



**HAL**  
open science

# Symmetry of optimizers of the Caffarelli-Kohn-Nirenberg inequalities

Jean Dolbeault, Maria J. Esteban, Michael Loss

► **To cite this version:**

Jean Dolbeault, Maria J. Esteban, Michael Loss. Symmetry of optimizers of the Caffarelli-Kohn-Nirenberg inequalities. 2016. hal-01286546

**HAL Id: hal-01286546**

**<https://hal.science/hal-01286546>**

Preprint submitted on 11 Mar 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Symmetry of optimizers of the Caffarelli-Kohn-Nirenberg inequalities

J. Dolbeault<sup>1</sup> and M.J. Esteban<sup>2</sup>

*CEREMADE (CNRS UMR n° 7534),  
PSL research university, Université Paris-Dauphine,  
Place de Lattre de Tassigny, 75775 Paris 16, France*

<sup>1</sup>*E-mail: [dolbeaul@ceremade.dauphine.fr](mailto:dolbeaul@ceremade.dauphine.fr)*

<sup>2</sup>*E-mail: [esteban@ceremade.dauphine.fr](mailto:esteban@ceremade.dauphine.fr)*

M. Loss

*School of Mathematics, Georgia Tech,  
686 Cherry St., Atlanta, GA, 30332-0160, USA  
E-mail: [loss@math.gatech.edu](mailto:loss@math.gatech.edu)*

In their simplest form, the Caffarelli-Kohn-Nirenberg inequalities are a two parameter family of inequalities. It has been known that there is a region in parameter space where the optimizers for the inequalities have broken symmetry. It has been shown recently that in the complement of this region the optimizers are radially symmetric. The ideas for the proof will be given.

*Keywords:* Caffarelli-Kohn-Nirenberg inequalities; symmetry; symmetry breaking; optimal constants; rigidity results; fast diffusion equation.

2010 Mathematics Subject Classification:35J20; 49K30; 53C21

## 1. Introduction

Symmetries of optimizers in variational problems is a central theme in the calculus of variations. Sophisticated methods like rearrangement inequalities, reflection methods and moving plane methods belong now to the standard repertoire of any analyst. There are, however, examples where these methods cannot be applied. Variational problems that depend on parameters very often cannot be treated by such methods, simply because, depending on the parameters, the optimizers are symmetric and sometimes not. Famous examples are the minimizers of the Ginzburg-Landau functional in superconductivity, where, depending on the strength of the quartic interaction the minimizers form a single, symmetric vortex or a vortex lattice. Clearly such problems cannot be treated by general methods. For certain parameters they ought to work while in others they cannot. Thus, rather special techniques, tailored to the problems at hand, have to be developed to prove symmetry in the desired regions.

One class of such examples is given by the Caffarelli-Kohn-Nirenberg inequali-

ties [3]. In these notes, we shall specifically consider the case of the inequality

$$\int_{\mathbb{R}^d} \frac{|\nabla w|^2}{|x|^{2a}} dx \geq C_{a,b}^d \left( \int_{\mathbb{R}^d} \frac{|w|^p}{|x|^{bp}} dx \right)^{2/p} \quad (\text{CKN})$$

with  $a \leq b \leq a + 1$  if  $d \geq 3$ ,  $a < b \leq a + 1$  if  $d = 2$ , and  $a < a_c$  where

$$a_c := \frac{d-2}{2}, \quad p = \frac{2d}{d-2+2(b-a)}.$$

The function  $w$  is in a suitable function space which contains, for instance, all smooth functions with compact support. The constant  $C_{a,b}^d$  is, by definition, the best possible constant. Rotating the function  $w$  does not change the value of the various expressions in (CKN), i.e., the inequality is rotationally invariant. The special case where  $a \geq 0$  has been treated by various authors (see the references in [7]). Rearrangement inequalities can be used to reduce the problem to the set of radial functions, for which the optimality issue can then be solved explicitly.

