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Jean-Philippe Monnier

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SEMI-ALGEBRAIC GEOMETRY WITH RATIONAL CONTINUOUS FUNCTIONS

JEAN-PHILIPPE MONNIER

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ABSTRACT. Let X be a real algebraic subset of \mathbb{R}^n . We investigate on the theory of algebraically constructible functions on X and the description of the semi-algebraic subsets of X when we replace the polynomial functions on X by some rational continuous functions on X . We also study the integral closure of some rings of rational continuous functions on X .

1. INTRODUCTION

The concept of rational continuous maps between smooth real algebraic sets was used the first time by W. Kucharz [8] in order to approximate continuous maps into spheres. In [12], rational continuous functions on smooth real algebraic sets are renamed by “regulous functions” and their systematic study is performed. A theory of vector bundles using these functions is done in [9].

J. Kollár and K. Nowak proved in [7] that the restriction of a regulous function to a real algebraic subset is still rational. It allows us to define the concept of regulous function on a possibly singular affine real algebraic set X by restriction from the ambient space. On X , we have two classes of functions: rational continuous functions and regulous functions. In cite [7] and [10], they give a condition for a rational continuous function to be regulous. In the first section of the present paper we continue the study of differences between these two classes of functions.

The second section is devoted to the study of the integral closure of the ring of regulous functions. In particular, we establish that the ring of regulous functions on an irreducible smooth real algebraic set is integrally closed.

The third section deals with the theory of algebraically constructible functions, due to C. McCrory and A. Parusiński [14]. This theory has been developed to study singular real algebraic sets. We prove that the theory of algebraically constructible functions can be done using only regulous objects (functions, maps, sets). In particular, we show that the sign of a regulous function is a sum of signs of polynomial functions and we investigate on the number of polynomial functions needed in such representation.

In the last section, we focus on the description of semi-algebraic sets when we replace polynomial functions by regulous functions.

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2. REGULOUS FUNCTIONS VERSUS RATIONAL CONTINUOUS FUNCTIONS

2.1. Regulous functions. Let $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{\infty\}$, we recall the definition of k -regulous functions on \mathbb{R}^n (see [12]).

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Definition 2.1. We say that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is k -regulous on \mathbb{R}^n if f is C^k on \mathbb{R}^n and f is a rational function on \mathbb{R}^n , i.e. there exists a non-empty Zariski open subset $U \subseteq \mathbb{R}^n$ such that $f|_U$ is regular (or f is the C^k -extension to \mathbb{R}^n of a rational function on \mathbb{R}^n).

A 0-regulous function on \mathbb{R}^n is simply called a regulous function on \mathbb{R}^n .

An equivalent definition of a k -regulous function on \mathbb{R}^n is given in [13, Thm. 2.15].

We denote by $\mathcal{R}^k(\mathbb{R}^n)$ the ring of k -regulous functions on \mathbb{R}^n . By Theorem 3.3 of [12] we know that $\mathcal{R}^\infty(\mathbb{R}^n)$ coincides with the ring $\mathcal{O}(\mathbb{R}^n)$ of regular functions on \mathbb{R}^n .

Denote by $\mathcal{Z}(f)$ the zero set of the real function f . For an integer k , the k -regulous topology of \mathbb{R}^n is defined to be the topology whose closed subsets are generated by the zero sets of regulous functions in $\mathcal{R}^k(\mathbb{R}^n)$. Although the k' -regulous topology is a priori finer than the k -regulous topology when $k' < k$, it has been proved in [12] that in fact they are the same. Hence, it is not necessary to specify the integer k to define the regulous topology on \mathbb{R}^n . By [12, Thm. 6.4], the regulous topology on \mathbb{R}^n is the algebraically constructible topology on \mathbb{R}^n (denoted by \mathcal{C} -topology). On \mathbb{R}^n , the euclidean topology is finer than the \mathcal{AR} -topology (the arc-symmetrical topology (see [12])) which is finer than the regulous topology which is the \mathcal{C} -topology which is finer than the Zariski topology.

We give now the definition of a regulous function on an affine algebraic variety [12, Cor. 5.38].

Definition 2.2. Let X be a real algebraic subset of \mathbb{R}^n . A k -regulous function on X is the restriction to X of a k -regulous function on \mathbb{R}^n . The ring of k -regulous functions on X , denoted by $\mathcal{R}^k(X)$, corresponds to

$$\mathcal{R}^k(X) = \mathcal{R}^k(\mathbb{R}^n) / \mathcal{I}_k(X)$$

where $\mathcal{I}_k(X)$ is the ideal of $\mathcal{R}^k(\mathbb{R}^n)$ of k -regulous functions on \mathbb{R}^n that vanish identically on X .

Remark 2.3. The previous definition can be extended to the case X is a closed regulous subset of \mathbb{R}^n [12].

Let $X \subset \mathbb{R}^n$ be a real algebraic set, we will denote by $\mathcal{O}(X)$ the ring of regular functions on X , by $\mathcal{P}(X)$ the ring of polynomial functions on X and by $\mathcal{K}(X)$ the ring of rational functions on X . By [7], a regulous function on X is always rational on X (coincides with a regular function on a dense Zariski open subset of X). Since the regulous topology on X is sometimes strictly finer than the Zariski topology on X , the ring $\mathcal{R}^0(X)$ is not always a subring of $\mathcal{K}(X)$ even if X is irreducible.

Example 2.4. Let X be the plane cubic with an isolated point $X = \mathcal{Z}(x^2 + y^2 - x^3)$. The curve X is irreducible in the Zariski topology but reducible in the \mathcal{C} -topology. The \mathcal{C} -irreducible components of X are F and $\{(0,0)\}$ where $F = \mathcal{Z}(f) \subset \mathbb{R}^2$, with $f = 1 - \frac{x^3}{x^2 + y^2}$, is the smooth branch of X . The

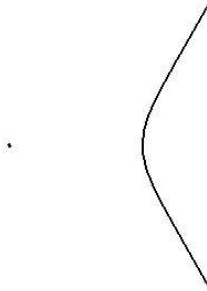


FIGURE 1. Cubic curve with an isolated point.

ring $\mathcal{R}^0(X)$ is the cartesian product $\mathcal{R}^0(F) \times \mathbb{R}$ and the class of f in $\mathcal{R}^0(X)$ is $(0,1)$. Remark that the ring $\mathcal{R}^0(X)$ is not an integral domain and consequently it is not a subring of $\mathcal{K}(X)$.

Let X be a real algebraic subset of \mathbb{R}^n . We have a natural ring morphism $\phi^0 : \mathcal{R}^0(X) \rightarrow \mathcal{K}(X)$ which send $f \in \mathcal{R}^0(X)$ to the class $(U, f|_U)$ in $\mathcal{K}(X)$, where U is a dense Zariski open subset of X and $f|_U$ is regular. We have seen that ϕ^0 is not always injective. In the following, we will denote by \overline{E}^τ the closure of the subset E of \mathbb{R}^n for the topology τ on \mathbb{R}^n . We prove now that ϕ^0 is injective when $\overline{X_{reg}}^{\mathcal{C}} = X$, X_{reg} denoting the closure of the smooth locus of X . This condition means that the Zariski irreducible components of X are also irreducible for the \mathcal{C} -topology.

Proposition 2.5. *Let X be a real algebraic subset of \mathbb{R}^n . The map $\phi^0 : \mathcal{R}^0(X) \rightarrow \mathcal{K}(X)$ is injective if and only if $\overline{X_{reg}}^{\mathcal{C}} = X$.*

Proof. Assume $\overline{X_{reg}}^{\mathcal{C}} = X$. Let $f_1, f_2 \in \mathcal{R}^0(X)$ be such that $\phi^0(f_1) = \phi^0(f_2)$. Let $\hat{f}_i \in \mathcal{R}^0(\mathbb{R}^n)$, $i = 1, 2$, be such that $\hat{f}_i|_F = f_i$. Since f_1 and f_2 are two continuous extensions to X of the same rational function on X , they coincide on X_{reg} . Hence $\hat{f}_1 - \hat{f}_2$ vanishes on X since X is the regulous closure of X_{reg} . It implies that $f_1 = f_2$.

Assume $\overline{X_{reg}}^{\mathcal{C}} \neq X$. By [12, Thm. 6.13], we may write $X = \overline{X_{reg}}^{\mathcal{C}} \cup F$ with F a non-empty regulous closed subset of \mathbb{R}^n such that $\dim F < \dim X$. Let $\hat{f} \in \mathcal{R}^0(\mathbb{R}^n)$ be such that $\mathcal{Z}(\hat{f}) = \overline{X_{reg}}^{\mathcal{C}}$ and let f denote the restriction of \hat{f} to X . We have $f \neq 0$ in $\mathcal{R}^0(X)$, $\phi^0(f) = 0$ in $\mathcal{K}(X)$ and thus ϕ^0 is non injective. \square

2.2. Rational continuous functions on central algebraic varieties. Let n be a positive integer and let $X \subset \mathbb{R}^n$ be a real algebraic set. Let $f \in \mathcal{K}(X)$ be a rational function on X . The domain of f , denoted by $\text{dom}(f)$, is the biggest dense Zariski open subset of X on which f is regular, namely $f = \frac{p}{q}$ where p and q are polynomial functions on \mathbb{R}^n such that q does not vanish on $\text{dom}(f)$. The indeterminacy locus of f is defined to be the Zariski closed set $\text{indet}(f) = X \setminus \text{dom}(f)$.

Definition 2.6. Let f be a real continuous function on X . We say that f is a rational continuous function on X if f is rational on X i.e there exists a dense Zariski open subset $U \subseteq X$ such that $f|_U$ is regular.

Let $\mathcal{R}_0(X)$ denote the ring of rational continuous functions on X . We have a natural ring morphism $\phi_0 : \mathcal{R}_0(X) \rightarrow \mathcal{K}(X)$ which send $f \in \mathcal{R}_0(X)$ to the class $(U, f|_U)$ in $\mathcal{K}(X)$, where U is a dense Zariski open subset of X and $f|_U$ is regular.

Remark 2.7. We have $\mathcal{R}_0(\mathbb{R}^n) = \mathcal{R}^0(\mathbb{R}^n)$.

Definition 2.8. We say that X is ‘‘central’’ if $\overline{X_{reg}}^{eucl} = X$.

Remark 2.9. The previous definition comes from the introduction of the the central locus of a real algebraic set made in [4, Def. 7.6.3].

The property to be central is the property of an algebraic set that ensures a rational continuous function on it to be the unique possible continuous extension of its associated rational function. The following example illustrates this fact.

Example 2.10. Let $X = \mathcal{Z}(zx^2 - y^2) \subset \mathbb{R}^3$ be the Whitney umbrella. By [12], X is irreducible in the \mathcal{C} -topology and we have

$$\overline{X_{reg}}^{AR} = \overline{X_{reg}}^{\mathcal{C}} = \overline{X_{reg}}^{Zar} = X.$$

The set $X \setminus \overline{X_{reg}}^{eucl}$ is the half of the stick. The class of the rational fraction $\frac{y^2}{x^2}|_X$ in $\mathcal{K}(X)$ (which is rational) has polar locus equal to the stick. It is also the class of the regular function z . This rational function can be extended continuously to X by z but also if we set it to be equal to z on $\overline{X_{reg}}^{eucl}$ and 0 on $X \setminus \overline{X_{reg}}^{eucl}$. Consequently, the map $\phi_0 : \mathcal{R}_0(X) \rightarrow \mathcal{K}(X)$ is not injective.

