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In a previous paper, we proposed a characterization of algebraic reasoning in primary education, based on the onto-semiotic approach to mathematical knowledge and instruction, where we distinguish three levels of algebraization. In defining these levels we took into account the types of representations used, the generalization processes involved and the analytical calculation performed in mathematical activity. In this paper, we extend this previous model by including three more advanced levels of algebraic reasoning in order to analyze the mathematical activity carried out in secondary education. These new levels are based on the following considerations: 1) using and processing parameters to represent families of equations and functions; 2) the study of algebraic structures themselves, their definitions and properties.

Key words: Algebraic reasoning, primary education, secondary education, onto-semiotic approach, teachers' education.

INTRODUCTION

Recognizing the characteristic features of algebraic thinking is an issue that has attracted many mathematics education researchers, because it is necessary to promote such reasoning at different levels of elementary and secondary education (Kieran, 2007; Filloy, Rojano, & Puig, 2008; Kaput, 2008). Depending on how the school algebra is conceived, decisions are taken concerning whether to introduce such algebra since early levels, or to delay its teaching until secondary education as well as to change the corresponding instructional strategies. In fact, the "early algebra" research and development program (Carraher & Schliemann, 2007; Cai & Knuth, 2011) is supported on a conception of algebra that recognizes signs of algebraic thinking in mathematical activities of initial educational levels, as shown in NCTM (2000). While there has been progress in the characterization of school algebra, the interconnection between primary and secondary education algebra is not completely solved.

In previous publications (Aké, Godino, Gonzato, & Wilhelmi, 2013; Godino, Aké, Gonzato, & Wilhelmi, 2014) we proposed a model of algebraic thinking for primary education, with three different levels of algebraic thinking. Furthermore, we established criteria to delimit these algebraic levels from 0 (arithmetic nature of mathematical activity) to 3 (clear algebraic activity), with two intermediate levels of proto-algebraic activity. The criteria to define these levels were based on the type of mathematical objects and processes involved in mathematical activity, according to the onto-semiotic approach (OSA) to mathematical knowledge (Godino, Batanero, & Font, 2007; Godino, Font, Wilhelmi, & Lurduy, 2011)[1]. These algebraization levels are assigned to the operative and discursive practices performed by a mathematical subject that solves a mathematical task, rather than to the task itself, which can be solved in different ways, and may bring into play different algebraic activity.

1 The Onto-Semiotic Approach of mathematical knowledge and instruction (OSA) is a theoretical framework that adopts semiotic and anthropological assumptions about mathematics, and socio-constructivist and interactionist principles for the study of teaching and learning processes. Due to space limitation, it is not possible to include a synthesis of the main theoretical tools that compose this framework; the readers might consult Godino and colleagues (2007) and Godino and colleagues (2011).

In this paper, we extend that model of algebraization levels to secondary school, mathematical activity. This extension is also supported by the onto-semiotic distinctions considered in the OSA; particularly by the presence, use and processing of functions and equations parameters. The work is organized in four sections. In the following section we summarize the features of algebraic reasoning levels in elementary education; next, we define three new levels of algebraization, include some illustrative examples and connect the new levels to the presence of discontinuities in the onto-semiotic configurations involved in mathematical practices.

LEVELS OF ALGEBRAIC REASONING IN PRIMARY EDUCATION

In Table 1 we summarize the essential features of the three preliminary algebraization levels described by Godino, Aké, Gonzato and Wilhelmi (2014), completed by level 0 (absence of algebraic characteristics). An example is also included to help understanding the distinction among levels. In summary, the definition of levels is based on the following onto-semiotic distinctions:

— Presence of *intensive* algebraic objects (i.e., entities of general or indeterminate character).

— Transformations (operations) based on structural properties applied to these objects.

— Type of used language (natural, iconic, gestural, symbolic).

Obviously, these levels do not exhaust the algebraization processes of school mathematical activity. Instead, they describe the gradual enrichment of solving problems tools with an increasing degree of symbolization in other contexts of use. These processes, in the end of primary school and junior secondary school, may evolve to higher algebraization levels. The criteria used to distinguish the different algebraic levels have been gradually refined through its application to the analysis of responses from different samples of student teachers (Aké et al., 2013; Godino et al., 2014).

