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# Pathwise integration with respect to paths of finite quadratic variation

Anna ANANOVA and Rama CONT

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## Abstract

We study a notion of pathwise integral, defined as the limit of non-anticipative Riemann sums, with respect to paths of finite quadratic variation. The construction allows to integrate 'gradient-type' integrands with respect to Hölder-continuous functions of Hölder index  $p < 1/2$ . We prove a pathwise isometry property for this integral, analogous to the well-known Ito isometry for stochastic integrals. This property is then used to represent the integral as a continuous map on an appropriately defined vector space of integrands and obtain a pathwise 'signal plus noise' decomposition for a large class of irregular paths obtained through functional transformations of a reference path with non-vanishing quadratic variation.

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In his seminal paper 'Calcul d'Ito sans probabilités' [11], Hans Föllmer proved a change of variable formula for smooth functions of paths with infinite variation, using the concept of *quadratic variation along a sequence of partitions*. A path  $x \in C^0([0, T], \mathbb{R})$  is said to have finite quadratic variation along the sequence of partitions  $\pi^m = (0 = t_0^m < t_1^m < \dots < t_{k(m)}^m = T)$  if for any  $t \in [0, T]$ , the limit

$$[x]_\pi(t) := \lim_{m \rightarrow \infty} \sum_{t_i^m \leq t} (x(t_{i+1}^m) - x(t_i^m))^2 < \infty \quad (1)$$

exists and defines a continuous increasing function  $[x]_\pi : [0, T] \rightarrow \mathbb{R}_+$  called the quadratic variation of  $x$  along  $\pi$ . Föllmer [11] defined an integral with respect to such paths, as a pathwise limit of Riemann sums along the partitions  $\pi^m$ , for integrands of the type  $\nabla f, f \in C^2(\mathbb{R}^d)$ .

This construction has been extended in various directions, to less regular functions [1, 8, 18] and path-dependent functionals [3, 6, 7]. In particular, Cont & Fournié [3] proved change of variable formulas for pathwise integrals of the type

$$\int_0^T \nabla_\omega F(t, \omega) \cdot d^\pi \omega$$

where  $F$  is a non-anticipative functional and  $\nabla_\omega F$  a directional derivative (Dupire derivative).

This paper is a systematic study of Föllmer's pathwise approach to stochastic integration and its extension to path-dependent functionals [3].

Our main result is a pathwise isometry formula for such integrals (Theorem 7): we give conditions on the integrand  $\phi$  and the path  $\omega \in C^0([0, T], \mathbb{R})$  under which

$$\left[ \int_0^\cdot \phi \cdot d^\pi \omega \right]_\pi(t) = \int_0^t |\phi|^2 d[\omega]_\pi.$$

Our conditions notably allow for Hölder-continuous paths with Hölder index strictly less than  $1/2$ , so our results apply to Brownian paths and paths of diffusion processes. Integrating our formula over the space of continuous functions with the Wiener measure, or any other measure under which the canonical process is a square-integrable martingale, then yields the well-known Ito isometry formula [14], showing that underlying the Ito isometry is an identity which holds pathwise.

This result is then used to identify a space of integrands on which the pathwise integral defines a continuous mapping for the 'quadratic variation' metric (Proposition 13). This continuity property, together with its interpretation as a limit of Riemann sums, distinguish our pathwise integral from other constructions which have one property or the other but not both [15, 16, 18].

A second result of our paper is to obtain a pathwise 'signal plus noise' decomposition for irregular paths, in the spirit of [12]: we show that, given a path  $\bar{\omega}$  with non-vanishing quadratic variation, any regular functional may be uniquely decomposed as the sum of a pathwise integral with respect to  $\bar{\omega}$  and a function with zero quadratic variation (Proposition 14).

Finally, we clarify the pathwise nature of the integral defined in [3]: we show (Theorem 16) that this integral is indeed a pathwise limit of non-anticipative Riemann sums, which is important for interpretation and in applications.