For the case where  $a < 0$  the problem is much more subtle. Nevertheless, Catrina and Wang [6], proved that the optimizers, i.e., the functions that yield equality in (CKN), exist in the open strip  $a < b < a + 1$ . This result establishes the existence of non-negative solutions  $w \in L^p(\mathbb{R}^d; |x|^{-bp} dx)$  of the equation

$$-\operatorname{div}(|x|^{-2a} \nabla w) = |x|^{-bp} w^{p-1}. \quad (1)$$

Moreover, in the same paper Catrina and Wang also showed that, in some region in the  $(a, b)$  plane, the rotational symmetry of the optimizers is broken. A more detailed analysis by Felli and Schneider [8] shows that the region where the optimizers have a broken symmetry contains the set  $\mathcal{R}_{\text{FS}} := \{(a, b) : a < 0, b < b_{\text{FS}}(a)\}$  where

$$b_{\text{FS}}(a) := \frac{d(a_c - a)}{2\sqrt{(a_c - a)^2 + d - 1}} + a - a_c.$$

We call this region  $\mathcal{R}_{\text{FS}}$  the *Felli-Schneider region*.

In [8] more is shown. The optimizers in the *radial class* can be determined explicitly which allows to compute the second variation operator about these solutions. The lowest eigenvalue of this operator is strictly negative for  $(a, b) \in \mathcal{R}_{\text{FS}}$ , equals zero on the curve  $b = b_{\text{FS}}(a)$  and is strictly positive in the open complement of the Felli-Schneider region: there, the radial optimizers are stable. Needless to say that positivity of the second variation does not imply the radial symmetry of the (global) optimizers for the (CKN) inequality. Thus, it is a natural question whether or not the optimizers possess rotational symmetry in the complement of  $\mathcal{R}_{\text{FS}}$ . Let  $2^* := \frac{2d}{d-2}$  if  $d \geq 3$  and  $2^* := \infty$  if  $d = 2$ . The following theorem is proved in [7]:

**Theorem 1.1.** *Let  $d \geq 2$ ,  $p \in (2, 2^*)$ ,  $a < 0$  and  $b$  in the complement of the Felli-Schneider region and such that  $p = \frac{2d}{d-2+2(b-a)}$ . Then any non-negative solution*

$w \in L^p(\mathbb{R}^d; |x|^{-bp} dx)$  of (1) must be of the form

$$(A + B|x|^{2\alpha})^{-\frac{n-2}{2}}$$

where  $A, B$  are positive constants,

$$\alpha = \frac{(1+a-b)(a_c-a)}{a_c-a+b} \quad (2)$$

and

$$n = \frac{2p}{p-2}. \quad (3)$$

In particular this holds for the optimizers of (CKN).

There are some interesting consequences. Using the change of variables

$$w(r, \omega) = r^{a-a_c} \phi(\log r, \omega),$$

equation (1) can be cast in the form

$$-\partial_z^2 \phi - \Delta_\omega \phi + \Lambda \phi = \phi^{p-1}. \quad (4)$$

Here,  $\frac{x}{|x|} = \omega \in \mathbb{S}^{d-1}$ ,  $r = |x|$ ,  $z = \log r$ ,  $\Delta_\omega$  is the Laplace-Beltrami operator on the sphere  $\mathbb{S}^{d-1}$  and

$$\Lambda = (a - a_c)^2.$$

Thus,  $\phi$  is a function on the cylinder  $\mathbb{R} \times \mathbb{S}^{d-1}$ . Moreover, as noticed in [6], (CKN) is transformed into

$$\|\partial_z \phi\|_{L^2(\mathbb{R} \times \mathbb{S}^{d-1})}^2 + \|\nabla_\omega \phi\|_{L^2(\mathbb{R} \times \mathbb{S}^{d-1})}^2 + \Lambda \|\phi\|_{L^2(\mathbb{R} \times \mathbb{S}^{d-1})}^2 \geq C_{a,b}^d \|\phi\|_{L^p(\mathbb{R} \times \mathbb{S}^{d-1})}^2. \quad (5)$$

**Corollary 1.1.** *Let  $d \geq 2$ ,  $p \in (2, 2^*)$ . Any non-negative solution  $\phi \in L^p(\mathbb{R} \times \mathbb{S}^{d-1}; dz d\omega)$  of (4) is, up to translations, of the form*

$$\phi_\Lambda(z) = \left( \frac{2}{p\Lambda} \cosh^2 \left( \frac{p-2}{2} \sqrt{\Lambda} z \right) \right)^{-\frac{1}{p-2}},$$

if and only if

$$\Lambda \leq 4 \frac{d-1}{p^2-4}.$$

In this range, equality in (5) is achieved if and only if  $\phi(z) = \phi_\Lambda(z + z_0)$  for some  $z_0 \in \mathbb{R}$ .