LEVELS OF ALGEBRAIC REASONING IN SECONDARY EDUCATION

In this section, we extend the model of algebraization levels to secondary and high school mathematical activity, in describing three additional algebraization levels for this educational stage.

The use and treatment of parameters is a criterion for defining higher levels of algebraization, as it is linked to the presence of equations and functions families, and, therefore it implies new "layers" or levels of generality (Radford, 2011). The intervention of parameters will be linked to the fourth and fifth algebraization levels, while the study of specific algebraic structures will mark a sixth algebraization level of mathematical activity.

Fourth algebraization level: using parameters

The use of parameter for expressing equations and function families is indicative of a higher level of algebraic reasoning, as compared to the third algebraization level considered by Aké and colleagues (2013), which is linked to operations with unknowns or variables. This "first encounter" with parameters and variable coefficients involves the discrimination of the domain and range of parametric functions, i.e. functions that assigns a specific function or equation to each value of the parameter. As suggested by Ely and Adams (2012, p. 22) "A significant conceptual shift must occur in order for students to be comfortable using placeholders in algebraic expressions rather than just numbers".

Example 1: The linear function

In the algebraic expression, $y = 2x$, the literal symbols x and y represent variables, symbols that can take any value from a previously established number set, usually R. The numerical values x and y co-vary in terms of each other, according to the rule laid down in the corresponding expression; in this case, y is twice the value assigned to x. The factor multiplying x can be generalized to any value in a certain domain; as we see in the expression $y = ax$. Here the letter a intervenes as a parameter: it can take different values within a certain domain, so that for each possible value a, we obtain a particular function. For example, for $a = 2$, we have $y = 2x$.

Consequently, a parameter is a literal symbol involved in an expression with other variables, such that for each particular value assigned to it, a function is obtained. We express such families of functions as $F = \{f(x) = ax/aR\}$, or more precisely, a family of functions that depend on the domain D of definition of the functions f: $F_D = \{f(x) = ax \mid aR; xD\}$.

<table>
<tr><td colspan="4">Task: Students either go by car or they walk to a certain school. There are 3 students walking for every 3 student going by car. If the school has 212 students, how many of them use each means of transportation?</td></tr>
<tr><td>LEVELS</td><td>OBJECTS</td><td>TRANSFORMATIONS</td><td>LANGUAGES</td></tr>
<tr><td rowspan="2">0</td><td>No intensive objects are involved.
In structural tasks unknown data can be used.</td><td>Operations are carried out with extensive objects.</td><td>Natural, numerical, iconic, gestural; symbols referring extensive objects or unknown data can take part.</td></tr>
<tr><td colspan="3">Example of resolution:
For every 3 students who walk, there is 1 going by car. Hence, in every group of 4 students (3 + 1) there is 1 going by car (a fourth of students). Thus, 50 out of 200 students go by car and 3 out of 12 students use the car. Therefore, 53 students use the car and three times that amount, that is, 159, walk to the school.</td></tr>
<tr><td rowspan="2">1</td><td>In structural tasks unknown data can be used.
In functional tasks intensive objects are recognized.</td><td>In structural tasks relations and properties of operations are applied.
In functional tasks calculation involve extensive objects.</td><td>Natural, numerical, iconic, gestural; symbols referring to intensive recognized can be used.</td></tr>
<tr><td colspan="3">Example of resolution:
For every 4 students there are 3 which walk. We write out the following proportion:
4 (children) ------> 3 walk
212 (children) ------> x walk
$\frac{4}{3} = \frac{212}{x}; x = 3 \times \frac{212}{4}; x = 159$
Once we obtain the number of children who walk to the school, the number of students going by car is easily obtained, 212 – 159 = 53.</td></tr>
<tr><td rowspan="2">2</td><td>Indeterminate or variables are involved.</td><td>In structural tasks equations are of the form $Ax \pm B = C$.
In functional tasks generality is recognized but operations with variables are not carried out to get canonical forms of expressions.</td><td>Symbolic – literal, used to refer the intensive recognized, although linked to the spatial, temporal and contextual information.</td></tr>
<tr><td colspan="3">Example of resolution:
$212 = x + 3x$
$212 = 4x; \quad x = 212 / 4; \quad x = 53$
53 children go by car and 212 – 53 = 159 walk.</td></tr>
<tr><td rowspan="2">3</td><td>Indeterminate or variables intervene.</td><td>In structural tasks equations are of the form $Ax \pm B = Cx \pm D$.
Operations with indeterminate or variables are carried out.</td><td>Symbolic – literal; symbols are used analytically, without referring to contextual information.</td></tr>
<tr><td colspan="3">Resolution example:
x = Children going by car
y = Children walking
$x + y = 212 \qquad x + 3x = 212;$
$y = 3x \qquad 4x = 212; x = 212/4 = 53$</td></tr>
</table>