**Outline** Section 1 recalls some key definitions and results on functional calculus from [3, 2] and recalls the definition of the Föllmer integral [11] and its extension to path-dependent integrands by Cont & Fournié [3]. The 'isometry formula' for this integral is derived in Section 2 (Theorem 7). We use this result in Section 3 to represent the integral as a continuous map (Proposition 13). Finally, in Section 4 we investigate the 'pathwise' nature of the integral and show that its value along a path may be represented as a limit of non-anticipative Riemann sums computed along the same path.

# 1 Non-anticipative functional calculus and integration of gradient functionals

Our approach relies on the Non-anticipative Functional Calculus [3, 2], a functional calculus which applies to non-anticipative functionals of cadlag paths with finite quadratic variation, in the sense of Föllmer [11]. We recall here some key concepts and results of this approach, following [2].

Let  $X$  be the canonical process on  $\Omega = D([0, T], \mathbb{R}^d)$ , and  $(\mathcal{F}_t^0)_{t \in [0, T]}$  be the filtration generated by  $X$ . We are interested in *causal*, or *non-anticipative functionals* of  $X$  [10], that is, functionals  $F : [0, T] \times D([0, T], \mathbb{R}^d) \mapsto \mathbb{R}$  such that

$$\forall \omega \in \Omega, \quad F(t, \omega) = F(t, \omega_t). \quad (2)$$

The process  $t \mapsto F(t, X_t)$  then only depends on the path of  $X$  up to  $t$  and is  $(\mathcal{F}_t^0)$ -adapted.

It is convenient to define such functionals on the space of stopped paths [2]: a stopped path is an equivalence class in  $[0, T] \times D([0, T], \mathbb{R}^d)$  for the following equivalence relation:

$$(t, \omega) \sim (t', \omega') \iff (t = t' \quad \text{and} \quad \omega(t \wedge \cdot) = \omega'(t' \wedge \cdot) \quad ). \quad (3)$$

The space of stopped paths is defined as the quotient of  $[0, T] \times D([0, T], \mathbb{R}^d)$  by the equivalence relation (3):

$$\Lambda_T = \{(t, \omega(t \wedge \cdot)), (t, \omega) \in [0, T] \times D([0, T], \mathbb{R}^d)\} = ([0, T] \times D([0, T], \mathbb{R}^d)) / \sim$$

We denote  $\mathcal{W}_T$  the subset of  $\Lambda_T$  consisting of continuous stopped paths. We endow this set with a metric space structure by defining the following distance:

$$d_\infty((t, \omega), (t', \omega')) = \sup_{u \in [0, T]} |\omega(u \wedge t) - \omega'(u \wedge t')| + |t - t'| = \|\omega_t - \omega'_t\|_\infty + |t - t'|$$

$(\Lambda_T, d_\infty)$  is then a complete metric space. Any map  $F : [0, T] \times D([0, T], \mathbb{R}^d) \rightarrow \mathbb{R}$  verifying the causality condition (2) can be equivalently viewed as a functional on the space  $\Lambda_T$  of stopped paths:

**Definition 1.** *A non-anticipative functional on  $D([0, T], \mathbb{R}^d)$  is a measurable map  $F : (\Lambda_T, d_\infty) \rightarrow \mathbb{R}$  on the space  $(\Lambda_T, d_\infty)$  of stopped paths.*

We denote by  $\mathbb{C}^{0,0}(\Lambda_T)$  the set of continuous maps  $F : (\Lambda_T, d_\infty) \mapsto \mathbb{R}$ . Some weaker notions of continuity for non-anticipative functionals turn out to be useful [4]:

**Definition 2.** *A non-anticipative functional  $F$  is said to be:*

- *continuous at fixed times if for any  $t \in [0, T]$ ,  $F(t, \cdot)$  is continuous with respect to the uniform norm  $\|\cdot\|_\infty$  in  $[0, T]$ , i.e.  $\forall \omega \in D([0, T], \mathbb{R}^d)$ ,  $\forall \epsilon > 0$ ,  $\exists \eta > 0$ ,  $\forall \omega' \in D([0, T], \mathbb{R}^d)$ ,*

$$\|\omega_t - \omega'_t\| < \eta \implies |F(t, \omega) - F(t, \omega')| < \epsilon$$

- *left-continuous if  $\forall (t, \omega) \in \Lambda_T$ ,  $\forall \epsilon > 0$ ,  $\exists \eta > 0$  such that  $\forall (t', \omega') \in \Lambda_T$ ,*

$$(t' < t \quad \text{and} \quad d_\infty((t, \omega), (t', \omega')) < \eta) \implies |F(t, \omega) - F(t', \omega')| < \epsilon$$

We denote by  $\mathbb{C}_l^{0,0}(\Lambda_T)$  the set of left-continuous functionals. Similarly, we can define the set  $\mathbb{C}_r^{0,0}(\Lambda_T)$  of right-continuous functionals.

We also introduce a notion of local boundedness for functionals.

**Definition 3.** A non-anticipative functional  $F$  is said to be boundedness-preserving if for every compact subset  $K$  of  $\mathbb{R}^d$ ,  $\forall t_0 \in [0, T]$ ,  $\exists C(K, t_0) > 0$  such that:

$$\forall t \in [0, t_0], \quad \forall (t, \omega) \in \Lambda_T, \quad \omega([0, t]) \subset K \implies |F(t, \omega)| < C(K, t_0).$$

We denote by  $\mathbb{B}(\Lambda_T)$  the set of boundedness-preserving functionals.

We now recall some notions of differentiability for functionals following [3, 2, 9]. For  $e \in \mathbb{R}^d$  and  $\omega \in D([0, T], \mathbb{R}^d)$ , we define the vertical perturbation  $\omega_t^e$  of  $(t, \omega)$  as the càdlàg path obtained by shifting the path by  $e$  after  $t$ :

$$\omega_t^e = \omega_t + e\mathbf{1}_{[t, T]}.$$

**Definition 4.** A non-anticipative functional  $F$  is said to be:

- horizontally differentiable at  $(t, \omega) \in \Lambda_T$  if

$$\mathcal{D}F(t, \omega) = \lim_{h \rightarrow 0^+} \frac{F(t+h, \omega_t) - F(t, \omega_t)}{h} \quad (4)$$

exists. If  $\mathcal{D}F(t, \omega)$  exists for all  $(t, \omega) \in \Lambda_T$ , then (4) defines a non-anticipative functional  $\mathcal{D}F$ , called the horizontal derivative of  $F$ .

- vertically differentiable at  $(t, \omega) \in \Lambda_T$  if the map:

$$\begin{array}{ccc} g_{(t, \omega)} : \mathbb{R}^d & \longrightarrow & \mathbb{R} \\ e & \mapsto & F(t, \omega_t + e\mathbf{1}_{[t, T]}) \end{array}$$

is differentiable at 0. Its gradient at 0 is called the Dupire derivative (or vertical derivative) of  $F$  at  $(t, \omega)$ :

$$\nabla_\omega F(t, \omega) = \nabla g_{(t, \omega)}(0) \in \mathbb{R}^d \quad (5)$$

i.e.  $\nabla_\omega F(t, \omega) = (\partial_i F(t, \omega), i = 1, \dots, d)$  with

$$\partial_i F(t, \omega) = \lim_{h \rightarrow 0} \frac{F(t, \omega_t + he_i\mathbf{1}_{[t, T]}) - F(t, \omega_t)}{h}$$

where  $(e_i, i = 1, \dots, d)$  is the canonical basis of  $\mathbb{R}^d$ . If  $F$  is vertically differentiable at all  $(t, \omega) \in \Lambda_T$ ,  $\nabla_\omega F : \Lambda_T \rightarrow \mathbb{R}^d$  defines a non-anticipative functional called the vertical derivative of  $F$ .