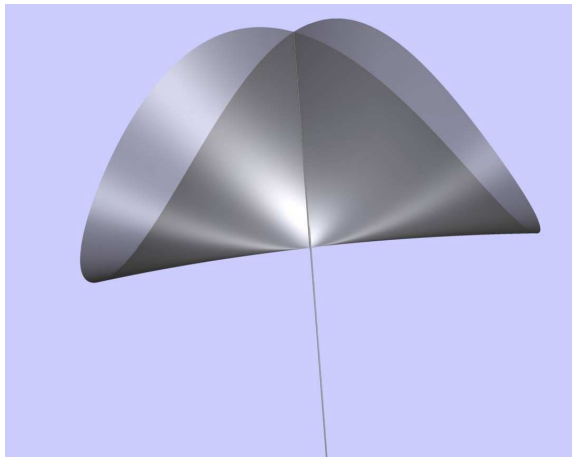


FIGURE 2. Whitney umbrella.

Proposition 2.11. *The map $\phi_0 : \mathcal{R}_0(X) \rightarrow \mathcal{K}(X)$ is injective if and only if X is central.*

Proof. With the hypothesis $X = \overline{X_{reg}}^{eucl}$, in case a rational function of $\mathcal{K}(X)$ has a continuous extension to X then this extension is the unique possible extension.

Assume X is not central. It is always possible to extend the null function on $\overline{X_{reg}}^{eucl}$ to a continuous function f on X such that f is not the null function on X . The function f is rational on X since it is a regular function on X_{reg} . Obviously, f is a non-trivial element of the kernel of ϕ_0 and the proof is done. \square

Obviously, any $f \in \mathcal{R}^0(X)$ can be identified with a unique function in $\mathcal{R}_0(X)$. Hence we get:

Proposition 2.12. *We have the following ring inclusion $\phi_0^0 : \mathcal{R}^0(X) \hookrightarrow \mathcal{R}_0(X)$ and moreover*

$$\phi^0 = \phi_0 \circ \phi_0^0.$$

Remark 2.13. Let X be a real algebraic subset of \mathbb{R}^n such that $\overline{X_{reg}}^c = X$ and X is not central (e.g the Whitney umbrella). By Propositions 2.12 and 2.11, we see that in this case the map ϕ_0^0 is not surjective i.e there is a rational continuous function on X which is not regulous.

In the following example, due to Kollár and Nowak [7, Ex. 2], we will see that, even if X is central, ϕ_0^0 may be not surjective.

Example 2.14. Let $X = \mathcal{Z}(x^3 - (1 + z^2)y^3) \subset \mathbb{R}^3$. Then X is a central singular surface with singular locus the z -axis. By [7, Ex. 2], the class of the rational function $\frac{x}{y}|_X$ in $\mathcal{K}(X)$ can be extended continuously to X (in a unique way) by the function $(1 + z^2)^{\frac{1}{3}}$ on the z -axis and gives an element $f \in \mathcal{R}_0(X)$. Moreover, f cannot be extended to an element of $\mathcal{R}_0(\mathbb{R}^3) = \mathcal{R}^0(\mathbb{R}^3)$ (the reason is that the restriction of f to the z -axis $(1 + z^2)^{\frac{1}{3}}$ is not rational) and thus f is not in $\mathcal{R}^0(X)$. Here the map $\phi_0^0 : \mathcal{R}^0(X) \hookrightarrow \mathcal{R}_0(X)$ is not surjective and the map $\phi_0 : \mathcal{R}_0(X) \rightarrow \mathcal{K}(X)$ is injective.

One of the goal of the paper [7] was to study the surjectivity of the map ϕ_0^0 when X is a central real algebraic set. Notice that “regulous functions” are named “hereditarily rational continuous functions” in [7].

We reformulate with our notation the three principal results of [7] with an improvement of the first one.

Lemma 2.15. ([7, Prop. 11])

Let $X \subset \mathbb{R}^n$ be a real algebraic set and let $f \in \mathcal{R}_0(X)$. Let $W = \text{indet}(f)$ be the polar locus of f in

X . If $f|_W \in \mathcal{R}^0(W)$ has the additional property that $f|_W$ is the restriction to W of $g \in \mathcal{R}^0(\mathbb{R}^n)$ such that g is regular on $\mathbb{R}^n \setminus W$ then

$$f \in \mathcal{R}^0(X).$$

Moreover, f has also the additional property that there exists $\hat{f} \in \mathcal{R}^0(\mathbb{R}^n)$ such that \hat{f} is regular on $\mathbb{R}^n \setminus W$ and $\hat{f}|_X = f$.

We improve Lemma 2.15 by removing the additional property from the hypotheses.

Lemma 2.16. *Let $X \subset \mathbb{R}^n$ be a real algebraic set and let $f \in \mathcal{R}_0(X)$. Let $W = \text{indet}(f)$ be the polar locus of f in X . If $f|_W \in \mathcal{R}^0(W)$ then*

$$f \in \mathcal{R}^0(X).$$

Proof. Assume $f|_W \in \mathcal{R}^0(W)$. By definition, there exists $g \in \mathcal{R}^0(\mathbb{R}^n)$ such that $g|_W = f|_W$. We denote by g_0 the regulous function $g|_W$. We consider the following sequence of regulous functions

$$(g_0, g_1 = (g_0)|_{\text{indet}(g_0)}, g_2 = (g_1)|_{\text{indet}(g_1)}, \dots)$$

on a sequence of Zariski closed subsets ($W_i = \text{indet}(g_{i-1})$) of W of dimension strictly decreasing and included one in another. The functions g_i are regulous since they are also a restriction of a regulous function on \mathbb{R}^n . We claim that there exists an integer m such that g_m is a regular function on W_m . Indeed, g_m is automatically regular if $\dim W_m = 0$ and we get the claim since $\dim W_{i+1} < \dim W_i$. By [4, Prop. 3.2.3], g_m is the restriction to $W_m = \text{indet}(g_{m-1})$ of regular function \hat{g}_m on \mathbb{R}^n . By Lemma 2.15 for $f = g_{m-1}$, $X = W_{m-1}$ and $W = W_m$, we get that g_{m-1} is the restriction to W_{m-1} of a regulous function \hat{g}_{m-1} on \mathbb{R}^n regular on $\mathbb{R}^n \setminus \text{indet}(g_{m-1})$. Using Lemma 2.15 several times, we get that $g_0 = f|_W$ is the restriction to W of a regulous function \hat{g}_0 on \mathbb{R}^n regular on $\mathbb{R}^n \setminus \text{indet}(f|_W)$. Since $\text{indet}(f|_W) \subset \text{indet}(f) = W$, using one last time Lemma 2.15, we get the proof. \square

Proposition 2.17. ([7, Prop. 8])

Let $X \subset \mathbb{R}^n$ be a real algebraic set and let $f \in \mathcal{R}_0(X)$. For any irreducible real algebraic subset $W \subset X$ not contained in the singular locus of X , we have

$$f|_W \in \mathcal{R}_0(W).$$

Theorem 2.18. ([7, Prop. 8, Thm. 10])

Let $X \subset \mathbb{R}^n$ be a smooth real algebraic set. Then the map $\phi_0^0 : \mathcal{R}^0(X) \leftrightarrow \mathcal{R}_0(X)$ is a ring isomorphism.

Proof. By [7, Prop. 8], a rational continuous function on a smooth real algebraic set is hereditarily rational. By [7, Thm. 10], a continuous hereditarily rational function on a non necessary smooth real algebraic set $X \subset \mathbb{R}^n$ is the restriction of rational continuous function on \mathbb{R}^n and thus “continuous hereditarily rational” means “regulous”. \square

We extend the result of Theorem 2.18 to real algebraic sets with isolated singularities using Lemma 2.16.

Theorem 2.19. *Let $X \subset \mathbb{R}^n$ be a real algebraic set with only isolated singularities. Then*

$$\mathcal{R}^0(X) = \mathcal{R}_0(X).$$

Proof. Let $f \in \mathcal{R}_0(X)$. Let $W \subset X$ be a real algebraic subset. If $\dim W = 0$ then $f|_W$ is regular and thus $f|_W \in \mathcal{R}_0(W)$. If W is irreducible and $\dim W \geq 1$ then $f|_W \in \mathcal{R}_0(W)$ by Proposition 2.17. It follows that $f|_W \in \mathcal{R}_0(W)$ without hypothesis on W . We consider the following sequence of continuous rational functions

$$(f_0 = f, f_1 = f|_{\text{indet}(f)}, f_2 = (f_1)|_{\text{indet}(f_1)}, \dots)$$

on a sequence of real algebraic subsets ($W_i = \text{indet}(f_{i-1})$) of X of dimension strictly decreasing and included one in another. There exists an integer m such that f_m is regular on W_m . Using several times Lemma 2.16, we get that $f \in \mathcal{R}^0(X)$. \square

Corollary 2.20. *Let $X \subset \mathbb{R}^n$ be a real algebraic curve. Then*

$$\mathcal{R}^0(X) = \mathcal{R}_0(X).$$

3. INTEGRAL CLOSURE OF THE RING OF RATIONAL CONTINUOUS FUNCTIONS

3.1. Blow-regular functions on central real algebraic sets. By [12, thm. 3.11], regulous functions on a smooth real algebraic set $X \subset \mathbb{R}^n$ coincide with blow-regular functions on X , it gives another equivalent definition for regulous functions on X .

Definition 3.1. Let $X \subset \mathbb{R}^n$ be a smooth real algebraic set. Let $f : X \rightarrow \mathbb{R}$ be a real function. We say that f is regular after blowings-up on X if there exists a composition $\pi : M \rightarrow X$ of successive blowings-up along smooth centers such that $f \circ \pi$ is regular on M . We denote by $\mathcal{B}(X)$ the ring of blow-regular functions of X .

Theorem 3.2. [12, thm. 3.11]

Let $X \subset \mathbb{R}^n$ be a smooth real algebraic set. We have $\mathcal{R}^0(X) = \mathcal{B}(X)$.

Now we will give a definition of blow-regular function for a non-necessarily smooth real algebraic set.

Definition 3.3. Let $X \subset \mathbb{R}^n$ be a central real algebraic set. Let $\mathcal{B}(X)$ denote the ring of real functions f defined on X such that, there exists a resolution of singularities $\pi : \tilde{X} \rightarrow X$ such that the composite $f \circ \pi$ is in $\mathcal{B}(\tilde{X}) = \mathcal{R}^0(\tilde{X}) = \mathcal{R}_0(\tilde{X})$. A $f \in \mathcal{B}(X)$ is called a “blow-regular function” on X .

Remark 3.4. Using the definition of blow-regular function on a smooth variety we get: $f \in \mathcal{B}(X)$ if and only if f is a real function defined on X such that there exists a resolution of singularities $\pi : \tilde{X} \rightarrow X$ such that $f \circ \pi$ is regular. This justifies the notation “blow-regular”.

Remark 3.5. In the definition 3.3 we can change \exists by \forall . It is not true in the equivalent definition of the remark 3.4.

We prove in the following that, even in the singular case, blow-regular functions and rational continuous functions coincide.

Proposition 3.6. *Let $X \subset \mathbb{R}^n$ be a central real algebraic set. We have*

$$\mathcal{B}(X) = \mathcal{R}_0(X).$$

Proof. Assume $f \in \mathcal{R}_0(X)$ and let $\pi : \tilde{X} \rightarrow X$ be a resolution of singularities. Then clearly $f \circ \pi$ is rational on \tilde{X} . Moreover, we have $\pi^{-1}(X) = \tilde{X}$ since $\pi^{-1}(X)$ is a closed set for the strong topology and contains the Zariski open subset $\pi^{-1}(X_{reg})$. We can conclude that $f \circ \pi$ is continuous on \tilde{X} and thus $f \circ \pi \in \mathcal{R}_0(\tilde{X})$.