To put this result in perspective we compare it with a result in [2].

**Theorem 1.2.** *Let  $d \geq 2$ ,  $p \in (2, 2^*)$ . On the sphere  $\mathbb{S}^d$  consider the equation*

$$\Delta u + \lambda u = u^{p-1}$$

with  $\lambda > 0$ . Here  $\Delta$  represents the Laplace-Beltrami operator on  $\mathbb{S}^d$ . Then the constant function  $u \equiv \lambda^{1/(p-2)}$  is the only non-negative solution if and only if

$$\lambda \leq \frac{d}{p-2}.$$

Thus, Corollary 1.1 can be viewed as an extension of the above mentioned *rigidity* result to the non-compact case of a cylinder. As a special case, this also allows to identify the equality case in the interpolation inequality (5) on the cylinder.

In the next sections some ideas about the proof are given: we start by the simple case of the standard Sobolev inequality in Section 2, explain in Section 3 how to recast (CKN) as a Sobolev type inequality in an artificial *dimension*  $n$ , where  $n$  is not necessarily an integer, and conclude by explaining how the main estimates can be produced using a *fast diffusion* flow.

## 2. Heuristics for the proof of Theorem 1.1

In order to avoid long computations it is best to explain the ideas in a ‘simple’ example. For any  $d \geq 3$ , the Sobolev inequality

$$\int_{\mathbb{R}^d} |\nabla u|^2 dx \geq C_d \left( \int_{\mathbb{R}^d} |u|^p dx \right)^{2/p}, \quad \text{with } p = 2^* = \frac{2d}{d-2} \quad (6)$$

is extremely well understood [1, 9, 10]. Once more  $C_d$  denotes the sharp constant. Note that this inequality appears as a special case of (CKN) if one sets  $a = b = 0$ , in which case  $C_d = C_{0,0}^d$ . There is equality in (6) if and only if  $u$  is a translate of the Aubin-Talenti function

$$\left( c_* \lambda + \frac{|x|^2}{\lambda} \right)^{-(d-2)/2},$$

where  $c_*$  and  $\lambda$  are positive constants. There have been some proofs using flow methods to understand this inequality [4, 5]. The flow used for the case at hand is a porous medium / fast diffusion flow. It is given by

$$\frac{\partial v}{\partial t} = \Delta v^{1-\frac{1}{d}} \quad (7)$$

and has the self-similar solutions

$$v_*(x, t) = \left( c_* t + \frac{|x|^2}{t} \right)^{-d}.$$

This function has slow decay in the  $x$  variable. The obvious similarity of the expressions of the Aubin-Talenti and self-similar functions suggests a reformulation of the Sobolev functional by setting

$$v = u^{\frac{2d}{d-2}}.$$

Let us define a *pressure variable*  $\mathbf{p}$  by

$$v = \mathbf{p}^{-d}.$$

A short computation shows

**Lemma 2.1.** *The Sobolev inequality, written in terms of  $v$  and  $\mathbf{p}$ , is given by*

$$a_c^2 \int_{\mathbb{R}^d} v |\nabla \mathbf{p}|^2 dx \geq C_d \left( \int_{\mathbb{R}^d} v dx \right)^{\frac{d-2}{d}}. \quad (8)$$

Assume now that  $v$  satisfies the fast diffusion equation (7). This implies that  $\mathbf{p}$  evolves by the equation

$$\frac{\partial \mathbf{p}}{\partial t} = \frac{d-1}{d} (\mathbf{p} \Delta \mathbf{p} - d |\nabla \mathbf{p}|^2).$$

The right side of (8) does not change if  $v$  evolves via (7). For the left side we have