We may repeat the same operation on  $\nabla_\omega F$  and define similarly  $\nabla_\omega^2 F, \nabla_\omega^3 F, \dots$ . This leads us to define the the following classes of smooth functionals:

**Definition 5.** We define  $\mathbb{C}_b^{1, k}(\Lambda_T)$  as the set of non-anticipative functionals  $F : (\Lambda_T, d_\infty) \rightarrow \mathbb{R}$  which are

- horizontally differentiable with  $\mathcal{D}F$  continuous at fixed times;
- $k$  times vertically differentiable with  $\nabla_\omega^j F \in \mathbb{C}_l^{0, 0}(\Lambda_T)$  for  $j = 0, \dots, k$ ;
- $\mathcal{D}F, \nabla_\omega F, \dots, \nabla_\omega^k F \in \mathbb{B}(\Lambda_T)$ .

We denote  $\mathbb{C}_b^{1,\infty}(\Lambda_T) = \cap_{k \geq 1} \mathbb{C}_b^{1,k}(\Lambda_T)$ .

Consider now a sequence  $\pi_n = (0 = t_0^n < t_1^n \dots < t_{k(n)}^n = T)$  of partitions of  $[0, T]$ .  $|\pi_n| = \sup\{|t_{i+1}^n - t_i^n|, i = 1..k(n)\} \rightarrow 0$  will denote the mesh size of the partition. A càdlàg path  $x \in D([0, T], \mathbb{R})$  is said to have finite quadratic variation along the sequence of partitions  $(\pi_n)_{n \geq 1}$  if for any  $t \in [0, T]$  the limit

$$[x](t) := \lim_{n \rightarrow \infty} \sum_{t_{i+1}^n \leq t} (x(t_{i+1}^n) - x(t_i^n))^2 < \infty \quad (6)$$

exists and the increasing function  $[x]$  has Lebesgue decomposition

$$[x]_\pi(t) = [x]_\pi^c(t) + \sum_{0 < s \leq t} |\Delta x(s)|^2$$

where  $[x]_\pi^c$  is a continuous, increasing function. We denote the set of such paths  $Q_\pi([0, T], \mathbb{R})$ .

A  $d$ -dimensional path  $x = (x^1, \dots, x^d) \in D([0, T], \mathbb{R}^d)$  is said to have finite quadratic variation along  $\pi$  if  $x^i \in Q_\pi([0, T], \mathbb{R})$  and  $x^i + x^j \in Q_\pi([0, T], \mathbb{R})$  for all  $i, j = 1..d$ . Then for any  $i, j = 1..d$  and  $t \in [0, T]$ , we have

$$\sum_{t_k^n \in \pi_n, t_k^n \leq t} (x^i(t_{k+1}^n) - x^i(t_k^n)) \cdot (x^j(t_{k+1}^n) - x^j(t_k^n)) \xrightarrow{n \rightarrow \infty} [x]_{ij}(t) = \frac{[x^i + x^j](t) - [x^i](t) - [x^j](t)}{2}.$$

The matrix-valued function  $[x] : [0, T] \rightarrow S_d^+$  whose elements are given by

$$[x]_{ij}(t) = \frac{[x^i + x^j](t) - [x^i](t) - [x^j](t)}{2}$$

is called the *quadratic covariation* of the path  $x$ . For further discussion of this concept, we refer to [2, 20].

