrm{Bd}(\mathcal{S}(f))$ then we get $\mathrm{Bd}(\mathcal{S}(f)) \subset \overline{\mathrm{Bd}(\mathcal{S}(f))}^{Zar} \subset \mathcal{Z}(p) \cap \mathcal{Z}(q)$. Looking at the signature of the anisotropic form $< p, q >$, it follows that $\mathcal{S}(f) \cap \overline{\mathrm{Bd}(\mathcal{S}(f))}^{Zar} = \emptyset$. By Theorem 5.7, the proof is done. $\square$

Theorem 6.25. *Let $X \subset \mathbb{R}^n$ be a central and irreducible real algebraic set. Let $f \in \mathcal{R}^0(X)$. Then $\mathcal{S}(f)$ is principal if and only if $\ell(\mathcal{S}(f)) \leq 2$.*

Proof. Proposition 6.24 gives one implication. One proves now the converse implication. Assume $\mathcal{S}(f) \neq \emptyset$ and there exists $p \in \mathcal{P}(X)$ such that $\mathcal{S}(f) = \mathcal{S}(p)$. If f is non-negative on X then $\ell(f) = \ell(\mathcal{S}(f)) = 1$ by Corollary 6.12. So we can assume $\mathcal{S}(-f) \neq \emptyset$. Since X is irreducible and central, it follows that $\dim \mathcal{S}(f) = \dim \mathcal{S}(-f) = \dim X$. By Proposition 6.22, we get $\mathrm{w}(\mathcal{S}(f)) = 1$. Since $\Lambda(< p, p^2 >) = 2\,\mathbf{1}_{\mathcal{S}(f)}$ then the proof is done. $\square$

Remark 6.26. The author cautions the reader that [1, Cor. 2.2, Ch. 5] is incorrect. Indeed, the width of a principal semi-algebraic set is not always equal to one: $\mathrm{w}(\mathcal{S}(p)) = 0$ when p is a non-negative polynomial function on a real algebraic set.

7. Signs of rational continuous functions

Throughout this section X will denote a real algebraic subset of dimension d of $\mathbb{R}^n$. In the following statements we will indicate when the hypothesis that X is central is needed. The goal of this section is to compare the signs of rational continuous functions and the signs of regulous functions on X when X is central.

The following statement is a regulous generalization of Lemma 4.3.

Lemma 7.1. *Let S be a closed semi-algebraic subset of X. Let $f, g \in \mathcal{R}^0(X)$. There exist $p \in \mathcal{P}(X)$ and $h \in \mathcal{R}^0(X)$ such that $p > 0$ on X, $h \geq 0$ on X, $\Lambda(< pf + hg >) = \Lambda(< f >)$ on S and $\mathcal{Z}(h) = \overline{\mathcal{Z}(f) \cap S}^{\mathcal{C}}$.*

Proof. As in the proof of Lemma 5.11, we may assume S is a closed semi-algebraic subset of $\mathbb{R}^n$ and $f, g \in \mathcal{R}^0(\mathbb{R}^n)$. Take $h \in \mathcal{R}^0(\mathbb{R}^n)$ such that $\mathcal{Z}(h) = \overline{\mathcal{Z}(f) \cap S}^{\mathcal{C}}$. By [4, Thm. 2.6.6], for a sufficiently big positive even integer N the function $h^N \dfrac{g}{f}$ extended by 0 on $\mathcal{Z}(f)$ is semi-algebraic and continuous on S. The end of the proof is the same as that of Lemma 5.11. $\square$

Lemma 7.2. *Let f be a continuous semi-algebraic function on X satisfying the following* 3 *conditions:*
• *there exists $g_1 \in \mathcal{R}^0(X)$ such that $\mathcal{S}(f) = \mathcal{S}(g_1)$,*
• *there exists $g_2 \in \mathcal{R}^0(X)$ such that $\mathcal{S}(-f) = \mathcal{S}(-g_2)$,*
• *there exists $g_3 \in \mathcal{R}^0(X)$ such that $\mathcal{Z}(f) = \mathcal{Z}(g_3)$.*
Then there exists $g \in \mathcal{R}^0(X)$ such that $\Lambda(< f >) = \Lambda(< g >)$ on X.