**Lemma 2.2.** *Assume that  $v$  evolves via (7). Then*

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} v |\nabla \mathbf{p}|^2 dx &= -2 \int_{\mathbb{R}^d} \left[ \frac{1}{2} \Delta |\nabla \mathbf{p}|^2 - \nabla \mathbf{p} \cdot \nabla \Delta \mathbf{p} - \frac{1}{d} (\Delta \mathbf{p})^2 \right] \mathbf{p}^{1-d} dx \\ &= -2 \int_{\mathbb{R}^d} \text{Tr} \left[ \mathbf{H}_{\mathbf{p}} - \frac{1}{d} (\text{Tr } \mathbf{H}_{\mathbf{p}}) \text{Id} \right]^2 \mathbf{p}^{1-d} dx \end{aligned}$$

where  $\mathbf{H}_{\mathbf{p}} = (\nabla \otimes \nabla) \mathbf{p}$  denotes the Hessian matrix of  $\mathbf{p}$ . Moreover,

$$\mathbf{H}_{\mathbf{p}} - \frac{1}{d} (\text{Tr } \mathbf{H}_{\mathbf{p}}) \text{Id} = 0$$

if and only if  $\mathbf{p}(x) = a + b \cdot x + c|x|^2$  for some  $(a, b, c) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}$ .

The proof is a somewhat longish but straightforward computation. Note, that it is precisely the particular choice of  $v$  and  $\mathbf{p}$  that renders the time derivative in such a simple form.

To summarize, while the right side of the Sobolev inequality stays fixed the left side diminishes under the flow. *The idea is to use the fast diffusion flow to drive the functional towards its optimal value.* Actually we use the fact that if  $v$  is optimal in (8), or if it is a critical point, the functional has to be stationary under the action of the flow, which allows to identify  $\mathbf{p}$ , hence  $v$ . To exploit this idea for the (CKN) inequality we have to rewrite it in the form of a Sobolev type inequality.

### 3. A modified Sobolev inequality

The first step in the proof is to rewrite the problem in a form that resembles the Sobolev inequality. If we write

$$w(r, \omega) = u(s, \omega) \quad \text{with} \quad s = r^\alpha,$$

the inequality (CKN) takes the form

$$\int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} \left[ \alpha^2 \left( \frac{\partial u}{\partial s} \right)^2 + \frac{|\nabla_\omega u|^2}{s^2} \right] s^{n-1} ds d\omega \geq C_{a,b}^d \alpha^{1-\frac{2}{p}} \left( \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} |u|^p s^{n-1} ds d\omega \right)^{\frac{2}{p}}$$

where  $d\omega$  denotes the uniform measure on the sphere  $\mathbb{S}^{d-1}$ ,  $\nabla_\omega$  denotes the gradient on  $\mathbb{S}^{d-1}$  and where  $\alpha$  and  $n$  are given by (2) and (3). We shall abbreviate

$$Du := \left( \alpha \frac{\partial u}{\partial s}, \frac{1}{s} \nabla_\omega u \right), \quad |Du|^2 = \alpha^2 \left( \frac{\partial u}{\partial s} \right)^2 + \frac{|\nabla_\omega u|^2}{s^2}.$$

Our inequality is therefore equivalent to a Sobolev type inequality and takes the form

$$\int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} |Du|^2 d\mu \geq C_{a,b}^d \alpha^{1-\frac{2}{p}} \left( \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} |u|^p d\mu \right)^{\frac{2}{p}}, \quad \text{with } p = \frac{2n}{n-2}. \quad (9)$$

This inequality generalizes (6). Here the measure  $d\mu$  is defined on  $\mathbb{R}^+ \times \mathbb{S}^{d-1}$  by

$$d\mu = s^{n-1} ds d\omega.$$

As in Section 2, we may consider  $v = u^p$  and define a pressure variable  $\mathbf{p}$  such that  $v = \mathbf{p}^{-n}$ , so that  $u = \mathbf{p}^{-(n-2)/2}$ . With these notations, (9) can be rewritten as

$$\frac{1}{4} (n-2)^2 \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} v |D\mathbf{p}|^2 d\mu \geq C_{a,b}^d \alpha^{1-\frac{2}{p}} \left( \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} v d\mu \right)^{\frac{2}{p}}. \quad (10)$$

With straightforward abuses of notations, we shall write  $\int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} f d\mu = \int_{\mathbb{R}^d} f dx$  and identify  $L^p(\mathbb{R}_+ \times \mathbb{S}^{d-1}; d\mu)$  with  $L^p(\mathbb{R}^d; |x|^{n-d} dx)$  or simply  $L^p(\mathbb{R}^d; d\mu)$ .