Assume $f \in \mathcal{B}(X)$ and let $\pi : \tilde{X} \rightarrow X$ be a resolution of singularities. Then $f \circ \pi \in \mathcal{R}_0(\tilde{X})$ and thus f is rational on X . Moreover, f is continuous on X since $f \circ \pi$ is continuous on \tilde{X} and since the strong topology on X is the quotient topology induced by the strong topology on \tilde{X} (the function $f \circ \pi$ is constant on the fibers of π and thus gives rise of a rational continuous function on X and this function is obviously f). \square

3.2. Integral closure. We begin by recalling some classical results on the integral closure of some ring of polynomial functions and regular functions on real algebraic sets.

In the following, we will denote by \overline{A} the integral closure of an integral domain A in its field of fractions.

Let $X \subset \mathbb{R}^n$ be a real algebraic set. We say that X is “geometrically smooth” if the associated complex algebraic set $X_{\mathbb{C}} = X \times_{\text{Spec } \mathbb{R}} \text{Spec } \mathbb{C}$ is smooth.

Proposition 3.7. *Let $X \subset \mathbb{R}^n$ be a geometrically smooth irreducible real algebraic set. Then $\mathcal{P}(X)$ is integrally closed in $\mathcal{K}(X)$.*

Proof. It is well-known that if $X_{\mathbb{C}}$ is smooth then $\mathcal{P}(X_{\mathbb{C}})$ is integrally closed in $\mathcal{K}(X_{\mathbb{C}})$ (see [17, thm. 1 Ch. 2 Sect. 5] for example). Hence $\mathcal{P}(X)$ is also integrally closed in $\mathcal{K}(X)$. \square

Example 3.8. The following real algebraic curve $X = \mathcal{Z}(y^2 - (x^2 + 1)^2(x - 1))$ is smooth but not geometrically smooth. The ring $\mathcal{P}(X)$ (instead of $\mathcal{O}(X)$) is not integrally closed in $\mathcal{K}(X)$ since the non-polynomial but regular function $f = \frac{y}{x^2 + 1}$ is integral on $\mathcal{P}(X)$ since $f^2 - (x - 1) = 0$.

Proposition 3.9. *Let $X \subset \mathbb{R}^n$ be a smooth irreducible real algebraic set. Then $\mathcal{O}(X)$ is integrally closed in $\mathcal{K}(X)$.*

Proof. By [4, 3.3.6, 3.3.7], for any $x \in X$ the local ring $\mathcal{O}_{X,x}$ of germs of regular functions at x is integrally closed in $\mathcal{K}(X)$. Therefore

$$\overline{\mathcal{O}(X)} = \bigcap_{x \in X} \overline{\mathcal{O}_{X,x}} \subset \bigcap_{x \in X} \overline{\mathcal{O}_{X,x}} = \bigcap_{x \in X} \mathcal{O}_{X,x} = \mathcal{O}(X).$$

\square

Proposition 3.10. *Let $X \subset \mathbb{R}^n$ be an irreducible real algebraic set. If $\mathcal{P}(X)$ is integrally closed in $\mathcal{K}(X)$ then $\mathcal{O}(X)$ is also integrally closed in $\mathcal{K}(X)$.*

Proof. By [4, prop. 3.2.3], $\mathcal{O}(X) = S^{-1}\mathcal{P}(X)$ where S is the multiplicative part of $\mathcal{P}(X)$ of polynomial functions that do not vanish on X . We get the proof using [2, prop. 5.12]. \square

Theorem 3.11. *Let $X \subset \mathbb{R}^n$ be a smooth irreducible real algebraic set. Then $\mathcal{R}_0(X) = \mathcal{R}^0(X)$ is integrally closed in $\mathcal{K}(X)$.*

Proof. Assume $f \in \mathcal{K}(X)^*$ and there exist $d \in \mathbb{N}^*$ and $a_i \in \mathcal{R}^0(X)$, $i = 0, \dots, d - 1$ such that

$$f^d + a_{d-1}f^{d-1} + \dots + a_0 = 0$$

in $\mathcal{K}(X)$. It means that there exists a non-empty Zariski open subset U of X such that $\forall x \in U$ we have $f^d(x) + a_{d-1}(x)f^{d-1}(x) + \dots + a_0(x) = 0$. By Proposition 3.6, there exists a composition of blowings-up with smooth centers $\pi : Y \rightarrow X$ such that $\tilde{a}_i = a_i \circ \pi$ is regular on Y for $i = 0, \dots, d - 1$. Then $f \circ \pi$ is a rational function on Y which is integral on $\mathcal{O}(Y)$. Since Y is smooth then $\mathcal{O}(Y)$ is integrally closed in $\mathcal{K}(Y)$ (Proposition 3.9) and thus $f \circ \pi$ can be extended to a regular function \tilde{f} on Y . Obviously, we have $\forall y \in Y$ $\tilde{f}^d(y) + \tilde{a}_{d-1}(y)\tilde{f}^{d-1}(y) + \dots + \tilde{a}_0(y) = 0$. Let $x \in X$. Since each \tilde{a}_i is constant on $\pi^{-1}(x)$ and $\pi^{-1}(x)$ is connected then \tilde{f} is constant on $\pi^{-1}(x)$ using the continuity of $\tilde{f}^d + \tilde{a}_{d-1}\tilde{f}^{d-1} + \dots + \tilde{a}_0$. Hence it gives a real continuous function g on X such that $\tilde{f} = g \circ \pi$ and g is a continuous extension to X of f . \square

We can extend the previous result to some singular cases.

Theorem 3.12. *Let $X \subset \mathbb{R}^n$ be a central irreducible real algebraic set such that there exists a resolution of singularities $\pi : \tilde{X} \rightarrow X$ such that $\forall x \in X$ the fiber $\pi^{-1}(x)$ is connected. Then $\mathcal{R}_0(X)$ is integrally closed in $\mathcal{K}(X)$.*

Proof. Assume $f \in \mathcal{K}(X)^*$ there exist $d \in \mathbb{N}^*$ and $a_i \in \mathcal{R}_0(X)$, $i = 0, \dots, d - 1$ such that

$$f^d + a_{d-1}f^{d-1} + \dots + a_0 = 0$$

in $\mathcal{K}(X)$. Let $\pi : \tilde{X} \rightarrow X$ a resolution of singularities such that $\forall x \in X$ the fiber $\pi^{-1}(x)$ is connected. By the previous theorem, $f \circ \pi$ can be extended continuously to \tilde{X} as a regulous function $\tilde{f} \in \mathcal{R}^0(\tilde{X})$. We get the proof since \tilde{f} will be constant on each fiber of π (the fibers are connected and $\forall i$ the regulous function $a_i \circ \pi$ is constant on each fiber). \square

Example 3.13. Let X be the cuspidal plane curve given by $y^2 - x^3 = 0$. By Theorem 3.12 we know that $\mathcal{R}^0(X) = \mathcal{R}_0(X)$ is integrally closed.

Example 3.14. Let X be the nodal plane curve given by $y^2 - (x+1)x^2 = 0$. The rational function $f = \frac{y}{x}$ is integral on $\mathcal{O}(X)$ (and also on $\mathcal{R}_0(X)$) since $f^2 - (x+1) = 0$ on $X \setminus \{(0,0)\}$. It is easy to see that f cannot be extended continuously to whole X . Hence $\mathcal{R}_0(X) = \mathcal{R}^0(X)$ is not integrally closed. Of course the fiber over the node is never connected when we solve the node.

Looking at the previous examples, we want to prove that for singular curves, the hypothesis on the fibers made in the statement of Theorem 3.12 is a necessary hypothesis to get that the ring of rational continuous functions is integrally closed.

Let $X \subset \mathbb{R}^n$ be an irreducible real algebraic set. Let $\mathcal{P}(\tilde{X})$ be the integral closure of the coordinate ring $\mathcal{P}(X)$ in $\mathcal{K}(X)$. So $\mathcal{P}(\tilde{X})$ is the coordinate ring of an irreducible normal real algebraic set \tilde{X} called the normalisation of X (we say that an irreducible real algebraic set $Y \subset \mathbb{R}^n$ is normal if $\mathcal{P}(Y)$ is integrally closed in $\mathcal{K}(Y)$). Obviously X and \tilde{X} are birationally equivalent real algebraic sets and we have a (regular) “normalization” map $\pi : \tilde{X} \rightarrow X$ induced by the injection $\mathcal{P}(X) \hookrightarrow \mathcal{P}(\tilde{X})$. Assume now X is central, the normalisation map induces an injective ring morphism

$$\pi_0 : \mathcal{R}_0(X) \hookrightarrow \mathcal{R}_0(\tilde{X}), f \mapsto f \circ \pi.$$

The hypothesis that X is central is necessary because it ensures that $\mathcal{R}_0(X)$ injects into $\mathcal{K}(X)$ (Proposition 2.5). Remark that, if in addition X is a curve, then the situation is simpler because then $\mathcal{R}_0(X) = \mathcal{R}^0(X)$ and $\mathcal{R}_0(\tilde{X}) = \mathcal{R}^0(\tilde{X}) = \mathcal{O}(\tilde{X})$ since \tilde{X} is smooth.

If a fiber of the normalisation map is not connected then we prove that the ring of rational continuous functions is not integrally closed.

Proposition 3.15. *Let $X \subset \mathbb{R}^n$ be a central irreducible real algebraic set. Let $\pi : \tilde{X} \rightarrow X$ be the normalization map. If there exists $x \in X$ such that the fiber $\pi^{-1}(x)$ is not connected then $\mathcal{R}_0(X)$ is not integrally closed in $\mathcal{K}(X)$ and moreover the injective map $\pi_0 : \mathcal{R}_0(X) \hookrightarrow \mathcal{R}_0(\tilde{X})$ is not surjective.*

Proof. We know that the fibers of π are finite since π is a finite map. Assume there exists $x \in X$ such that we have $\{y_1, y_2\} \subset \pi^{-1}(x)$ and $y_1 \neq y_2$. There exists $p \in \mathcal{P}(\tilde{X})$ such that $p(y_1) \neq p(y_2)$. Obviously p is integral on $\mathcal{R}_0(X)$ and moreover $p \notin \pi_0(\mathcal{R}_0(X))$ since p is not constant on the fibers of π . \square

Theorem 3.16. *Let $X \subset \mathbb{R}^n$ be a central irreducible real algebraic set. Let $\pi : \tilde{X} \rightarrow X$ be the normalization map. The map $\pi_0 : \mathcal{R}_0(X) \hookrightarrow \mathcal{R}_0(\tilde{X})$ is an isomorphism if and only if $\forall x \in X$ the fiber $\pi^{-1}(x)$ is a point.*

Proof. Assume that $\forall x \in X$ the fiber $\pi^{-1}(x)$ is a point. Then any element of $\mathcal{R}_0(\tilde{X})$ is rational on X , continuous on \tilde{X} and constant on the fibers of π and so it gives a rational function on X that can be extended continuously to X . Thus π_0 is surjective. Using Proposition 3.15, the proof is done. \square

For curves, we know exactly when the ring of rational continuous functions is integrally closed (use Theorem 3.12 and Theorem 3.16).