Proof. Let S denote the set $\bar{\mathcal{S}}(f)$. The set S is closed and semi-algebraic since f is respectively continuous and semi-algebraic. Remark that $\mathcal{S}(f) = \mathcal{S}(g_1 g_3^2)$ and $\mathcal{Z}(f) \subset \mathcal{Z}(g_1 g_3^2)$ and thus we get $\Lambda(< f >) = \Lambda(< g_1 g_3^2 >)$ on S. By Lemma 7.1, there exist $p \in \mathcal{P}(X)$ and $h \in \mathcal{R}^0(X)$ such that $p > 0$ on X, $h \geq 0$ on X, $\Lambda(< p g_1 g_3^2 + h g_2 >) = \Lambda(< g_1 g_3^2 >) = \Lambda(< f >)$ on S and $\mathcal{Z}(h) = \overline{\mathcal{Z}(g_1 g_3^2) \cap S}^{\mathcal{C}}$. We denote by g the regulous function $p g_1 g_3^2 + h g_2$. We are left to prove that $\Lambda(< g >) = \Lambda(< f >)$ on $\mathcal{S}(-f)$. Let $x \notin S$ i.e $f(x) < 0$. Since $g_2(x) < 0$ and $g_1(x) \leq 0$, it is sufficient to prove that $h(x) > 0$. We have $S \cap \mathcal{Z}(g_1 g_3^2) \subset \mathcal{Z}(f) \cap \mathcal{Z}(g_1 g_3^2) \subset \mathcal{Z}(f) = \mathcal{Z}(g_3)$ and thus $\mathcal{Z}(h) = \overline{\mathcal{Z}(g_1 g_3^2) \cap S}^{\mathcal{C}} \subset \overline{\mathcal{Z}(g_3)}^{\mathcal{C}} = \mathcal{Z}(g_3) = \mathcal{Z}(f)$. It follows that $h(x) > 0$ and the proof is done. □

Proposition 7.3. *Let $X \subset \mathbb{R}^n$ be a central real algebraic set and let $f \in \mathcal{R}_0(X)$. There exists $g \in \mathcal{R}^0(X)$ such that $\Lambda(< f >) = \Lambda(< g >)$ on X if and only if the semi-algebraic sets $\mathcal{S}(f) \cap \operatorname{indet}(f)$ and $\mathcal{S}(-f) \cap \operatorname{indet}(f)$ are $\mathcal{R}^0(\operatorname{indet}(f))$-principal.*

Proof. Let $f \in \mathcal{R}_0(X)$, there exist $p, q \in \mathcal{P}(X)$ such that $f = \dfrac{p}{q}$ on $X \setminus \operatorname{indet}(f)$ and $\mathcal{Z}(q) = \operatorname{indet}(f)$.

If there exists $g \in \mathcal{R}^0(X)$ such that $\Lambda(< f >) = \Lambda(< g >)$ on X then clearly the semi-algebraic sets $\mathcal{S}(f) \cap \operatorname{indet}(f)$ and $\mathcal{S}(-f) \cap \operatorname{indet}(f)$ are $\mathcal{R}^0(\operatorname{indet}(f))$-principal.