One should note that  $n$  is, in general, not an integer and the above inequality reduces to Sobolev's inequality only if  $n = d$ . Of particular significance is that the curve

$$b = b_{\text{FS}}(a),$$

when represented in the new variables  $\alpha$  and  $n$ , is given by the equation  $\alpha = \alpha_{\text{FS}}$  with

$$\alpha_{\text{FS}} := \sqrt{\frac{d-1}{n-1}}.$$

Thus, for  $\alpha > \alpha_{\text{FS}}$  the minimizers are not radial. The equation (1) transforms into the equation

$$-\mathcal{L} u = u^{p-1}, \quad (11)$$

where  $\mathcal{L}$  is the Laplacian associated with the quadratic form given by the left side of (9), i.e.,  $\mathcal{L} = -D^* \cdot D$ . Theorem 1.1 can be reformulated as

**Theorem 3.1.** *Let  $d \geq 2$ ,  $p \in (2, 2^*)$ ,  $n = \frac{2p}{p-2} > d$  and  $\alpha \leq \alpha_{\text{FS}}$ . Then any non-negative solution  $u \in L^p(\mathbb{R}^d; d\mu)$  of (11) must be of the form*

$$(A + B|x|^2)^{-\frac{n-2}{2}} \quad (12)$$

where  $A, B$  are positive constants, and  $n$  is given by (3). As a special case, equality in (10) is achieved if and only if  $u$  is given by (12).

The upshot of this work can be summarized in the following fashion: *Any optimizer in the radial class that is not unstable under small perturbations is in fact a global minimizer for the (CKN) inequality.*

#### 4. The flow

We consider the fast diffusion flow

$$\frac{\partial v}{\partial t} = \mathcal{L} v^{1-\frac{1}{n}}. \quad (13)$$

It is easily seen that the flow (13) has the self-similar solutions

$$v_*(t; s, \omega) = t^{-n} \left( c_* + \frac{s^2}{2(n-1)\alpha^2 t^2} \right)^{-n}.$$

The basic idea is now quite simple. We consider a non-negative solution  $u \in L^p(\mathbb{R}^d; d\mu)$  of (11) and set  $v = u^p$ . We also consider the pressure variable  $\mathbf{p}$  such that  $v = \mathbf{p}^{-n}$ . The first thing to note is that the right side of (10) does not change if we evolve  $v$  and hence  $u$  under the flow (13). Further, if we differentiate the left side of (10) along the flow we obtain

$$\frac{d}{dt} \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} v |\mathbf{D}\mathbf{p}|^2 d\mu = -2 \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} \left[ \frac{1}{2} \mathcal{L} |\mathbf{D}\mathbf{p}|^2 - \mathbf{D}\mathbf{p} \cdot \mathbf{D}\mathcal{L}\mathbf{p} - \frac{1}{n} (\mathcal{L}\mathbf{p})^2 \right] d\mu.$$

On the other hand simple computations show that

$$\frac{1}{4} (n-2)^2 \frac{d}{dt} \left( \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} v |\mathbf{D}\mathbf{p}|^2 d\mu \right) \Big|_{t=0} = -2 \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} (\mathcal{L}u) u^{1-p} (\mathcal{L}u^{p(n-1)/n}) d\mu \quad (14)$$

when expressed in terms of  $u$ . Now we take  $v = u^p$ , where  $u$  is the solution to (11), as initial datum for (1). With this choice, the right side in (14) is actually zero. Indeed, by multiplying both sides of (11) by  $u^{1-p} (\mathcal{L}u^{p(n-1)/n})$  one obtains

$$\int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} (\mathcal{L}u) u^{1-p} (\mathcal{L}u^{p(n-1)/n}) d\mu = \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} u^{p-1} u^{1-p} (\mathcal{L}u^{p(n-1)/n}) d\mu = 0.$$

