A study of the preparation of the function concept

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Function is a basic concept of mathematics, in particular, mathematical analysis. With an appropriate development of a function approach, it becomes possible for students to use function models to describe mathematical and non-mathematical problems. After an analysis of the function concept development process, I propose a model of rule following and rule recognition skills development that combines features of the van Hiele levels and the levels of language about function (Isoda, 1996). Using this model I investigate students' rule following and rule recognition skills from the viewpoint of the preparation for the function concept of sixth grade students (11–12 years old) in the Ukrainian education system.

Keywords: Function concept, Ukrainian secondary education, features of van Hiele levels, rule following and rule recognition skills.

INTRODUCTION

The function concept interweaves the whole teaching of mathematics. Functions are incorporated in the concepts of numbers, equations, inequalities, ratio, proportionality, geometrical transformations, etc. Through the teaching of functions, it is also possible for students to develop creativity, functional thinking, and other cognitive strategies (Czeglédy, Orosz, Szalontai, & Szilák, 1994).

In her study, Sierpinska (1992) sets out the conditions of understanding the notion of function. These conditions illustrate that it takes time to reach a thorough understanding of the function concept. There is a long journey between beginning to develop an understanding of the links between the elements of sets to the robust function concept. In this study I examined the portion of this journey that happens during the fifth – sixth grade, which is the period before learning the definition of the function (preparation period). The study was based on the analyses of the Ukrainian curriculum framework. The study revealed that the development of rule following and rule recognition (hereafter referred to as RF and RR) skills are missing from the curriculum. Dreyfus and Vinner (1989), however, point out that a function can also be defined as a rule, and the rule is an element of the function concept (Kwari, 2007). The present study examines RF and RR skills that are necessary in the formation of the function concept and in the construction of function tables, which help children to figure out the relationship between quantities (Blanton & Kaput, 2011). The participants were a class of sixth grade (11–12 years old) students, studying in the Ukrainian education system.

THEORETICAL BACKGROUND

Definition plays an important role in mathematics. According to Skemp (1987), definitions have their specific places in mathematical concept development. Concepts of a higher order than those which people already have an understanding of, cannot be communicated to them by a definition, but rather by presenting to them a suitable connection of examples. Since in mathematics these examples are almost invariably other concepts, it must first be ensured that these are already formed in the mind of the learner (Skemp, 1987).

The modern definition of function that frames this study is the Dirichlet-Bourbaki definition, which is "a correspondence between two nonempty sets that assigns to every element in the first set (the domain) exactly one element in the second set (the codomain)" (Vinner & Dreyfus, 1989, p. 357). So in order to develop a concept of function, the knowledge of both simple and higher level concepts is necessary, and this formation is a long process.

Vinner and Dreyfus (1989) discuss the notions of function put forth by secondary school students after being given the definition of function. The authors, drawing on Vinner (1983), categorized students' definitions of function into six categories: (A) correspondence (the Dirichlet-Bourbaki definition); (B) dependence relation (dependence between two variables); (C) rule (a function is a rule; a rule is expected to have some regularity, whereas a correspondence may be "arbitrary"); (D) operation (a function is an operation or manipulation); (E) formula (a function is a formula, an algebric expression, or an equation); and, (F) representation (graphical or symbolic representation) (Vinner & Dreyfus, 1989, p. 360).

Taking into account these categories and the above mentioned studies, it can be highlighted that the function concept has many elements. Sierpinska (1992) described the "worlds" that the study of functions should focus on: the world of changes or changing objects; the world of relationships or processes; and, the world of rules, patterns, and laws. Likewise, Kwari (2007) showed the constitutive elements or aspects of the function concept that should be developed: change and what changes; relationships (attribute — builds rules to determine a unique y-value from any given x-value); rules (symbolically given by, e.g., $f(x) = ax + b$); representation; and, language/notation. Of the skills that could be linked to the above listed aspects, possession of the rule recognition and rule following skills are significant for the students in this study in order to recognise and express function-like relations. So, this study concentrates on the investigation of these skills and only touches upon the question of symbolisation (the articulation of rule by arithmetic operations), but does not intend to go further (to write function-like links with symbols, e.g., $f(x) = ax + b$).

As a *skill* is considered to be the psychic feature of an individual, that evolves by the practice of some kind of activity, and is manifested in the doing of that activity, then the mentioned skills can also be developed by cognitive operations. The recognition of a rule (regularity), the following of the rule, and in some cases, the appropriate application of the rule, presumes the execution of a series of cognitive operations (categorisation, selection, and link-recognition).