Theorem 3.17. *Let $X \subset \mathbb{R}^n$ be a central irreducible real algebraic curve. Let $\pi : \tilde{X} \rightarrow X$ be the normalization map. The ring $\mathcal{R}_0(X)$ is integrally closed in $\mathcal{K}(X)$ if and only if $\forall x \in X$ the fiber $\pi^{-1}(x)$ is a point.*

Proposition 3.18. *Let $X \subset \mathbb{R}^n$ be a central irreducible real algebraic curve. Let $\pi : \tilde{X} \rightarrow X$ be the normalization map. The integral closure of $\mathcal{R}^0(X)$ in $\mathcal{K}(X)$ is $\mathcal{R}^0(\tilde{X}) = \mathcal{O}(\tilde{X})$.*

Proof. If $f \in \mathcal{O}(\tilde{X})$ then f is integral over $\mathcal{O}(X)$ [2, prop. 5.12] and thus over $\mathcal{R}^0(X)$. The proof follows from the fact that $\mathcal{O}(\tilde{X})$ is integrally closed in $\mathcal{K}(X)$ (Proposition 3.9). \square

To end the section, we ask some questions:

Q_1 : Let $X \subset \mathbb{R}^n$ be a central irreducible real algebraic set. Is the injective map $\phi_0^0 : \mathcal{R}^0(X) \rightarrow \mathcal{R}_0(X)$

integral?

Q_2 : Let $X \subset \mathbb{R}^n$ be a central normal irreducible real algebraic set. Is the ring $\mathcal{R}_0(X)$ integrally closed?

Concerning the question Q_2 in the two-dimensional case, we know by Theorem 3.12 that the ring of rational continuous functions is integrally closed if the fibers over the singular points of a resolution of singularities are connected. However, as we shall see in the following example, it may happen that the exceptional set is not connected (for the strong topology) when we solve a singular point of a normal surface.

Example 3.19. Let X be the real algebraic surface in \mathbb{R}^3 defined by $X = \mathcal{Z}(z^2 - xy(x^2 - y^2))$. The surface X is the double cover of the plane ramified along the union of four lines passing through the origin. The point $(0, 0, 0)$ is the only singular point of $X_{\mathbb{C}}$ and thus X is normal. Let $\varphi : W \rightarrow \mathbb{R}^3$ be the blowing-up of \mathbb{R}^3 at the origin. We have

$$W = \{((x, y, z)(u : v : w) \in \mathbb{R}^3 \times \mathbb{P}^2(\mathbb{R}) \mid xv = yu, xw = zu, yw = zv)\}.$$

Let Y denote the strict transform of X by φ and $E = \varphi^{-1}((0, 0, 0)) \cap Y$. In the chart $u \neq 0$ i.e. $u = 1$,

Y is given by
$$\begin{cases} y = xv \\ z = xw \\ w^2 - vx^2(1 - v^2) = 0 \end{cases}$$
 and E is the line $w = x = y = z = 0$. Jacobi criterium gives

that E is the singular locus of Y . Let $\psi : V \rightarrow W$ be the blowing-up of W at E . The strict transform \tilde{X} of Y by ψ is a resolution of X . Let F denote the exceptional curve $\psi^{-1} \circ \varphi^{-1}((0, 0, 0)) \cap \tilde{X}$. We can also see \tilde{X} as the double cover of the blowing-up of the plane at the origin ramified along the strict transforms of the four lines. Hence $F_{\mathbb{C}}$ (or its smooth completion) is an elliptic curve as the double cover of \mathbb{P}^1 at four points and it is easy to see that F has two connected components.

4. ALGEBRAICALLY CONSTRUCTIBLE FUNCTIONS

We make reminders on the theory of constructible and algebraically constructible functions due to C. McCrory and A. Parusiński (see [14], [15]). This theory was used to study singular real algebraic sets. We follow the definitions and the results given in [6].

Let S be a semi-algebraic set. A constructible function on S is a function $f : S \rightarrow \mathbb{Z}$ that can be written as a finite sum

$$\varphi = \sum_{i \in I} m_i \mathbf{1}_{S_i}$$

where for each $i \in I$, m_i is an integer and $\mathbf{1}_{S_i}$ is the characteristic function of a semi-algebraic subset S_i of S . The set of constructible functions on S provided with the sum and the product form a commutative ring denoted by $F(S)$. If $\varphi = \sum_{i \in I} m_i \mathbf{1}_{S_i}$ is a constructible function then the Euler integral of φ on S is

$$\int_S \varphi d\chi = \sum_{i \in I} m_i \chi(\mathbf{1}_{S_i})$$

where χ is the Euler characteristic with compact support. Let $f : S \rightarrow T$ be a continuous semi-algebraic map and $\varphi \in F(S)$. The pushforward $f_*\varphi$ of φ along f is the function from T to \mathbb{Z} defined by

$$f_*\varphi(y) = \int_{f^{-1}(y)} \varphi d\chi.$$

It is known that $f_*\varphi \in F(T)$ and that $f_* : F(S) \rightarrow F(T)$ is a morphism of additive groups.

Let $X \subset \mathbb{R}^n$ be a real algebraic set. We say that a constructible function φ on X is algebraically constructible if it can be written as a finite sum

$$\varphi = \sum_{i \in I} m_i f_{i*}(\mathbf{1}_{X_i})$$

where f_i are regular maps from real algebraic sets X_i to X . Algebraically constructible functions on X form a ring denoted by $A(X)$. We say that a constructible function φ on X is strongly algebraically constructible if it can be written as a finite sum

$$\varphi = \sum_{i \in I} m_i \mathbf{1}_{X_i}$$

where X_i are real algebraic subsets of X . Strongly algebraically constructible functions on X form a subring of $A(X)$ denoted by $AS(X)$. Let A be a ring of semi-algebraic functions on X . A (quadratic) form on A is a finite family of elements of A denoted by $\langle f_1, \dots, f_n \rangle$ with $f_i \in A$. We denote by $FQ(A)$ the ring of forms on A provided with the two operations \perp and \otimes :

$$\langle f_1, \dots, f_n \rangle \perp \langle g_1, \dots, g_m \rangle = \langle f_1, \dots, f_n, g_1, \dots, g_m \rangle,$$

$$\langle f_1, \dots, f_n \rangle \otimes \langle g_1, \dots, g_m \rangle = \langle f_1 g_1, \dots, f_n g_1, f_1 g_2, \dots, f_n g_2, \dots, f_n g_m \rangle.$$

Since A is a ring of semi-algebraic functions on X , we have a signature map

$$\Lambda : FQ(A) \rightarrow F(X)$$

$$\langle f_1, \dots, f_n \rangle \mapsto (x \mapsto \text{sign}(f_1(x)) + \dots + \text{sign}(f_n(x)))$$

which is a morphism of rings.

Parusiński and Szafraniec have proved that algebraically constructible functions correspond to sums of signs of polynomial functions.

Theorem 4.1. [16, Thm. 6.1]

Let $X \subset \mathbb{R}^n$ be a real algebraic set. Then

$$A(X) = \Lambda(FQ(\mathcal{P}(X))) = \Lambda(FQ(\mathcal{O}(X))).$$

We prove now that algebraically constructible functions correspond to sums of signs of regulous functions. It is a very natural result since the topology generated by zero sets of regulous functions is the algebraically constructible topology.

Theorem 4.2. Let $X \subset \mathbb{R}^n$ be a real algebraic set. Then

$$A(X) = \Lambda(FQ(\mathcal{R}^0(X))).$$

Proof. We make an induction on the dimension of X . If $\dim(X) = 0$ then there is nothing to do since regulous means regular.

Assume $\dim(X) > 0$ and let $f \in \mathcal{R}^0(X)$. Let W denote $\text{indet}(f)$. There exist $p, q \in \mathcal{P}(X)$ such that $f = \frac{p}{q}$ on $\text{dom}(f)$ and $\mathcal{Z}(q) = W$. We have $f|_W \in \mathcal{R}^0(W)$ and by induction there exists $h_1, \dots, h_n \in \mathcal{P}(W)$ such that $\Lambda(\langle f|_W \rangle) = \Lambda(\langle h_1, \dots, h_n \rangle)$. The polynomial functions h_i are restrictions of polynomial functions on X again denoted by h_i . The proof is done since

$$\Lambda(\langle f \rangle) = \Lambda(\langle pq \rangle \perp \langle 1, -q^2 \rangle \otimes \langle h_1, \dots, h_n \rangle).$$

□

In the next section, we will count the number of signs of polynomial functions we need in the sum to be the sign of a regulous function.

We prove now that strongly algebraically constructible functions are exactly finite sums of characteristic functions of regulous closed sets.

Proposition 4.3. *Let $X \subset \mathbb{R}^n$ be a real algebraic set. Then*

$$AS(X) = \left\{ \sum_{i \in I} m_i \mathbf{1}_{W_i}, I \text{ finite}, m_i \in \mathbb{Z}, W_i \subset X \text{ regulous closed} \right\}.$$

Proof. Let W be a closed regulous subset of X . By [12, Thm. 4.1], there exist a finite stratification $W = \coprod_{i \in I} S_i$ with S_i Zariski locally closed subsets of X . It means that $S_i = Z_i \cap (X \setminus Z'_i)$ where Z_i and Z'_i are real algebraic subsets of X . Then

$$\mathbf{1}_W = \sum_{i \in I} \mathbf{1}_{S_i} = \sum_{i \in I} (\mathbf{1}_{Z_i} (\mathbf{1}_X - \mathbf{1}_{Z'_i})) = \sum_{i \in I} (\mathbf{1}_{Z_i} - \mathbf{1}_{Z_i \cap Z'_i}) \in AS(X).$$

□

We characterize algebraically constructible functions using regulous closed sets and regulous maps.

Theorem 4.4. *Let $X \subset \mathbb{R}^n$ be a real algebraic set. Then*

$$A(X) = \left\{ \sum_{i \in I} m_i f_{i*}(\mathbf{1}_{W_i}), I \text{ finite}, m_i \in \mathbb{Z}, W_i \text{ regulous closed}, f_i : W_i \rightarrow X \text{ regulous map} \right\}.$$

Proof. By Proposition 4.3 and since f_* is additive, it is sufficient to prove that $f_*(\mathbf{1}_Y) \in A(X)$ when $f : Y \rightarrow X$ is a regulous map between two real algebraic sets. We make an induction on the dimension of Y . If $\dim(Y) = 0$ then f is regular and there is nothing to prove. Assume $\dim(Y) > 0$. We may also assume that Y is irreducible. By [12, Thm. 3.11], there exists a regular birational map $\pi : \tilde{Y} \rightarrow Y$ such that $f \circ \pi$ is a regular map (we solve the singularities of Y and then we use [12, Thm. 3.11]). The birational map π is biregular from $\tilde{Y} \setminus \pi^{-1}(Z)$ to $Y \setminus Z$ with Z a real algebraic subset of Y of positive codimension. Then

$$f_*(\mathbf{1}_Y) = (f \circ \pi)_*(\mathbf{1}_{\tilde{Y}}) - (f \circ \pi)_*(\mathbf{1}_{\pi^{-1}(Z)}) + f_*(\mathbf{1}_Z)$$

and $f_*(\mathbf{1}_Z) \in A(X)$ by the induction hypothesis. □

Now we look at sum of signs of rational continuous functions.

The proof of the next result is due to G. Fichou.

Proposition 4.5. *Let $X \subset \mathbb{R}^n$ be a central real algebraic set. Let $f \in \mathcal{R}_0(X)$ be a rational continuous function on X . Then $\mathcal{Z}(f) \in \mathcal{C}$.*

Proof. Remark that $\mathcal{Z}(f)$ is clearly a semi-algebraic set. Denote by A the \mathcal{C} -closure of $\mathcal{Z}(f)$ in X . We want to show that $A = \mathcal{Z}(f)$.