Consider now a path  $\omega \in Q_\pi([0, T], \mathbb{R}^d)$  with finite quadratic variation along  $\pi$ . Since  $\omega$  has at most a countable set of jump times, we may assume that the partition 'exhausts' the jump times in the sense that

$$\sup_{t \in [0, T] - \pi_n} |\omega(t) - \omega(t-)| \xrightarrow{n \rightarrow \infty} 0. \quad (7)$$

Then the piecewise-constant approximation

$$\omega^n(t) = \sum_{i=0}^{k(n)-1} \omega(t_{i+1}^n -) 1_{[t_i, t_{i+1}^n)}(t) + \omega(T) 1_{\{T\}}(t) \quad (8)$$

converges uniformly to  $\omega$ :  $\sup_{t \in [0, T]} \|\omega^n(t) - \omega(t)\| \xrightarrow{n \rightarrow \infty} 0$ . Note that with the notation (8),  $\omega^n(t_i^n -) = \omega(t_i^n -)$  but  $\omega^n(t_i^n) = \omega(t_{i+1}^n -)$ . If we define

$$\omega_{t_i^n}^{n, \Delta \omega(t_i^n)} = \omega^n + \Delta \omega(t_i^n) 1_{[t_i^n, T]}, \quad \text{then} \quad \omega_{t_i^n -}^{n, \Delta \omega(t_i^n)}(t_i^n) = \omega(t_i^n).$$

Approximating the variations of the functional by vertical and horizontal increments along the partition  $\pi_n$ , we obtain the following pathwise change of variable formula for  $\mathbb{C}^{1,2}(\Lambda_T)$  functionals, derived in [3] under more general assumptions:

**Theorem 6** (Pathwise change of variable formula for  $\mathbb{C}^{1,2}$  functionals [3]). *Let  $\omega \in Q_\pi([0, T], \mathbb{R}^d)$  verifying (7). Then for any  $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$  the limit*

$$\int_0^T \nabla_\omega F(t, \omega_{t-}) d^\pi \omega := \lim_{n \rightarrow \infty} \sum_{i=0}^{k(n)-1} \nabla_\omega F \left( t_i^n, \omega_{t_i^n-}^{n, \Delta \omega(t_i^n)} \right) \cdot (\omega(t_{i+1}^n) - \omega(t_i^n)) \quad (9)$$

exists and

$$\begin{aligned} F(T, \omega_T) - F(0, \omega_0) &= \int_0^T \mathcal{D}F(t, \omega_t) dt + \int_0^T \frac{1}{2} \text{tr} \left( {}^t \nabla_\omega^2 F(t, \omega_{t-}) d[\omega]^c(t) \right) \\ &+ \int_0^T \nabla_\omega F(t, \omega_{t-}) d^\pi \omega + \sum_{t \in ]0, T[} [F(t, \omega_t) - F(t, \omega_{t-}) - \nabla_\omega F(t, \omega_{t-}) \cdot \Delta \omega(t)]. \end{aligned} \quad (10)$$

A consequence of this theorem is the ability to define the pathwise integral  $\int_0^\cdot \nabla_\omega F(t, \omega_{t-}) d^\pi \omega$  as a limit of "Riemann sums" computed along the sequence of partitions  $\pi$ . For a continuous path  $\omega \in C^0([0, T], \mathbb{R}^d)$  this simplifies to:

$$\int_0^T \nabla_\omega F(t, \omega) d^\pi \omega := \lim_{n \rightarrow \infty} \sum_{i=0}^{k(n)-1} \nabla_\omega F \left( t_i^n, \omega_{t_i^n-}^n \right) \cdot (\omega(t_{i+1}^n) - \omega(t_i^n)) \quad (11)$$

This integral, first constructed in [3], extends H. Föllmer's construction [11] for integrands of the form  $\nabla f$ ,  $f \in C^2(\mathbb{R}^d)$  to (path-dependent) integrands of the form

$$\nabla_\omega F, \quad F \in \mathbb{C}_b^{1,2}(\Lambda_T).$$

The goal of this paper is to explore the properties of this integral.