The information acquisition process is strongly influenced by the development of students' cognitive operations. Two Dutch didacticians, Pierre van Hiele and Dina van Hiele-Geldof developed a pedagogical theory in 1957 for the understanding of the process of geometric thinking, which differentiates between five levels of geometric thinking: visualization; analysis; informal deduction; deduction; and, rigor (as cited in Herendiné Kónya, 2003, p. 51).

Freudenthal (1973) and Isoda (1996) extend the van Hiele levels from geometry to other areas. Freudenthal viewed progressive mathematization as the main goal of school mathematics. For this ongoing task, he provided a framework by recursively defined levels: the activity of the lower level, that is the organizing activity by the means of this level, becomes an object of analysis on the higher level. Freudenthal's theoretical approach rests on the Van Hiele levels. Van Hiele, himself, has written about levels in arithmetic and algebra (van Hiele, 2002). He observed 'a change in level' from the act of counting to the concept of number. Isoda's paper (1996) points out features of van Hiele levels and shows that they are also characteristics of the proposed levels of language about function. These features include: (1) Language hierarchy. Each level has its own language and the levels are hierarchical; (2) The existence of untranslatable concepts. The corresponding contents of different levels sometimes conflict; (3) Duality of object and method. The thinking of each level has its own inquiring object (subject matter) and inquiring method (the way of learning); (4) Mathematical language and student thinking in context. While the levels are distinguished as sets of mathematical language, the actual thinking of each student varies depending on the teaching and learning context.

Isoda (1996) first discusses the levels of function from the point of view of language, and shows the duality between object and method in van Hiele's levels (the levels of geometry) and in the levels of function. These levels of language are: Level 1. Level of everyday language (students describe relation in phenomena using everyday language obscurely: students explore phenomena (object) using obscure relations or variation (method)); Level 2. Level of arithmetic (students describe the rules of relations using tables. They make and explore tables with arithmetic: students explore the relations using rules); Level 3. Level of algebra and geometry (students describe function using equations and graphs: students explore the rules using notations of function); Level 4. Level of calculus (students describe function using calculus); Level 5. Level of analysis (an example of language for description is functional analysis which is a metatheory of calculus).

The present study examines the preparatory part of the notion of function in the sixth grade in light of the state framework curriculum, using features (1) and (3) of van Hiele levels and the first three of the five levels of function described by Isoda (1996). I set out the levels of the cognitive operations that are crucial for the possession of RF and RR skills and the criteria for categorising activity forms into levels. Noticing an analogy between these levels and the van Hiele levels, I used the names of the van Hiele levels for the marking of the discussed levels. The levels which I created by joining the features of van Hiele levels and Isoda's levels and using them to develop a deeper understanding in (sixth grade) students' development of the function concept, are the following:

Level 1 (visualization): Students recognise some kind of rule (functional relation) (method) between the element pairs (object) and follow the recognised rule *(level of everyday language)*.

Level 2 (analysis): Students are able to phrase the recognised rule (they can argue in favor of the recognised links between the cohesive element pairs) and follow the rule which is given by words or by simple formulas *(level of everyday language and level of arithmetic)*.

Level 3 (informal deduction): At this level the harmony of the simple (2–3 steps) rule-making and its description with formula develops *(level of arithmetic)*.

METHODOLOGY

Sample

Participants were 26 sixth grade students (11–12 years old), with moderate abilities, in a school with Hungarian as the language of instruction in Ukraine. The students had four classes of mathematics a week, according to the state curriculum framework. The Hungarian version of the mathematics textbook is used at this level and is approved by the Ukrainian Ministry of Education and Science. As the research was carried out in March, during the second semester of the sixth grade, students were already familiar with the natural numbers, fractions (common fractions and decimals), and had learned arithmetic operations with rational numbers. The introduction of proportional amounts and linear relationships occurred during this period, with the practical application in the initial phase.

Background

In the Ukrainian education system, function as a mathematical concept is defined at the seventh grade of the secondary school. In the lower classes, students are prepared with the use of different materials for introduction of the function concept. Analysing the curriculum and the textbooks for the fifth and sixth grade from the point of view of topics and their content that are supposed to support the development of the function concept, major deficiencies come to the surface in the requirements for developing RF and RR skills (in the lower classes it does not exist at all). In the development requirements of the themes of the curriculum, rule recognition and rule following skills are not mentioned. Prior research (cf., studies cited above), however, suggest that they are neces-

Class	Themes	Development requirements
5.	Number line	The representation of natural numbers on the number line.
	Letter expressions	The recognition of number- and letter expressions and the illustration with examples.
6.	Linear relationship	Illustration of proportional amounts with examples, the definition of the concept of linear relationship, finding the unknown element of the proportion, defining the proportion between amounts.
	Diagrams	Editing column- and circle diagrams.
	Cartesian coordinate system	Finding the coordinate of the point in the coordinate plane and representing the given coordinate point.
	Graphs	Representing correlations between quantities by graphs and analysing these graphs. The student is able to read the data from the graphs.