The interesting point, and the heart of the argument, is that

$$0 = \int_{\mathbb{R}_+ \times \mathbb{S}^{d-1}} \left[ \frac{1}{2} \mathcal{L} |\mathbf{D}\mathbf{p}|^2 - \mathbf{D}\mathbf{p} \cdot \mathbf{D}\mathcal{L}\mathbf{p} - \frac{1}{n} (\mathcal{L}\mathbf{p})^2 \right] d\mu$$



can be written as a sum of non-negative terms precisely when  $\alpha \leq \alpha_{\text{FS}}$ , and the vanishing of these terms shows that  $u$  must be of the form  $(A + B s^2)^{-(n-2)/2}$ . In this way one obtains a classification of the non-negative solutions of (11) provided they are in  $L^p(\mathbb{R}^d; d\mu)$ . To simplify notations, we shall omit the index  $\omega$ , so that from now on  $\nabla$  and  $\Delta$  respectively refer to the gradient and to the Laplace-Beltrami operator on  $\mathbb{S}^{d-1}$ . With the notation  $' = \partial_s$ , our identity can be reworked as follows.

**Lemma 4.1.** *Assume that  $d \geq 3$ ,  $n > d$  and let  $\mathbf{p}$  be a positive function in  $C^3(\mathbb{S}^{d-1})$ . Then*

$$\begin{aligned} & \frac{1}{2} \mathcal{L} |\text{D}\mathbf{p}|^2 - \text{D}\mathbf{p} \cdot \text{D}\mathcal{L}\mathbf{p} - \frac{1}{n} (\mathcal{L}\mathbf{p})^2 \\ &= \alpha^4 \frac{n-1}{n} \left[ \mathbf{p}'' - \frac{\mathbf{p}'}{r} - \frac{\Delta\mathbf{p}}{\alpha^2(n-1)r^2} \right]^2 + \frac{2\alpha^2}{r^2} \left| \nabla\mathbf{p}' - \frac{\nabla\mathbf{p}}{r} \right|^2 \\ &+ \frac{1}{r^4} \left[ \frac{1}{2} \Delta |\nabla\mathbf{p}|^2 - \nabla\mathbf{p} \cdot \nabla\Delta\mathbf{p} - \frac{1}{n-1} (\Delta\mathbf{p})^2 - (n-2)\alpha^2 |\nabla\mathbf{p}|^2 \right]. \end{aligned}$$

The only term in Lemma 4.1 that does not have a sign is the last one. When integrated against  $\mathbf{p}^{1-n}$  over  $\mathbb{S}^{d-1}$ , however, this term can be written as a sum of squares. The following lemma holds for  $d \geq 3$ . For the case  $d = 2$  we refer the reader to [7].

**Lemma 4.2.** *Assume that  $d \geq 3$  and that  $\mathbf{p}$  is a positive function in  $C^3(\mathbb{S}^{d-1})$ . Then*

$$\begin{aligned} & \int_{\mathbb{S}^{d-1}} \left[ \frac{1}{2} \Delta |\nabla\mathbf{p}|^2 - \nabla\mathbf{p} \cdot \nabla\Delta\mathbf{p} - \frac{1}{n-1} (\Delta\mathbf{p})^2 - (n-2)\alpha^2 |\nabla\mathbf{p}|^2 \right] \mathbf{p}^{1-n} d\omega \\ &= \frac{(n-2)(d-1)}{(n-1)(d-2)} \int_{\mathbb{S}^{d-1}} \left\| \text{L}\mathbf{p} - \frac{3(n-1)(n-d)}{2(n-2)(d+1)} \text{M}\mathbf{p} \right\|^2 \mathbf{p}^{1-n} d\omega \\ &+ \frac{n-d}{2(d+1)} \left[ \frac{n+3}{2} + \frac{3(n-1)(n+1)(d-2)}{2(n-2)(d+1)} \right] \int_{\mathbb{S}^{d-1}} \frac{|\nabla\mathbf{p}|^4}{\mathbf{p}^2} \mathbf{p}^{1-n} d\omega \\ &+ (n-2) [\alpha_{\text{FS}}^2 - \alpha^2] \int_{\mathbb{S}^{d-1}} |\nabla\mathbf{p}|^2 \mathbf{p}^{1-n} d\omega \end{aligned}$$

where  $\text{L}\mathbf{p} := (\nabla \otimes \nabla)\mathbf{p} - \frac{1}{d-1} (\Delta\mathbf{p})g$  and  $\text{M}\mathbf{p} := \frac{\nabla\mathbf{p} \otimes \nabla\mathbf{p}}{\mathbf{p}} - \frac{1}{d-1} \frac{|\nabla\mathbf{p}|^2}{\mathbf{p}} g$ . Here  $g$  is the standard metric on  $\mathbb{S}^{d-1}$  and  $\text{L}\mathbf{p}$  denotes the trace free Hessian of  $\mathbf{p}$ .