Let B be an irreducible \mathcal{C} -component of A of maximal dimension, i.e of the same dimension d as $\mathcal{Z}(f)$. We are going to show that $\overline{\text{Reg } B}^{\text{eucl}} \subset \mathcal{Z}(f)$ where $\text{Reg } B = \{x \in B : \dim_x B = d \text{ and } B \text{ is smooth at } x\}$. First, note that the dimension of $\mathcal{Z}(f) \cap B$ is also d . Actually

$$\overline{\mathcal{Z}(f) \cap B}^{\mathcal{C}} \cup \overline{\mathcal{Z}(f) \setminus B}^{\mathcal{C}} = \overline{\mathcal{Z}(f)}^{\mathcal{C}},$$

and B is not included in $\overline{\mathcal{Z}(f) \setminus B}^{\mathcal{C}}$, so that B is included in $\overline{\mathcal{Z}(f) \cap B}^{\mathcal{C}}$. As a consequence the dimension of $\overline{\mathcal{Z}(f) \cap B}^{\mathcal{C}}$ is also d , and therefore the same holds for $\mathcal{Z}(f) \cap B$ (cf. [11], Remark 2.4).

Now let $\sigma : \tilde{X} \rightarrow X$ be a resolution of the singularities of X such that $f \circ \sigma$ is regular on \tilde{X} , as in Proposition 3.6. We can assume moreover, performing more blowings-up if necessary, that σ restricts to a resolution of the singularities of the Zariski closure of B in X . This is possible because B is included in X , meaning that σ is surjective on X . By [11], there exists a connected component \tilde{B} of \tilde{X} such that $\sigma(\tilde{B})$ is equal to the Euclidean closure of $\text{Reg } B$. Note moreover that f vanishes on a semi-algebraic subset of B of maximal dimension, so the same is true for $f \circ \sigma$ on \tilde{B} . Since $f \circ \sigma$ is regular and \tilde{B} is irreducible (smooth and connected), then $f \circ \sigma$ vanishes on \tilde{B} . As a consequence $\overline{\text{Reg } B}^{\text{eucl}} \subset \mathcal{Z}(f)$.

So the difference between $\mathcal{Z}(f)$ and its \mathcal{C} -closure A is of dimension strictly less than the dimension d of $\mathcal{Z}(f)$. Denote by C the \mathcal{C} -closure of that difference, so that $A = \mathcal{Z}(f) \cup C$, where C is a \mathcal{C} -set of dimension strictly less than d . Now, let D be one of the irreducible \mathcal{C} -components of C of maximal dimension (i.e. of dimension $\dim C$).

Note that $\dim D \cap \mathcal{Z}(f) = \dim D$. Otherwise (as previously) one may replace in C the component D by the \mathcal{C} -closure of $D \cap \mathcal{Z}(f)$ which would be strictly smaller (notably in dimension), in contradiction with the fact that $\dim D$ is maximal. Let us play the same game as previously: since $D \subset X$, there exists a resolution of the singularities $\sigma : \tilde{X} \rightarrow X$ of X such that $f \circ \sigma$ is regular on \tilde{X} , and σ restricts to a resolution of the singularities of the Zariski closure of D in X . By [11], there exists a smooth connected real algebraic subset \tilde{D} in \tilde{X} such that $\sigma(\tilde{D}) = \overline{\text{Reg } D}^{\text{eucl}}$.

For the same reason as before, $f \circ \sigma$ vanishes on \tilde{D} so f vanishes on $\overline{\text{Reg } D}^{\text{eucl}}$, in contradiction with the fact that D is an irreducible \mathcal{C} -component of maximal dimension not included in $\mathcal{Z}(f)$. Therefore $C = \emptyset$ and the proof is achieved. \square

Theorem 4.6. *Let $X \subset \mathbb{R}^n$ be a central real algebraic set. Then*

$$A(X) = \Lambda(FQ(\mathcal{R}_0(X))).$$

Proof. Let $f \in \mathcal{R}_0(X)$, we have to prove $\Lambda(\langle f \rangle) \in A(X)$. Let $Y = \{(x, t) \in X \times \mathbb{R} \mid f(x) = t^2\}$. By Proposition 4.5, $Y \in \mathcal{C}$ i.e Y is a closed regulous subset of the central real algebraic set $X \times \mathbb{R}$. By Proposition 4.3, $\mathbf{1}_Y \in AS(X \times \mathbb{R})$. We get

$$\Lambda(\langle f \rangle) = \pi_*(\mathbf{1}_Y) - \mathbf{1}_X \in A(X)$$

where $\pi : X \times \mathbb{R} \rightarrow X$ is the projection. \square

Example 4.7. We go back to Example 2.14. We have $f = \frac{x}{y} \in \mathcal{R}_0(X) \setminus \mathcal{R}^0(X)$ but $f^3 \in \mathcal{R}^0(X)$. We have $\Lambda(\langle f \rangle) = \Lambda(\langle f^3 \rangle) \in A(X)$.

5. SIGNS OF REGULOUS FUNCTIONS

Throughout this section X will denote a real algebraic subset of dimension d of \mathbb{R}^n . By Theorem 4.4, the sign of a regulous function on X can be written as a sum of signs of polynomial functions on X . The goal of this section is to give an upper bound depending on d , on the number of polynomial functions needed in such representation.

Let f_1, \dots, f_m be semi-algebraic functions on X . In the rest of the paper, we will use the following notation:

$$\begin{aligned} \mathcal{S}(f_1, \dots, f_m) &= \{x \in X \mid f_1(x) > 0, \dots, f_m(x) > 0\} \\ \bar{\mathcal{S}}(f_1, \dots, f_m) &= \{x \in X \mid f_1(x) \geq 0, \dots, f_m(x) \geq 0\}. \end{aligned}$$

If all the functions f_i lie in a ring A of semi-algebraic functions, the set $\mathcal{S}(f_1, \dots, f_m)$ (resp. $\bar{\mathcal{S}}(f_1, \dots, f_m)$) is called A -open basic (resp. A -closed basic). If $m = 1$, we replace ‘‘basic’’ by ‘‘principal’’. If $A = \mathcal{P}(X)$ then we omit A .

The zero-dimensional case is trivial since regulous means regular. In the following example, we prove that for curves the sign of a regulous function is not always the sign of a polynomial function.

Example 5.1. (Example 2.4)

Let $X = \mathcal{Z}(y^2 - x^2(x-1))$ and let f be the restriction to X of the plane regulous function $1 - \frac{x^3}{x^2 + y^2}$. The function f is zero on the one-dimensional connected component of X and has value 1 on the isolated point of X . Clearly if a polynomial function g has the sign of f on the one-dimensional connected component of X then g vanishes on whole X since X is Zariski irreducible. However the sign of f is the sum of signs of two polynomial functions on X , more precisely we have $\Lambda(\langle f \rangle) = \Lambda(\langle 1, -(x^2 + y^2) \rangle)$ on X .

The pathology of the previous example is the fact that the curve is Zariski irreducible but \mathcal{C} -reducible i.e the fact that X is not central since $\dim(X) = 1$ (see [12]). We will prove now that, under the hypothesis that the curve is central, the sign of a regulous function coincide with the sign of a polynomial function.

We will use several times the following lemma which is a consequence of Łojasiewicz inequality.

Lemma 5.2. [3, Lem. 7.7.10]

Let S be a closed semi-algebraic subset of X . Let $f, g \in \mathcal{P}(X)$. There exist $p, q \in \mathcal{P}(X)$ such that $p > 0$ on X , $q \geq 0$ on X , $\Lambda(\langle pf + qg \rangle) = \Lambda(\langle f \rangle)$ on S and $\mathcal{Z}(q) = \overline{\mathcal{Z}(f) \cap S}^{\text{Zar}}$.

Proposition 5.3. Assume $\dim(X) = 1$ and X is central. Let $f \in \mathcal{R}^0(X)$. There exists $h \in \mathcal{P}(X)$ such that $\Lambda(\langle h \rangle) = \Lambda(\langle f \rangle)$ on X .

Proof. By [5], any open semi-algebraic subset of X is principal and thus there exists $p_1 \in \mathcal{P}(X)$ such that $\mathcal{S}(f) = \mathcal{S}(p_1)$. Since X is central then the Zariski irreducible components of X correspond exactly to the \mathcal{C} -irreducible components of X and thus $\mathcal{Z}(f)$ is Zariski closed (see [12], a \mathcal{C} -irreducible component of dimension 1 of $\mathcal{Z}(f)$ is necessarily Zariski irreducible). So we can multiply p_1 by the square of a polynomial equation of $\mathcal{Z}(f)$ and we get:

$$\mathcal{S}(f) = \mathcal{S}(p_1) \text{ and } \mathcal{Z}(f) \subset \mathcal{Z}(p_1).$$

Similarly there exists $p_2 \in \mathcal{P}(X)$ such that $\mathcal{S}(-f) = \mathcal{S}(-p_2)$. Let S denote the closed semi-algebraic set $\bar{\mathcal{S}}(f)$. By Lemma 5.2, there exist $p, q \in \mathcal{P}(X)$ such that $p > 0$ on X , $q \geq 0$ on X , $\Lambda(\langle pp_1 + qp_2 \rangle) = \Lambda(\langle p_1 \rangle)$ on S and $\mathcal{Z}(q) = \overline{\mathcal{Z}(p_1) \cap S}^{\text{Zar}}$. Let h denote the polynomial function $pp_1 + qp_2$. We want to prove that $\Lambda(\langle h \rangle) = \Lambda(\langle f \rangle)$ on X . We have $\Lambda(\langle h \rangle) = \Lambda(\langle p_1 \rangle) = \Lambda(\langle f \rangle)$ on S since $\mathcal{S}(f) = \mathcal{S}(p_1)$ and since $\mathcal{Z}(f) \subset \mathcal{Z}(p_1)$. Assume now $x \notin S$. So $f(x) < 0$ and thus $p_2(x) < 0$ (since $\mathcal{S}(-f) = \mathcal{S}(-p_2)$), $p_1(x) \leq 0$ (since $\bar{\mathcal{S}}(-f) = \bar{\mathcal{S}}(-p_1)$). The proof is done if we prove that $q(x) > 0$ since in that case we would have $h(x) < 0$. We have $S \cap \mathcal{Z}(p_1) = \bar{\mathcal{S}}(f) \cap \mathcal{Z}(p_1) \subset \mathcal{Z}(f) \cap \mathcal{Z}(p_1) = \mathcal{Z}(f)$ since $\mathcal{S}(f) = \mathcal{S}(p_1)$ (you can not have simultaneously $f(y) > 0$ and $p_1(y) = 0$). We have already noticed that $\mathcal{Z}(f)$ is Zariski closed, hence

$$\mathcal{Z}(q) = \overline{\mathcal{Z}(p_1) \cap S}^{\text{Zar}} \subset \overline{\mathcal{Z}(f)}^{\text{Zar}} = \mathcal{Z}(f)$$

and it follows that $x \notin \mathcal{Z}(q)$. □

Remark 5.4. We have the same result if we replace “ X is central” by “ $\mathcal{Z}(f)$ is Zariski closed” in the hypotheses of Proposition 5.3. Look at Theorem 6.8 for a generalization of Proposition 5.3 in any dimension and in the case $\mathcal{Z}(f)$ is Zariski closed.

Example 5.5. Let $X = \mathcal{Z}(x^2 - y^3) \subset \mathbb{R}^2$ be the cuspidal curve and let $f = \frac{y^2}{x}|_X$. We have $f \in \mathcal{R}^0(X) \setminus \mathcal{P}(X)$ but $\Lambda(\langle f \rangle) = \Lambda(\langle x \rangle)$ on X .

Proposition 5.6. Assume $\dim(X) = 1$ and let $f \in \mathcal{R}^0(X)$. There exist $h_1, h_2 \in \mathcal{P}(X)$ such that $\Lambda(\langle h_1, h_2 \rangle) = \Lambda(\langle f \rangle)$ on X .