Table 1: Themes preparing the function concept in the Ukrainian textbooks[1] and curricula[2]

sary for the development of the function concept. In Table 1, I summarized the textbook themes that could support the preparation of the function concept. It can clearly be seen in the table that RF and RR skills are not amongst the development requirements of the materials.

Based on these aspects, in this study I am looking for the answers to the following questions: *At the end of the 6th grade, what level do students reach in their RF and RR skills? What are the typical mistakes students make when carrying out activities at each level and what might explain these errors?*

The questionnaire

A written test was used in order to investigate the RF and RR skills of students. Students worked independently and had 30 minutes to complete the test. The test contained five tasks that were based on the recognition and application of the relationship between the cohesive elements (assignment rules), as well as on the expression of the recognised rule, including as a formula. I was interested in students' possession of the necessary skills for the preparation of the function concept. In some exercises, the cohesive element pairs did not clearly make a function, so more rules might be possible. In the direction to the test, however, I tried to make it clear that I wanted students to find only one adequate rule. When constructing the test, I included tasks for levels 1, 2 and 3. When choosing the tasks, I predominately relied on the literature and used some of them without any alterations.

1) *Find a rule between the first and second row of the table. Fill in the table according to the rule* (Level 1)*! Write down the recognised rule in words* (Level 2).

pék	*tér*	*ló*	*bál*	*görög*		
kép	*rét*	*ól*			*derék*	*savas*

Figure 1

2) *Find a rule for the numbers in the columns and fill in the blank places of the table according to that rule* (Level 1). *Write down the recognised rule in words* (Level 2).

40	80	12	60	44	100	160				
10	20	3					1	13	31	95

Figure 2

I indicate the level of the task, parenthetically, within the instructions (see figures below). The first two tasks (Figure 1 and Figure 2) targeted the recognition, application and verbal expression of the relationship between cohesive elements (words and numbers). The filling in of the blank places of the tables assessed the application of the rule. Although the correct solution of both tasks assumes the same level of cognitive operations and activity forms (level 1 and level 2), the difference can be found in the context of the tasks: While in the first task the cohesive element pairs are words, in the second they are numbers. Because function relationships do not only occur between numbers, it is crucial that students recognise this relationship, as well.

The aim of the third task (Figure 3) was to make students recognise the relationship between the elements, apply it, and to express it with both words and symbols. In order to reach the first level, it is necessary to recognise some kind of relationship between the cohesive elements (x and y), but unlike in the first two tasks, the table is extended by an extra (first) column which serves as a hint to record the recognised relationship in the language of arithmetic (2nd level). When asking students to express the relationship with a formula, I touch upon the question of symbolisation (3rd level), but I do not intend to examine it deeper in this study. The "end product" (y value) should be found with the help of the given "raw material" (x value) according to the recognised rule, while in the previous two tasks knowing the "end product" and using the recognised rule, the raw material should be found.

The fourth task (Figure 4) was aimed at the interpretation and following of a predefined rule. In order to solve the task, the student needed to possess the activity forms of the two levels in order to interpret (analyse) the given formula. A correct completion

3) *Find a relationship between the x and y values of the columns and based on it, complete the table with the missing elements* (Level 1)*! Write down the relationship with words* (Level 2) *and as an expression* (Level 3)*!*

x	1	10	7	0	9	20	38
y	5	23	17				

y=__________

Figure 3

x	-3	4	44	48	-20	0
y						

4) Fill in the table according to the following rule: y=2x+3. Write down the rule in words (Level 2).

Figure 4

5) 2 litres/second of water flows from a tap to a tank. How much water is in the tank at:

a) 1 s,	*c) 5 s,*	*e) 16 s, (Level 2)*
b) 2 s,	*d) 10 s,*	*f) x s (Level 3)*

later if the tank was empty at the beginning? Illustrate the relationship between the amounts in a table.

Figure 5

of the table indicated a correct interpretation of the symbolic rule.

In the fifth task (Figure 5) I examined rule recognition and its mode of illustration during the solution of a task given in context. In this case, the rule is given verbally, in context. I take students' correct responses for parts (a) through (e) (2nd level) as an indication that the student had correctly interpreted the rule. A correct response to part (f) indicated that students' understanding of function had reached the third level, since the student was able to generalise the task, i.e. write down the relationship using a formula.

RESULTS

Analysis of students' answers

All of the students filled in the table in Task 1 correctly. This indicated that the students could recognise some kind of regularity between the first and the second row of the table, and they could apply the recognised rule. This means that when the cohesive element pairs are words, students can recognise the relationship between them. Writing down the recognised rule in words, however, was difficult for eight students. Some students skipped this part of the task or gave a rule that was not supported by the completed table. Some examples of correct responses for recognised rules: "Words should be read backwards"; "If we change the first and the last letters we get another meaningful word".