Table 1: Characteristic features of elementary algebraic reasoning levels

The symbols x and y ($f(x)$) are variables indicative of a first level of generality; their definition domains and range are the numeric sets in which they are defined. The symbol a is also a variable; however a second level of generality is involved, since its definition domain could either be (D) as before or just another number set, and the range of values is the family of functions F_D.

Example 2: Quadratic equation

Parameters are used not only to express and operate with function families, but also with equation families (Ely & Adams, 2012). For example, $ax^2 + bx + c = 0$ ($a \neq 0$) is the general expression for the quadratic equations family. There is only one unknown, x. The letters a, b, c, usually considered as variable coefficients, take

specific values within a set of possible values (real numbers and a ≠ 0) to produce a particular equation.

Therefore, a parameter is a variable that is used with two or more other variables to specify a family of functions or equations. For families of equations the parameter is commonly named coefficient. In some way, the parameter plays the role of independent variable in a function whose domain is the set in which the parameter takes its values and whose rank is a set of functions. For each value assigned to the parameter a function image is obtained. Therefore, the expression $y = ax^2+bx+c$, is not a function but a family of functions, though it is usually referred to as "the quadratic function." It is an expression in which three parameters indicated by the letters *a, b, c* are involved. Giving a particular value to each of the parameters a specific quadratic function is obtained.

Fifth level of algebraization: treatment of parameters

We can assign a higher level of algebraization to mathematical activity displayed, when analytical (syntactic) calculations are carried out involving one or more parameters. Operations with parameter involve a higher semiotic complexity level, since the objects emerging from these systems of practices are built on algebraic objects of the previous level (equations or functions families).

Example 3: Obtaining the general formula for quadratic equations

To obtain the general formula for quadratic equations we perform symbolic manipulation and use successive equivalences. Assuming the director coefficient a is not 0 (a ≠ 0) – otherwise the equation would not be quadratic – we have:

$$ax^2 + bx + c = 0 \Leftrightarrow x^2 + \frac{b}{a}x + \frac{b}{a} = 0 \Leftrightarrow x2+ \frac{b}{a}x = -\frac{c}{a} \Leftrightarrow$$

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \Leftrightarrow x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{4ac}{4a^2} + \frac{b^2}{4a^2} \Leftrightarrow$$

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} \Leftrightarrow x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}} \Leftrightarrow x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \Leftrightarrow$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Thus, in this case the solution is written in terms of the parameters linked by rational operations (addition, subtraction, multiplication, division) and square roots.

Example 4: Geometric progressions

We define the general term of a geometric progression (Figure 1) by discursive practices in which two parameters, a_1 (first term of the sequence) and *r* (progression ratio) are involved. The sequence is a function with domain *N* and range *R*; therefore the parameters a_1 and *r* define a family of functions (sequences), and consequently this discursive practice uses an algebraization level 4. The description and proof of the sum of the first *n* terms of a geometric progression (r ≠ 0) involves a computation with parameters, as shown in Figure 1; therefore it implies the algebraization level 5.

Sixth level of algebraization

The introduction of certain algebraic structures (such as vector spaces, or groups) and the study of functional algebra (addition, subtraction, division, multiplication, and composition) start at high school. These practices bring into play higher level algebraic objects and processes according to its onto-semiotic complexity than those considered at level five. It may be useful, therefore, to characterize a sixth algebraization level to focus our attention on the specific nature of the mathematical activity involved. High school

36.8 Finite Geometric Series

When we sum a known number of terms in a geometric sequence, we get a finite geometric series. We know that we can write out each term of a geometric sequence in the general form:

$a_n = a_1 \cdot r^{n-1}$

where

- n is the index of the sequence;
- a_n is the nth-term of the sequence;
- a_1 is the first term;
- r is the common ratio (the ratio of any term to the previous term).