The key device used for the proof of this lemma is the Bochner-Lichnerowicz-Weitzenböck formula. If  $\mathcal{M}$  is a compact Riemannian manifold, then for any smooth function  $f : \mathcal{M} \rightarrow \mathbb{R}$  we have

$$\frac{1}{2} \Delta |\nabla f|^2 = \|\text{H}_f\|^2 + \text{Ric}(\nabla f, \nabla f)$$

where  $\|\text{H}_f\|^2$  is the trace of the square of the Hessian of  $f$  and  $\text{Ric}(\nabla f, \nabla f)$  is the Ricci curvature tensor contracted against  $\nabla f \otimes \nabla f$ . If  $\mathcal{M} = \mathbb{S}^{d-1}$ , then  $\text{Ric}(\nabla f, \nabla f) = (d-2)|\nabla f|^2$ . The main point in Lemma 4.2 is that, provided  $\alpha \leq \alpha_{\text{FS}}$ , all terms are non-negative.

It is quite easy to see that the vanishing of these terms entails that  $p$  can only depend on the variable  $s = |x|$  and must be of the form (12).

While the formal computations are straightforward there is the perennial issue of the boundary terms that occur in all the integration by parts. This is due to the fact that one is dealing with solutions of (11) and it is not at all clear that the boundary terms vanish. This requires a detailed regularity analysis of the solutions of (11). The task is non-trivial because the exponent  $p$  is critical for the scaling in the  $s$  variable. The reader may consult [7] for details.

The computations outlined above can be carried over to the case where  $\mathbb{S}^{d-1}$  is replaced by a compact Riemannian manifold  $\mathcal{M}$  of dimension  $d-1$ . The results are then expressed in terms of the Ricci curvature of the manifold. Again the reader may consult [7] for details.

**Acknowledgment:** This work has been partially supported by the Projects STAB and Kibord (J.D.) of the French National Research Agency (ANR). M.L. has been partially supported by NSF Grant DMS-1301555 and the Humboldt Foundation.

© 2016 by the authors. This paper may be reproduced, in its entirety, for non-commercial purposes.

## References

- [1] T. AUBIN, *Problèmes isopérimétriques et espaces de Sobolev*, J. Differential Geometry, 11 (1976), pp. 573–598.
- [2] M.-F. BIDAUT-VÉRON AND L. VÉRON, *Nonlinear elliptic equations on compact Riemannian manifolds and asymptotics of Emden equations*, Invent. Math., 106 (1991), pp. 489–539.
- [3] L. CAFFARELLI, R. KOHN, AND L. NIRENBERG, *First order interpolation inequalities with weights*, Compositio Math., 53 (1984), pp. 259–275.
- [4] E. A. CARLEN AND M. LOSS, *Extremals of functionals with competing symmetries*, J. Funct. Anal., 88 (1990), pp. 437–456.
- [5] J. A. CARRILLO AND G. TOSCANI, *Asymptotic  $L^1$ -decay of solutions of the porous medium equation to self-similarity*, Indiana Univ. Math. J., 49 (2000), pp. 113–142.
- [6] F. CATRINA AND Z.-Q. WANG, *On the Caffarelli-Kohn-Nirenberg inequalities: sharp constants, existence (and nonexistence), and symmetry of extremal functions*, Comm. Pure Appl. Math., 54 (2001), pp. 229–258.
- [7] J. DOLBEAULT, M. J. ESTEBAN, AND M. LOSS, *Rigidity versus symmetry breaking via nonlinear flows on cylinders and Euclidean spaces*. Preprint, to appear in *Inventiones Mathematicae*, 2016.
- [8] V. FELLI AND M. SCHNEIDER, *Perturbation results of critical elliptic equations of Caffarelli-Kohn-Nirenberg type*, J. Differential Equations, 191 (2003), pp. 121–142.
- [9] E. H. LIEB, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. (2), 118 (1983), pp. 349–374.
- [10] G. TALENTI, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl. (4), 110 (1976), pp. 353–372.