In the second task, where the cohesive element pairs were numbers, only 18 students' gave a correct solution, while 14 students were able to give the recognised rule in words. The other students made one of the following mistakes: (1) they filled in the blank squares in the second row of the table according to a recognised rule, but in the first row they filled in the blank squares using another rule; that is, they did not apply the inverse of the recognised rule and interpreted this part of the table separately. From the point of view of the function concept, these mistakes indicated issues in recognising and differentiating between the basic set and the image set; (2) students tried to find different rules for each column and filled in the squares according to it. This could be the consequence of being unfamiliar with the table illustration of cohesive amounts. Here are some examples of correct responses for recognised rules:

> "If the square in the second row is empty the number above it has to be divided by four, and where the first row square is empty the second row number has to be multiplied by four" ; "Numbers of the first row are the fourfold of the lower row".

Only two students completed the third task, while other students did not give any indication of their thinking. This let me conclude that those students who possess the skill of one step rule recognition and rule creation may have difficulty with two step rule recognition.

In the fourth task, the rule was given symbolically. Students had to understand the symbolic rule and fill in the table accordingly. The given rule could have been familiar to the students, as letter expressions were from the fifth grade mathematics material, when they had to define the value of the letter expression along the certain values of the variable, but the values were not given in table form. Presumably, this new situation confused many students. Only twelve students could solve the task with only minor calculation mistakes.

Only ten students could answer all of the sub questions of the fifth task, including the last (f), so they could generalise the rule of calculating the amount of water in the tank if the elapsed time was unknown, and they could illustrate the relationship between the results with a table. Eight students could calculate with concrete numbers (parts (a) through (e)), but failed to complete the (f) question.

By analysing the responses of the students in the tasks according to the criteria of the set out levels it can be said that a student reached level 1 if he/she could complete at least one of Tasks 1, 2, or 3. I considered that a student had reached Level 2 when he/she correctly provided the rule in at least three tasks out of the five. The student reached Level 3 if he/she gave the correct answer to all of the questions of the third task. In some tasks students made calculation mistakes (such as in Tasks 4 and 5), but I did not take these into consideration if the student demonstrated the correct reasoning.

The students' answers were analysed based on the levels at which the various parts of the tasks were categorised. The results are summarised in Table 2.

Based on the analysis, the students were most successful at demonstrating a Level 1 understanding in the first task since every student correctly completed it. However, the part of the same task, which was categorised as Level 2, was completed by fewer students (18). It can be concluded, however, that in the case of each task, the highest results were reached on Level 2, as compared to the other levels. The third task was the most difficult. Only two students gave a complete solution.

Based on these aspects, out of 26 students, 14 are on the first level, ten are on the second level, and only two students are on the third level. So, most of the students can recognise some kind of rule between the element pairs and can follow it, but to write these rules down with words cause them difficulties. In addition, interpreting the rules given by symbols and making multistep rules also seems to be problematic. This study also confirmed the hierarchy of the levels. There was no student who could meet the requirements of Level 3, but not Level 1 or Level 2.

CONCLUSIONS

The goal of this paper was to investigate the RF and RR skills of sixth grade students studying in the Ukrainian education system, from the point of view of the development of the function concept. The results showed that students certainly reached level 1, indicating that they can recognise the relationship between simple elements. In many cases, however, I could see that some students fulfilled the requirements of level 1, but could not get to level 2 due to possible deficiencies in the area of communication in the language of mathematics. Many could also not successfully use the table as a tool for displaying cohesive elements. I suspect that students' deficiencies are not only age-specific, but are also related to the absence of tables from the curriculum requirements and from the textbook tasks. Students' lack of success in correctly completing Tasks 4 and 5, which entailed the use of already known concepts (letter expressions and linear relationship) in new situations (problem solving), indicated that this was also a problematic area for the students.

Finally, I would highlight a component of van Hiele's theory: that reaching a level does not only depend on the age of the student, but also on the teaching methods used and the quality of student learning. Based on this suggestion, with appropriate teaching methods and practical exercises integrated into the teaching/learning process, we can ensure that students possess the adequate skills for at least the first two levels in order to make the introduction of the abstract concepts easier in the seventh grade.

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Task / Levels number	Level 1		Level 2		Level 3	
	Number of students	%	Number of students	%	Number of students	%
1.	26	100	18	69		
2.	18	69	14	54		
3.	2	8	2	8	2	8
4.			12	46		
5.			17	65	10	38

Table 2: Counts and percentages of correct solutions to the tasks according to levels

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