By simply adding together the first n terms, we are actually writing out the series

$S_n = a_1 + a_1 r + a_1 r^2 + \ldots + a_1 r^{n-2} + a_1 r^{n-1}$

We may multiply the above equation by r on both sides, giving us

$rS_n = a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n-1} + a_1 r^n$

....

Dividing by (r – 1) on both sides, we arrive at the general form of a geometric series:

$$S_n = \sum_{i=1}^{n} a_1 \cdot r^{i-1} = \frac{a_1 (r^n - 1)}{r - 1}$$

Figure 1: Finite geometric series (Free High School Science Texts, Mathematics Grades 10 – 12, p. 469, 2008)

books include texts and activities corresponding to this sixth algebraization level:

Example 5: Vector space
Figure 2 shows a general formulation for the vector space algebraic structure. In this first encounter with this algebraic structure a set of mathematical objects (vectors) are defined on which operations satisfying a set of specific properties are carried out. An initial "structural study" of vectors is required, since in this type of (axiomatic) presentation, the properties of the vector addition and multiplication by numbers have to be established.

Example 6: Composition of functions
In Figure 3 the notion of function is used in all its generality, in replacing a particular family of functions by any function. Operations are carried out over functions to produce new functions, whose properties will be studied in general. For example, properties such as "the composition of functions is not commutative" would arise. In fact, a set of functions (polynomials, for example) satisfying certain operations (addition, multiplication, etc.) is an "algebra".

Algebraization levels and strands of algebraic reasoning

In various studies Kaput has proposed a model of algebraic reasoning as a complex composite organized around five interrelated forms, or strands of reasoning listed below (Kaput & Blanton, 2001; Kaput, 2008):

1) Algebra as Generalizing and Formalizing Patterns & Constraints,
2) Algebra as Syntactically-Guided Manipulation of Formalisms.
3) Algebra as the Study of Structures and Systems Abstracted from Computations and Relations.
4) Algebra as the Study of Functions, Relations, and Joint Variation
5) Algebra as a Cluster of Modeling and Phenomena-Controlling Languages

The algebraization levels we propose are related to strands 1 and 2. Strand 1 is specified in our model by levels 1 and 2 of proto-algebraic reasoning, while strand 2 is associated with level 3, where algebra is already consolidated. Strands 3, 4 and 5 basically correspond to fields or areas of school algebra (generalized arithmetic, study of abstract structures, functions, modeling).

Our algebrization levels of primary and secondary school mathematical activity can be identified in each mathematical content strands, and involve a progressive epistemic and cognitive complexity degree due to the level of generality of mathematical objects, ostensive representations and syntactic calculation used. The presence and manipulation of parameters associated with levels 4 and 5 take place within the strands "Algebra as the study of functions", and "Algebra as a cluster of modeling of phenomena". Kaput's (2008) algebraic reasoning model is oriented mainly to characterize algebra as institutionalized mathematical content, while our model attempts to

Imagining a vector idea as an arrow help conceive the vector space: sets of vectors among which some operations satisfying certain properties are defined. But there are other mathematic entities with the same operations and properties. So, the definition of vector space is much broader and open than collections of "arrows". We have a set, V; among their elements (called vectors) two operations are defined:

SUM OF TWO ELEMENTS OF V: if $\vec{u}, \vec{v} \in V$, then $\vec{u}, \vec{v} \in V$

PRODUCT BY A REAL NUMBER: if $a \in R$ and $\vec{u} \in V$, then $a \cdot \vec{u} \in V$

If $(V, +, \cdot)$ satisfies the following properties then is a vector space on R.

	SUM OF VECTORS
ASSOCIATIVE	$(\vec{u}+\vec{v})+\vec{w}=\vec{u}+(\vec{v}+\vec{w})$
COMMUTATIVE	$\vec{u}+\vec{v}=\vec{v}+\vec{u}$
NULL VECTOR	It is a vector called $\vec{0}$ such that if $\vec{v} \in V$ fulfils: $\vec{v}+0=\vec{v}$
OPPOSITE VECTOR	All v has its opposite $-\vec{v}$: $\vec{v}+(-\vec{v})=0$
MULTIPLYING A VECTOR BY A NUMBER	
ASSOCIATIVE	$(a \cdot b) \cdot \vec{v}=a \cdot (b \cdot \vec{v})$
DISTRIBUTIVE I	$(a+b) \cdot \vec{v}=a \cdot \vec{v}+b \cdot \vec{v}$
DISTRIBUTIVE II	$a \cdot (\vec{u}+\vec{v})=a \cdot \vec{u}+a \cdot \vec{v}$
PRODUCT BY 1	If $\vec{v} \in V$ then $1 \cdot \vec{v}=\vec{v}$