Assume for the rest of the proof that the semi-algebraic sets $\mathcal{S}(f) \cap \operatorname{indet}(f)$ and $\mathcal{S}(-f) \cap \operatorname{indet}(f)$ are $\mathcal{R}^0(\operatorname{indet}(f))$-principal. Since the restriction map $\mathcal{R}^0(X) \to \mathcal{R}^0(\operatorname{indet}(f))$ is surjective there exist $g_1, g_2 \in \mathcal{R}^0(X)$ such that $\mathcal{S}(f) \cap \operatorname{indet}(f) = \mathcal{S}(g_1) \cap \operatorname{indet}(f)$ and $\mathcal{S}(-f) \cap \operatorname{indet}(f) = \mathcal{S}(-g_2) \cap \operatorname{indet}(f)$. By Proposition 3.5, there exists $g_3 \in \mathcal{R}^0(X)$ such that $\mathcal{Z}(g_3) = \mathcal{Z}(f)$. We have $\mathcal{S}(f) \cap \overline{\operatorname{Bd}(\mathcal{S}(f))}^{\mathcal{C}} \subset \mathcal{S}(f) \cap \overline{\mathcal{Z}(f)}^{\mathcal{C}} = \mathcal{S}(f) \cap \overline{\mathcal{Z}(g_3)}^{\mathcal{C}} = \mathcal{S}(f) \cap \mathcal{Z}(g_3) = \mathcal{S}(f) \cap \mathcal{Z}(f) = \emptyset$. Since $\mathcal{S}(f) \setminus \operatorname{indet}(f) = \mathcal{S}(pq) \setminus \operatorname{indet}(f)$, it follows from Proposition 5.12 that there exists $h_1 \in \mathcal{R}^0(X)$ such that $\mathcal{S}(f) = \mathcal{S}(h_1)$. The same reasoning gives $h_2 \in \mathcal{R}^0(X)$ such that $\mathcal{S}(-f) = \mathcal{S}(-h_2)$. Since X is central then the function f is semi-algebraic. By Lemma 7.2 the proof is done. □

Corollary 7.4. *Let $X \subset \mathbb{R}^n$ be a central real algebraic set. Let $f \in \mathcal{R}_0(X)$ such that $\dim(\operatorname{indet}(f)) \leq 1$ (it is automatically the case if $\dim X \leq 2$). There exists $g \in \mathcal{R}^0(X)$ such that $\Lambda(< f >) = \Lambda(< g >)$ on X.*

Proof. The restriction of f to $\operatorname{indet}(f)$ is a continuous semi-algebraic function. So the sets $\mathcal{S}(f) \cap \operatorname{indet}(f)$ and $\mathcal{S}(-f) \cap \operatorname{indet}(f)$ are open semi-algebraic subsets of $\operatorname{indet}(f)$. Now since $\dim(\operatorname{indet}(f)) \leq 1$ then the sets $\mathcal{S}(f) \cap \operatorname{indet}(f)$ and $\mathcal{S}(-f) \cap \operatorname{indet}(f)$ are principal by [7]. By Proposition 7.3 the proof is complete. □

Proposition 7.5. *Let $X \subset \mathbb{R}^n$ be a central real algebraic set and let $f \in \mathcal{R}_0(X)$. There exists $g \in \mathcal{R}^0(X)$ such that $\Lambda(< f >) = \Lambda(< g >)$ on X if and only if for any algebraic subset V of X the semi-algebraic sets $\mathcal{S}(f) \cap V$ and $\mathcal{S}(-f) \cap V$ are generically principal.*

Proof. We only prove the "if" part of the proof. Assume that for any algebraic subset V of X the semi-algebraic sets $\mathcal{S}(f) \cap V$ and $\mathcal{S}(-f) \cap V$ are generically principal. Since $\mathcal{S}(f) \cap \overline{\operatorname{Bd}(\mathcal{S}(f))}^{\mathcal{C}} = \emptyset$ and $\mathcal{S}(-f) \cap \overline{\operatorname{Bd}(\mathcal{S}(-f))}^{\mathcal{C}} = \emptyset$ (see the proof of Proposition 7.3, it is a consequence of Proposition 3.5), it follows from Proposition 5.13 that there exist $g_1, g_2 \in \mathcal{R}^0(X)$ such that $\mathcal{S}(f) = \mathcal{S}(g_1)$ and $\mathcal{S}(-f) = \mathcal{S}(-g_2)$. To end the proof use Proposition 3.5 and Lemma 7.2. □

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