Proof. By the previous results we may assume X is not central and $\mathcal{Z}(f)$ is not Zariski closed. We also assume that X is irreducible to simplify the proof. By [12], $X = F \amalg \{x_1, \dots, x_m\}$ where $F = \overline{X_{\text{reg}}}^{\text{eucl}}$ is the one-dimensional irreducible regulous component of X and x_1, \dots, x_m are the isolated points of X . Since $\mathcal{Z}(f)$ is not Zariski closed, we must have $\dim \mathcal{Z}(f) = 1$ and thus $F \subset \mathcal{Z}(f)$ (see [12]). For each x_i let $p_i \in \mathcal{P}(X)$ such that $p_i \geq 0$ on X and $\mathcal{Z}(p_i) = \{x_i\}$. We set h_1 to be the product of the p_i such that $f(x_i) \leq 0$ and h_2 to be the $(-1) \times$ the product of the p_i such that $f(x_i) \geq 0$. For this choice of h_1 and h_2 , we get the proof. □

Definition 5.7. Let $f \in \mathcal{R}^0(X)$.

We set $f_0 = f$, $X_0 = X$ and $X_1 = \text{indet}(f_0)$.

If $X_1 \neq \emptyset$ i.e if f_0 is not regular on X_0 then we set $f_1 = f_0|_{X_1} \in \mathcal{R}^0(X_1)$ and $X_2 = \text{indet}(f_1)$. Repeating the process it will stop after at most d steps since $\dim(X_{i+1}) < \dim(X_i)$ and $X_{i+1} = \emptyset$ if $\dim X_i = 0$.

At the step of index i we associate to the regular function f_i on X_i a rational representation $(p_i, q_i) \in \mathcal{P}(X) \times \mathcal{P}(X)$ such that $f_i = \frac{p_i}{q_i}$ on $X_i \setminus X_{i+1}$.

The following sequence

$$((f_0, X_0, p_0, q_0), \dots, (f_k, X_k, p_k, q_k))$$

is called a ‘‘polar sequence’’ associated to f . We have $X_i \neq \emptyset$ for $i = 1, \dots, k$ and $X_{k+1} = \emptyset$ i.e f_k is regular on X_k .

The number k of the previous sequence is called the ‘‘polar depth’’ of f and denoted by $\text{poldepth}(f)$.

Remark 5.8. If $f \in \mathcal{R}^0(X)$ then obviously $\text{poldepth}(f) \leq d$.

Proposition 5.9. *If $f \in \mathcal{R}^0(X)$ then $\text{codim}(\text{indet}(f) \setminus \text{Sing}(X)) \geq 2$.*

Proof. We may assume X is irreducible and suppose $\dim(\text{indet}(f) \setminus \text{Sing}(X)) = d - 1$. Under this hypothesis there exists a resolution of singularities $\pi : \tilde{X} \rightarrow X$ of X and also of $\text{indet}(f)$ such that $\tilde{f} = f \circ \pi \in \mathcal{R}^0(\tilde{X})$, $\text{indet}(\tilde{f}) = Z$ where Z is the strict transform of $\text{indet}(f)$ and $\dim Z = d - 1$. Let W be an irreducible component of Z of dimension $d - 1$. Since the local ring $\mathcal{O}_{\tilde{X}, W}$ is a discrete valuation ring, we may write the rational function $\tilde{f} = t^m u$ with t an uniformizing parameter of $\mathcal{O}_{\tilde{X}, W}$, $m < 0$ and u a unit of $\mathcal{O}_{\tilde{X}, W}$. There exists a Zariski open subset U of W where u doesn't vanish and thus it is impossible to extend continuously the rational function $t^m u$ to W , it gives a contradiction. \square

Proposition 5.10. *Let $X \subset \mathbb{R}^n$ be a real algebraic set of dimension d . Let $f \in \mathcal{R}^0(X)$, $k = \text{poldepth}(f)$ and $((f_0, X_0, p_0, q_0), \dots, (f_k, X_k, p_k, q_k))$ a ‘‘polar sequence’’ associated to f . Then*

$$\Lambda(\langle f \rangle) = \Lambda(\langle p_0 q_0 \rangle \perp_{i=1}^k (\langle 1, -q_{i-1}^2 \rangle \otimes \langle p_i q_i \rangle)).$$

Therefore, we may write the sign of f on X as the sum of l signs of polynomial functions on X with

$$l \leq 1 + 2 \text{poldepth}(f).$$

Proof. The proof is straightforward since we have

$$\Lambda(\langle f \rangle) = \Lambda(\langle p_0 q_0 \rangle \perp_{i=1}^j (\langle 1, -q_{i-1}^2 \rangle \otimes \langle p_i q_i \rangle))$$

on $X \setminus X_{j+1}$ for $j = 1, \dots, k$ and $X_{k+1} = \emptyset$. \square

It follows from Proposition 5.10 and Proposition 5.6:

Theorem 5.11. *Let $X \subset \mathbb{R}^n$ be a real algebraic set of dimension d . Let $f \in \mathcal{R}^0(X)$. We may write the sign of f on X as the sum of l signs of polynomial functions on X with*

$$\begin{aligned} l &= 1 \text{ if } d = 0, \\ l &\leq 2 \text{ if } d = 1, \\ l &\leq 2d + 1 \text{ else.} \end{aligned}$$

In the next section, we will improve the result of Theorem 5.11 in the two dimensional case.

It follows from Proposition 5.10, Proposition 5.3 and Proposition 5.9:

Theorem 5.12. *Let $X \subset \mathbb{R}^n$ be a real algebraic set of dimension d such that $\text{codim}(\text{Sing}(X)) > 1$. Let $f \in \mathcal{R}^0(X)$. We may write the sign of f on X as the sum of l signs of polynomial functions on X with*

$$\begin{aligned} l &= 1 \text{ if } d = 0 \text{ or } 1, \\ l &\leq 2d - 1 \text{ else.} \end{aligned}$$

In the next section, we will improve the result of Theorem 5.12 in the three dimensional case.

Example 5.13. We prove the optimality of the bound given in Theorem 5.12 for $X = \mathbb{R}^2$. Consider the regulous function $f = -1 + \frac{x^3}{x^2 + y^2}$. We have a partition $\mathbb{R}^2 = E_{-1} \amalg E_0 \amalg E_1$ of \mathbb{R}^2 where E_i is the semi-algebraic subset of \mathbb{R}^2 where $\Lambda(\langle f \rangle) = i$. We can not write $\Lambda(\langle f \rangle) = \Lambda(\langle h \rangle)$ with $h \in \mathbb{R}[x, y]$ since E_0 is not Zariski closed.

We can not write $\Lambda(\langle f \rangle) = \Lambda(\langle h_1, h_2 \rangle)$ with $h_1, h_2 \in \mathbb{R}[x, y]$ since it would imply that $h_1 h_2$ vanishes on $E_{-1} \cup E_1$ and thus vanishes on whole \mathbb{R}^2 .

Following the statement of Proposition 5.10, we get

$$\Lambda(\langle f \rangle) = \Lambda(\langle -x^2 - y^2 + x^3, -1, x^2 + y^2 \rangle).$$

6. REGULOUS PRINCIPAL SEMI-ALGEBRAIC SETS

Let $X \subset \mathbb{R}^n$ be a real algebraic set of dimension d . In this section we study the following question: Is a regulous principal open semi-algebraic subset of X a principal open semi-algebraic subset of X ? Remark first that it is equivalent to same question but for a closed subset by taking the complementary set. If $d = 0$ the answer is trivially “yes”. For $d = 1$ the answer is also “yes” by [5] since in this case any open (resp. closed) semi-algebraic subset of X is principal.

For $d = 2$ the answer can be negative:

Example 6.1. As usual consider $X = \mathbb{R}^2$ and $f = 1 - \frac{x^3}{x^2 + y^2}$. Let $S = \mathcal{S}(f)$. Since $S \cap \overline{\text{Bd}(S)}^{\text{Zar}} = \{(0, 0)\} \neq \emptyset$ then S cannot be basic [5, Prop. 2.2] ($\text{Bd}(S) = \overline{S}^{\text{eucl}} \setminus S$).

In the following we will prove that under the topological condition “ $S \cap \overline{\text{Bd}(S)}^{\text{Zar}} = \emptyset$ ”, the answer to the previous question, for the regulous principal open semi-algebraic set S , is “yes”.

Remark 6.2. Let $f \in \mathcal{R}^0(X)$. Set $S = \mathcal{S}(f)$ and assume $f = \frac{p}{q}$ on $\text{dom}(f)$ with $p, q \in \mathcal{P}(X)$ and $\mathcal{Z}(q) = \text{indet}(f)$. If we assume in addition that $S \cap \overline{\text{Bd}(S)}^{\text{Zar}} = \emptyset$, we will prove in the following that there exists $r \in \mathcal{P}(X)$ such that $S = \mathcal{S}(r)$ but it may happen that we can not choose r to be equal to pq . Consider $X = \mathbb{R}^2$, $f = \frac{y^2 + x^2(1-x)^2}{x^2 + y^2} = \frac{p}{q}$. We have $S = \mathcal{S}(f) = \mathbb{R}^2 \setminus \{(1, 0)\}$, $\overline{\text{Bd}(S)}^{\text{Zar}} = \{(1, 0)\}$ and $\mathcal{S}(pq) = \mathbb{R}^2 \setminus \{(1, 0), (0, 0)\}$.

We will use several times the following consequence of Hörmander-Łojasiewicz inequality.

Lemma 6.3. [1, Prop. 1.16, Chap. 2]

Let C be a closed semi-algebraic subset of X and let $f, g \in \mathcal{P}(X)$ such that $\mathcal{Z}(f) \cap C \subset \mathcal{Z}(g)$. There exist $h \in \mathcal{P}(X)$ and $l \in \mathbb{N}$ odd such that

$$\Lambda(\langle (1 + h^2)f + g^l \rangle) = \Lambda(\langle f \rangle)$$

on C .

Theorem 6.4. Let $f \in \mathcal{R}^0(X)$ and $S = \mathcal{S}(f)$. There exists $r \in \mathcal{P}(X)$ such that

$$\mathcal{S}(r) \subset S \text{ and } S \setminus \mathcal{S}(r) \subset \overline{\text{Bd}(S)}^{\text{Zar}} \cap \text{indet}(f).$$

More precisely, if $((f_0, X_0, p_0, q_0), \dots, (f_k, X_k, p_k, q_k))$ is a polar sequence associated to f then, for $i = 0, \dots, k$, there exists $r_i \in \mathcal{P}(X)$ such that

$$\mathcal{S}(r_i) \cap X_i \subset S \cap X_i \text{ and } (S \setminus \mathcal{S}(r_i)) \cap X_i \subset \overline{\text{Bd}(S)}^{\text{Zar}} \cap X_{i+1}.$$

Proof. We set $S_i = S \cap X_i$ for $i = 0, \dots, k$. We make the proof by decreasing induction on $i = k, \dots, 0$.

- For $i = k$ there is nothing to do since f_k is regular on X_k .
- Assume $i \in \{0, \dots, k-1\}$ and there exists $r_{i+1} \in \mathcal{P}(X)$ such that

$$\mathcal{S}(r_{i+1}) \cap X_{i+1} \subset S \cap X_{i+1} \text{ and } (S \setminus \mathcal{S}(r_{i+1})) \cap X_{i+1} \subset \overline{\text{Bd}(S)}^{\text{Zar}} \cap X_{i+2}.$$

Let F denote the closed semi-algebraic subset of X_i defined by $F = \overline{\mathcal{S}(r_{i+1}) \cap X_i}^{\text{eucl}} \cap (X_i \setminus S_i)$.