Figure 2: Vector space (Colera & Oliveira, 2009, p. 62)

It is possible to combine two functions by adding, subtracting, multiplying or dividing two given functions.
There is another way to combine two functions to create a new function. It is called composition of two functions. It is a process through which we will substitute an entire function into another function.
First let's get acquainted with the notation that is used for composition of functions. When we want to find the composition of two functions we use the notation $(f \circ g)(x)$.
Another way to write this is $(f(g(x))$. This is probably the more practical notation although the first notation is what appears most often in books.

Figure 3: Composition of functions (AlgebraLAB. Project Manager. Mainland High School)

characterize the algebraic activity performed by the individuals solving mathematical tasks. Therefore both theoretical school algebra models are compatible and complementary.

ALGEBRAIZATION LEVELS AND ONTO-SEMIOTIC DISCONTINUITIES

Algebraization levels are basically generality levels, combining various registers of semiotic representation (RSR), theirs transformations and conversions (Duval, 1995). Under the OSA these levels can be characterized by the presence of different types of onto-semiotic configurations (Godino, Font, Wilhelmi, & Lurduy, 2011) which involve practices, objects and processes implying new levels of generality or syntactic calculus, supported by symbolic representations of the corresponding objects. Furthermore, they imply unitization, materialization and reification processes involved in generalization and representation (Godino et al., 2014).

Considering algebraization levels of mathematical activity can help raise awareness of gaps or discontinuities in didactical trajectories. These gaps involve the use of different registers of semiotic representation, their treatment and conversion, as well as the establishment of relations between conceptual, propositional, procedural and argumentative objects of higher generality. In other words, these gaps can be explained by analyzing how the numerical-iconic and analytical – algebraic onto-semiotic configurations involved are articulated, and not only by the treatment or conversion of RSR. Strømskag (2015) referring several studies emphasizes that it is not generalization tasks that are difficult for students but they are related to the way tasks are designed. We think that taking into account the levels of algebraization could help in selecting and design tasks that increase students' opportunities to learn algebra.

SYNTHESIS AND IMPLICATIONS FOR TEACHER EDUCATION

In this work we complemented the work by Ake and colleagues (2013) and Godino and colleagues (2014) on the identification of algebraization levels of mathematical activity in primary education, including three new levels that characterize secondary mathematics. As a summary we propose the following six levels of algebraic thinking in primary and secondary education (along with level 0, indicating absence of algebraization):

- Level 0: Operations with particular objects using natural, numerical, iconic, gestural languages are carried out.
- Level 1: Use of intensive objects (generic entities), the algebraic structure properties of N and the algebraic equality (equivalence).
- Level 2: Use of symbolic – alphanumeric representations to refer the intensive recognized, although linked to the spatial, temporal and contextual information; solving equations of the form $Ax \pm B = C$.
- Level 3: Symbols are used analytically, without referring to contextual information. Operations with indeterminate quantities or variables are carried out.
- Level 4: Studying families of equations and functions using parameters and coefficients.
- Level 5: Analytical (syntactic) calculations are carried out involving one or more parameters.

Level 6: Study of algebraic structures themselves, their definitions and structural properties.

These algebraic reasoning levels have implications for teacher training, both in primary and secondary education. In addition to develop curricular proposals (NCTM, 2000) including algebra from the earliest levels of education, the teacher need to act as the main agent of change in the introduction and development of algebraic reasoning in elementary classrooms, and its progression in secondary education. Reflecting on the recognition of algebraic thinking objects and processes can help identify the features of mathematical practices on which the teachers can intervene to gradually increase the algebraization levels of students' mathematical activity.

Consequently, recognizing the algebraization levels 4, 5 and 6 by secondary school teachers, along with its articulation with the previous levels, can help raise their awareness of the gaps or onto-semiotic discontinuities which may appear when carrying out tasks proposed to their students.

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