## 2 Pathwise isometry formula

Let  $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$  and let  $\omega \in Q_\pi([0, T], \mathbb{R}^d)$  be a given path with finite quadratic variation along the sequence of partitions  $\pi = \{\pi^n\}$ . Our goal is to provide conditions under which the the following identity holds:

$$[F(\cdot, \omega)](t) = \int_0^t \langle \nabla_\omega F(s, \omega_{s-}), {}^t \nabla_\omega F(s, \omega_{s-}), d[\omega](s) \rangle. \quad (12)$$

This relation was first noted in [2] for piecewise-constant integrands. To extend this property to a more general setting we assume a regularity condition on the functional  $F$  and a Hölder-type regularity condition for the path  $\omega$ .

**Assumption 1.**  $\exists K > 0, q > 0, \quad \forall \omega, \omega' \in D([0, T], \mathbb{R}^d),$

$$|F(t, \omega) - F(t, \omega')| \leq K \|\omega_t - \omega'_t\|_\infty^q.$$

**Assumption 2** (Horizontal local Lipschitz property for  $\nabla_\omega F$ ).

$\forall \omega \in D([0, T], \mathbb{R}^d), \exists C > 0, \eta > 0, \forall h \geq 0, \forall t \leq T - h, \forall \omega' \in D([0, T], \mathbb{R}^d):$

$$\|\omega_t - \omega'_t\|_\infty < \eta, \Rightarrow |\nabla_\omega F(t + h, \omega'_t) - \nabla_\omega F(t, \omega'_t)| \leq Ch.$$

Consider a nested sequence of partitions  $\pi_n = \{t_i^n, i = 0..m(n)\}$  of  $[0, T]$ . We require that the sequence  $\pi$  is *well-balanced* in the following sense:

**Assumption 3** (Well-balanced sequence of partitions).

Let  $\underline{\pi}_n = \inf_{i=0..m(n)-1} |t_{i+1}^n - t_i^n|$ . We call the sequence of partitions  $(\pi_n)_{n \geq 1}$  *well-balanced* if

$$\exists c > 0, \quad \frac{|\pi_n|}{\underline{\pi}_n} \leq c. \quad (13)$$

This condition means that the intervals in the partition  $\pi_n$  are asymptotically comparable. Note that since  $\underline{\pi}_n m(n) \leq T$ , for a well-balanced sequence of partitions we have

$$|\pi_n| \leq c \underline{\pi}_n \leq \frac{cT}{m(n)}.$$

We further assume that:

**Assumption 4.** The sequence  $\frac{m(n+1)}{m(n)}$  is bounded.

Assumptions 3 and 4 are verified for instance by the dyadic partition, and, more generally, any nested sequence of partitions constructed by refining the partition iteratively by splitting the intervals into a bounded (but not necessarily constant) number of subintervals.

Consider  $\kappa > 1$  and define:

$$l_n := \inf \{ k \geq n : m(k) \geq m(n)^\kappa \}.$$

Then  $m(l_n - 1) < m(n)^\kappa \leq m(l_n)$ , hence  $1 \leq \frac{m(l_n)}{m(n)^\kappa} < \frac{m(l_n)}{m(l_n - 1)}$ .

Thus, Assumption 4 implies that, for any  $\kappa > 1$ , there exists a sequence  $(l_n, n \geq 1)$  such that  $m(l_n) = \mathcal{O}(m(n)^\kappa)$  and  $m(n)^\kappa = \mathcal{O}(m(l_n))$ . We summarize this scaling relation with the following notation:  $m(l_n) \simeq m(n)^\kappa$ .