We have

$$(1) \quad X_{i+1} \cap F \subset \mathcal{Z}(r_{i+1}) \cap X_i.$$

If $x \in X_{i+1} \cap F$ then $x \in X_{i+1}$ and $x \notin S_i \cap X_{i+1} = S_{i+1}$. By induction hypothesis we have $\mathcal{S}(r_{i+1}) \cap X_{i+1} \subset S_{i+1}$ and thus $r_{i+1}(x) \leq 0$. Since $x \in \overline{\mathcal{S}(r_{i+1}) \cap X_i}^{\text{eucl}}$ then $x \in \overline{\mathcal{S}(r_{i+1}) \cap X_i}^{\text{eucl}} \setminus (\mathcal{S}(r_{i+1}) \cap X_i) = \text{Bd}(\mathcal{S}(r_{i+1}) \cap X_i)$ i.e. $r_{i+1}(x) = 0$ and it proves (1).

Since $X_{i+1} = \mathcal{Z}(-q_i^2)$ then Lemma 6.3 provides us $h' \in \mathcal{P}(X)$, l' an odd positive integer such that $r'_{i+1} = (1 + h'^2)(-q_i^2) + r_{i+1}^{l'}$ verifies $\Lambda(\langle r'_{i+1} \rangle) = \Lambda(\langle -q_i^2 \rangle)$ on F . Since $r'_{i+1} = r_{i+1}$ on X_{i+1} then r'_{i+1} satisfies the same induction hypotheses than r_{i+1} namely

$$(2) \quad \mathcal{S}(r'_{i+1}) \cap X_{i+1} \subset S_{i+1}$$

and

$$(3) \quad (S_{i+1} \setminus \mathcal{S}(r'_{i+1})) \subset \overline{\text{Bd}(S)}^{\text{Zar}} \cap X_{i+2}.$$

We claim that r'_{i+1} satisfies the third property

$$(4) \quad \mathcal{S}(r'_{i+1}) \cap X_i \subset S_i.$$

If $x \in \mathcal{S}(r'_{i+1}) \cap X_i$ then $r_{i+1}(x)$ must be > 0 and if $x \notin S_i$ then $x \in F$ and the sign of $r'_{i+1}(x)$ is the sign of $-q_i^2(x)$, which is impossible. We have proved (4).

We set $C = \overline{S_i}^{\text{eucl}} \setminus (\mathcal{S}(r'_{i+1}) \cap X_i)$. Let $t \in \mathcal{P}(X)$ such that $\mathcal{Z}(t) = \overline{\mathcal{Z}(p_i)} \cap C^{\text{Zar}}$. Since $f_i \in \mathcal{R}^0(X_i)$ then $\mathcal{Z}(q_i) \cap X_i \subset \mathcal{Z}(p_i) \cap X_i$ [12, Prop. 3.5] and thus we get $\mathcal{Z}(p_i q_i) \cap C \subset \mathcal{Z}(t) \subset \mathcal{Z}(t^2 r'_{i+1})$. By Lemma 6.3, there exist $h \in \mathcal{P}(X)$ and l an odd positive integer such that $r_i = (1 + h^2)p_i q_i + t^{2l} r'_{i+1}$ verifies $\Lambda(\langle r_i \rangle) = \Lambda(\langle p_i q_i \rangle)$ on C . We prove now that r_i is the function we are looking for.

Assume $x \in X_i \setminus S_i$. If $x \in X_{i+1}$ then $p_i(x)q_i(x) = 0$, else $x \in X_i \setminus (S_i \cup X_{i+1})$ and the sign of $p_i(x)q_i(x)$ is the sign of $f_i(x)$; thus $p_i(x)q_i(x) \leq 0$. By (4) we get $r'_{i+1}(x) \leq 0$ and thus $r_i(x) \leq 0$. We have proved that

$$(5) \quad \mathcal{S}(r_i) \cap X_i \subset S_i.$$

We have to prove

$$(6) \quad (S_i \setminus \mathcal{S}(r_i)) \cap X_i \subset \overline{\text{Bd}(S)}^{\text{Zar}} \cap X_{i+1}.$$

Assume $x \in S_i \setminus X_{i+1}$. We have $f_i(x) = \frac{p_i(x)}{q_i(x)}$ and thus $p_i(x)q_i(x) > 0$. If $r'_{i+1}(x) \geq 0$ then $r_i(x) > 0$.

If $r'_{i+1}(x) < 0$ then $x \in C$ and we know that the sign of $r_i(x)$ is the sign of $p_i(x)q_i(x)$. We have proved that $S_i \setminus X_{i+1} \subset \mathcal{S}(r_i) \cap (X_i \setminus X_{i+1})$ and by (5) then $S_i \setminus (\mathcal{S}(r_i) \cap X_i) \subset X_{i+1}$. So in order to prove (6) we are left to prove

$$(7) \quad S_{i+1} \setminus (\mathcal{S}(r_i) \cap X_{i+1}) \subset \overline{\text{Bd}(S)}^{\text{Zar}} \cap X_{i+1}.$$

We first establish

$$(8) \quad \mathcal{Z}(p_i) \cap C \cap X_{i+1} \subset \overline{\text{Bd}(S)}^{\text{Zar}} \cap X_{i+1}.$$

Assume $y \in \mathcal{Z}(p_i) \cap C \cap X_{i+1}$. Then $p_i(x)q_i(x) = 0$, $y \in \overline{S_i}^{eucl} \cap X_{i+1}$ and $r'_{i+1}(y) \leq 0$. Hence if $y \in S_{i+1}$ then $y \in S_{i+1} \setminus (\mathcal{S}(r'_{i+1}) \cap X_{i+1}) \subset \overline{\text{Bd}(S)}^{Zar} \cap X_{i+2}$. If $y \notin S_{i+1}$ then $y \notin S_i$ and thus $y \in (\overline{S_i}^{eucl} \setminus S_i) \cap X_{i+1} \subset \text{Bd}(S_i) \cap X_{i+1} \subset \overline{\text{Bd}(S)}^{Zar} \cap X_{i+1}$. We get (8).

Finally we prove (7). Assume $x \in S_{i+1} \setminus (\mathcal{S}(r_i) \cap X_{i+1})$. On X_{i+1} we have $r_i = t^{2l} r'_{i+1}$. If $x \notin \mathcal{Z}(t)$ then $x \in S_{i+1} \setminus (\mathcal{S}(r'_{i+1}) \cap X_{i+1})$ and thus we get $x \in \overline{\text{Bd}(S)}^{Zar} \cap X_{i+2} \subset \overline{\text{Bd}(S)}^{Zar} \cap X_{i+1}$ using (3). So we assume $x \in \mathcal{Z}(t)$. By (8) it follows that $\mathcal{Z}(t) \cap X_{i+1} \subset \overline{\text{Bd}(S)}^{Zar} \cap X_{i+1}$ and thus $x \in \overline{\text{Bd}(S)}^{Zar} \cap X_{i+1}$. \square

Remark 6.5. In the previous theorem, it is not difficult to see that we also have

$$S(f) \setminus \mathcal{S}(r) \subset \text{Bd}(\mathcal{S}(r)) \subset \mathcal{Z}(r)$$

and

$$\overline{\mathcal{S}(f)}^{eucl} = \overline{\mathcal{S}(r)}^{eucl}.$$

It follows from Theorem 6.4 that:

Theorem 6.6. *Let $f \in \mathcal{R}^0(X)$ and $S = \mathcal{S}(f)$. Then S is a principal open semi-algebraic set if and only if $S \cap \overline{\text{Bd}(S)}^{Zar} = \emptyset$.*

Corollary 6.7. *Let $f \in \mathcal{R}^0(X)$ such that $\mathcal{Z}(f)$ is Zariski closed. Then $\mathcal{S}(f)$ is a principal semi-algebraic set.*

Proof. Assume $\mathcal{Z}(f)$ is Zariski closed. Since $\text{Bd}(\mathcal{S}(f)) \subset \mathcal{Z}(f)$, we get $\mathcal{S}(f) \cap \overline{\text{Bd}(\mathcal{S}(f))}^{Zar} \subset \mathcal{S}(f) \cap \overline{\mathcal{Z}(f)}^{Zar} = \mathcal{S}(f) \cap \mathcal{Z}(f) = \emptyset$. We get the conclusion using Theorem 6.6. \square

We can use Corollary 6.15 to improve some of the results of the previous section concerning the sum of signs of regulous functions.

We extend the result of Proposition 5.3 (see Remark 5.4), which concerns curves, to any real algebraic set of any dimension.

Theorem 6.8. *Let $f \in \mathcal{R}^0(X)$. Then $\mathcal{Z}(f)$ is Zariski closed if and only if there exists $p \in \mathcal{P}(X)$ such that $\Lambda(\langle f \rangle) = \Lambda(\langle p \rangle)$ on X .*

Proof. The proof of the “if” is trivial.

Assume $\mathcal{Z}(f)$ is Zariski closed. By Corollary 6.15, there exist p_1, p_2 in $\mathcal{P}(X)$ such that $\mathcal{S}(f) = \mathcal{S}(p_1)$ and $\mathcal{S}(-f) = \mathcal{S}(p_2)$. Copy now the end of the proof of Proposition 5.3 to conclude. \square

Corollary 6.9. *Let $f \in \mathcal{R}^0(X)$, $k = \text{poldepth}(f)$ and $((f_0, X_0, p_0, q_0), \dots, (f_k, X_k, p_k, q_k))$ a “polar sequence” associated to f . Let*

$$t = \min\{i \in \{0, \dots, k\} \mid \mathcal{Z}(f) \cap X_i \text{ is Zariski closed}\}.$$

Therefore, we may write the sign of f on X as the sum of l signs of polynomial functions on X with

$$l \leq 1 + 2t.$$

Proof. The proof is straightforward using Proposition 5.10 and Theorem 6.8. \square

Example 6.10. It may happen that the zero set of a regulous function is not Zariski closed but its intersection with the polar locus is Zariski closed. Consider $f = z - \frac{x^3}{x^2 + y^2} \in \mathcal{R}^0(\mathbb{R}^3)$. So

$\mathcal{Z}(z - \frac{x^3}{x^2 + y^2}) \subset \mathbb{R}^3$ is the “canopy” of the Cartan umbrella $V = \mathcal{Z}(z(x^2 + y^2) - x^3) \subset \mathbb{R}^3$. Moreover, $\text{indet}(f)$ is the stick of the umbrella and $\mathcal{Z}(f) \cap \text{indet}(f) = \{(0, 0, 0)\}$.

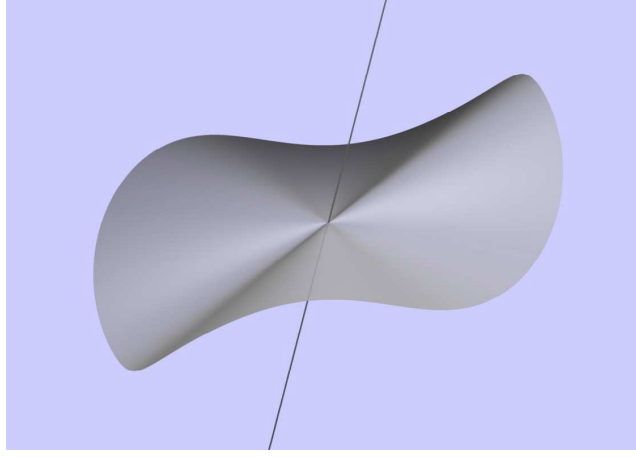


FIGURE 3. Cartan umbrella.

In the following proposition, we improve the result of Theorem 5.11 in the two dimensional case and the result of Theorem 5.12 in the three dimensional case.

Proposition 6.11. *Assume $d = \dim(X) \leq 3$ and moreover that $\text{codim}(\text{Sing}(X)) > 1$ in the case $d = 3$. Let $f \in \mathcal{R}^0(X)$. We may write the sign of f on X as the sum of at most 3 signs of polynomial functions on X .*

Proof. By Theorems 5.11 and 5.12, we may assume that $d = 2$ or $d = 3$ and also that $\text{codim}(\text{Sing}(X)) > 1$ in the case $d = 3$. By Proposition 5.9, we have $\dim(\text{indet}(f)) \leq 1$. We get the proof, using Corollary 6.9, if $\dim(\text{indet}(f)) < 1$ or if $\mathcal{Z}(f) \cap \text{indet}(f)$ is Zariski closed. So we assume $\dim(\text{indet}(f)) = 1$ and $\mathcal{Z}(f) \cap \text{indet}(f)$ is not Zariski closed. We write $f = \frac{p}{q}$ on $\text{dom}(f)$ with $p, q \in \mathcal{P}(X)$ and $\mathcal{Z}(q) = \text{indet}(f)$. We decompose $\mathcal{Z}(q) = \text{indet}(f)$ as a union $C_1 \cup \dots \cup C_t \cup W$ where the C_i are irreducible real algebraic curves and $\dim(W) = 0$. For each curve C_i , we denote by F_i the regulous closed set $\overline{(C_i)_{\text{reg}}}$. By [12, Thm. 6.7], the sets F_i are \mathcal{C} -irreducible and $C_i \setminus F_i$ is empty or a finite set of points. Since $\mathcal{Z}(f) \cap \text{indet}(f)$ is not Zariski closed, we have $\dim(\mathcal{Z}(f) \cap \text{indet}(f)) = 1$. Since the F_i are \mathcal{C} -irreducible, we get that $F_i \subset \mathcal{Z}(f)$ if and only if $\dim(\mathcal{Z}(f) \cap C_i) = 1$. Remark that there exists at least one F_i such that $F_i \subset \mathcal{Z}(f)$ but $C_i \not\subset \mathcal{Z}(f)$ since $\mathcal{Z}(f) \cap \text{indet}(f)$ is not Zariski closed. If $F_i \subset \mathcal{Z}(f)$ then $\Lambda(\langle f \rangle) = \Lambda(\langle pq \rangle)$ on C_i outside a finite number of points. If $F_i \not\subset \mathcal{Z}(f)$ then $\mathcal{Z}(f) \cap C_i$ is Zariski closed. It follows that there exists a real algebraic subset Y of $\text{indet}(f)$ such that $\mathcal{Z}(f) \cap Y$ is Zariski closed and such that $\Lambda(\langle f \rangle) = \Lambda(\langle pq \rangle)$ on $X \setminus Y$. By Theorem 6.8, there exists $h \in \mathcal{P}(X)$ such that $\Lambda(\langle f \rangle) = \Lambda(\langle h \rangle)$ on Y . Let $r \in \mathcal{P}(X)$ be such that $\mathcal{Z}(r) = Y$. The proof is done since

$$\Lambda(\langle f \rangle) = \Lambda(\langle pq \rangle \perp \langle 1, -r^2 \rangle \otimes \langle h \rangle) \text{ on } X.$$

□

To end this section, we will give an answer to the following question: Under which conditions an open semi-algebraic set is regulous principal?

Definition 6.12. A semi-algebraic subset S of X is said to be generically principal on X if S coincides with a principal open semi-algebraic subset of X outside a real algebraic subset of X of positive codimension.

The next result is a regulous version of Lemma 6.3.

Lemma 6.13. *Let C be a closed semi-algebraic subset of X and let $f, g \in \mathcal{R}^0(X)$ such that $\mathcal{Z}(f) \cap C \subset \mathcal{Z}(g)$. There exist $h \in \mathcal{P}(X)$ and $l \in \mathbb{N}$ odd such that $h > 0$ on X and*

$$\Lambda(\langle hf + g^l \rangle) = \Lambda(\langle f \rangle)$$

on C .

Proof. We can see C as a closed semi-algebraic subset of \mathbb{R}^n and $f, g \in \mathcal{R}^0(\mathbb{R}^n)$ by definition of regulous functions on X . By [4, Thm. 2.6.6], for a sufficiently big positive odd integer l the function $\frac{g^l}{f}$ is semi-algebraic and continuous on C . By [4, Thm. 2.6.2], $|\frac{g^l}{f}|$ is bounded on C by a polynomial function h which is positive definite on \mathbb{R}^n . The proof is done by restricting these functions to X . \square

Proposition 6.14. *Let S be a semi-algebraic subset of X such that:*

$$1) S \cap \overline{\text{Bd}(S)}^C = \emptyset,$$

and there exists an algebraic subset W of X of positive codimension such that:

$$2) \text{ there exists } p \in \mathcal{P}(X) \text{ such that } S \setminus W = \mathcal{S}(p) \setminus W,$$

$$3) \text{ there exists } g \in \mathcal{R}^0(X) \text{ such that } S \cap W = \mathcal{S}(g) \cap W.$$

Then there exists $f \in \mathcal{R}^0(X)$ such that $S = \mathcal{S}(f)$.

Proof. We may assume $W \subset \mathcal{Z}(p)$ changing p by q^2p where $q \in \mathcal{P}(X)$ satisfies $W = \mathcal{Z}(q)$.

Set $F = \overline{\mathcal{S}(g)}^{\text{eucl}} \setminus S$. Assume $x \in W \cap F$ then $x \in W \setminus (S \cap W)$ and thus $g(x) \leq 0$. Then $x \in \text{Bd}(\mathcal{S}(g)) \subset \mathcal{Z}(g)$. We have proved that $\mathcal{Z}(-q^2) \cap F \subset \mathcal{Z}(g)$. By Lemma 6.13 there exist $h \in \mathcal{P}(X)$, $l \in \mathbb{N}$ odd and $g' \in \mathcal{R}^0(X)$ such that $h > 0$ on X , $g' = -hq^2 + g^l$ and $\Lambda(\langle g' \rangle) = \Lambda(\langle -q^2 \rangle)$ on F . Clearly, the function g' satisfies again the property 3) of the proposition, namely

$$(9) \quad S \cap W = \mathcal{S}(g') \cap W.$$

The function g' satisfies in addition the following property

$$(10) \quad \mathcal{S}(g') \subset S.$$

Assume $g'(x) > 0$ then $g(x) > 0$ and moreover if $x \notin S$ then $x \in F$ and we get a contradiction since then the sign of $g'(x)$ would be the sign of $-q^2(x)$. We have proved (10).

Set $C = \overline{\mathcal{S}(g')}^{\text{eucl}} \setminus \mathcal{S}(g')$. Let $t \in \mathcal{R}^0(X)$ be such that $\mathcal{Z}(t) = \overline{\mathcal{Z}(p) \cap C}^C$. We clearly have $\mathcal{Z}(p) \cap C \subset \mathcal{Z}(t^2g')$. By Lemma 6.13, there exist $p' \in \mathcal{P}(X)$ positive definite on X and a positive odd integer l' such that $f = p'p + t^{2l'}g'^{l'}$ satisfies $\Lambda(\langle f \rangle) = \Lambda(\langle p \rangle)$ on C .

Assume $x \notin S$. We have $p(x) \leq 0$ since $W \subset \mathcal{Z}(p)$. We have $g'(x) \leq 0$ by (10). Hence $f(x) \leq 0$ and we have proved that

$$(11) \quad \mathcal{S}(f) \subset S.$$

Assume $x \in S \setminus W$. If $g'(x) > 0$ then clearly $f(x) > 0$. If $g'(x) \leq 0$ then $x \in C$ and $f(x) > 0$ since $\Lambda(\langle f \rangle) = \Lambda(\langle p \rangle)$ on C . We have proved that

$$(12) \quad S \setminus W \subset \mathcal{S}(f) \setminus W.$$

Since $W \subset \mathcal{Z}(p)$ and using (9) it follows that

$$(13) \quad (S \cap W) \setminus (S(f) \cap W) \subset \mathcal{Z}(t) = \overline{\mathcal{Z}(p) \cap C}^C.$$

We prove now that

$$(14) \quad \mathcal{Z}(p) \cap C \subset \text{Bd}(S).$$

Assume $y \in \mathcal{Z}(p) \cap C \cap W = W \cap C$. We have $p(y) = 0$, $y \in \overline{\mathcal{S}}^{\text{eucl}} \cap W$ and $g'(y) \leq 0$. We have $y \notin S \cap W$ by (9). Hence $y \in \text{Bd}(S) \cap W$. Assume $y \in \mathcal{Z}(p) \cap C$ and $y \notin W$. Since $p(y) = 0$ and $y \notin W$ then $y \notin S$. We get $y \in \overline{\mathcal{S}}^{\text{eucl}}$ since $y \in C$ and it proves (14).

From (12), (13) and (14) it follows that

$$S \setminus \mathcal{S}(f) \subset \overline{\mathcal{Z}(p) \cap C^c} \cap W \subset \overline{\text{Bd}(S)}^c \cap W.$$

Since $S \cap \overline{\text{Bd}(S)}^c = \emptyset$ we finally get

$$S = \mathcal{S}(f).$$

□

Theorem 6.15. *Let $X \subset \mathbb{R}^n$ be a real algebraic set and let S be a semi-algebraic subset of X . Then S is regulous open principal i.e there exists $f \in \mathcal{R}^0(X)$ such that $S = \mathcal{S}(f)$ if and only if we have:*

1) *for any real algebraic subset V of X then $S \cap V$ is generically principal,*

and

2) $S \cap \overline{\text{Bd}(S)}^c = \emptyset$.

Proof. If $S = \mathcal{S}(f)$ with $f \in \mathcal{R}^0(X)$ then $S \cap \overline{\text{Bd}(S)}^c = \emptyset$ since $\overline{\text{Bd}(S)}^c \subset \mathcal{Z}(f)$.

Assume now the set S satisfies the conditions 1) and 2) of the theorem. We denote the set X by X_0 and S by S_0 . Since S_0 is generically principal there exist $p_0 \in \mathcal{P}(X_0)$ and an algebraic subset X_1 of X_0 of positive codimension such that $S_0 \setminus X_1 = S(p_0) \setminus X_1$. If $X_1 = \emptyset$ then we are done since S is even algebraic open principal. If $X_1 \neq \emptyset$ then we denote by S_1 the set $S_0 \cap X_1$. Remark that S_1 satisfies the conditions 1) and 2) of the theorem as an open semi-algebraic subset of X_1 and we can repeat the process used for S_0 but here for the set S_1 . So we build a finite sequence

$$((X_0, S_0, p_0), \dots, (X_k, S_k, p_k))$$

such that for $i = 0, \dots, k-1$, X_{i+1} is an algebraic subset of X_i of positive codimension, $S_i = S \cap X_i$ satisfies the conditions 1) and 2), $p_i \in \mathcal{P}(X_i)$, $S_i \setminus X_{i+1} = (S(p_i) \cap X_i) \setminus X_{i+1}$ and $S_k = S \cap X_k = \mathcal{S}(p_k) \cap X_k$ with $p_k \in \mathcal{P}(X_k)$. By Lemma 6.14, there exists $g_{k-1} \in \mathcal{R}^0(X)$ such that $S_{k-1} = \mathcal{S}(g_{k-1}) \cap X_{k-1}$. By successive application of Lemma 6.14, there exists $g_i \in \mathcal{R}^0(X)$ such that $S_i = \mathcal{S}(g_i) \cap X_i$ for $i = k-2, \dots, 0$ i.e S is in particular regulous open principal. □

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JEAN-PHILIPPE MONNIER, LUNAM UNIVERSITÉ, LAREMA, UNIVERSITÉ D'ANGERS
E-mail address: `jean-philippe.monnier@univ-angers.fr`