For  $0 < p < 1$ , denote by  $C^p([0, T])$  the space of  $p$ -Hölder continuous functions:

$$C^p([0, T], \mathbb{R}^d) = \{ f \in C^0([0, T]), \sup_{(t,s) \in [0,T]^2, t \neq s} \frac{\|f(t) - f(s)\|}{|t - s|^p} < \infty \}.$$

For  $\omega \in C^p([0, T])$ , the following piecewise constant approximation

$$\omega^n := \sum_{i=0}^{m(n)-1} \omega(t_{i+1}^n -) \mathbf{1}_{[t_i^n, t_{i+1}^n)} + \omega(T) \mathbf{1}_{\{T\}} \quad (14)$$

satisfies:  $\|\omega - \omega^n\|_\infty \leq |\pi^n|^p$ . For all the results below it is sufficient to have  $1/2 > p > 0$ . This is satisfied almost-surely by paths of Brownian motion and diffusion process for any  $p < 1/2$  [19].

The main result of this section is the following 'pathwise isometry formula' :

**Theorem 7** (Pathwise Isometry formula). *Let  $\pi = (\pi_n)_{n \geq 1}$  be a sequence of partitions of  $[0, T]$  satisfying Assumptions 3 and 4 and  $\omega \in Q^\pi([0, T], \mathbb{R}^d) \cap C^p([0, T], \mathbb{R}^d)$  for some  $p > 0$ . If  $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$  satisfies Assumption 2 and Assumption 1 for some  $q > \frac{1}{2p(2p+1)}$ , then*

$$[F(\cdot, \omega)](t) = \left[ \int_0^\cdot \nabla_\omega F(s, \omega_{s-}) \cdot d^\pi \omega \right](t) = \int_0^t \langle {}^t \nabla_\omega F(s, \omega_{s-}), \nabla_\omega F(s, \omega_{s-}), d[\omega] \rangle. \quad (15)$$



**Remark 8.** We note that for typical paths of a Brownian diffusion or continuous semimartingale with non-degenerate local martingale component, the assumptions of Theorem 7 hold almost surely as soon as Assumption 1 is satisfied for some  $q > 1/2$ .

*Proof.* Our goal is to show that

$$\sum_{\pi_n} \left( F(t_{i+1}^n, \omega_{t_{i+1}^n}) - F(t_i^n, \omega_{t_i^n}) \right)^2 \xrightarrow{n \rightarrow \infty} \int_0^T \langle {}^t \nabla_{\omega} F(t, \omega_{t-}) \nabla_{\omega} F(t, \omega_{t-}) d[\omega] \rangle. \quad (16)$$

For this we approximate the increments of  $F$  for the path  $\omega$  by the respective increments along the piecewise constant path  $\omega^l$  defined in (14) with  $l > n$ , choosing  $l$  such that the sum

$$\sum_{\pi_n} \left( F(t_{i+1}^n, \omega_{t_{i+1}^n}^l) - F(t_i^n, \omega_{t_i^n}^l) \right)^2,$$

approximates the left-hand side of (16), with a remainder which converges to zero. To simplify notations in the rest of the proof, we shall detail the arguments in the case  $d = 1$ , the vector case being a straightforward extension.

By Assumption 4, we can choose  $l(n) \geq n$  such

$$m(n)^\kappa \leq m(l) \leq cm(n)^\kappa \quad \text{for some } \kappa > 1, c > 0.$$

We decompose the increment of  $F(t, \omega)$  along  $\pi_n$  as follows:

$$\begin{aligned} F(t_{i+1}^n, \omega_{t_{i+1}^n}) - F(t_i^n, \omega_{t_i^n}) &= F(t_{i+1}^n, \omega_{t_{i+1}^n}^{l(n)-}) - F(t_i^n, \omega_{t_i^n}^{l(n)-}) \\ &\quad + \underbrace{F(t_{i+1}^n, \omega_{t_{i+1}^n}^{l(n)-}) - F(t_{i+1}^n, \omega_{t_{i+1}^n}^{l(n)})}_{R_i^{n,1}} + F(t_i^n, \omega_{t_i^n}^{l(n)}) - F(t_i^n, \omega_{t_i^n}) \end{aligned} \quad (17)$$

Since  $\omega \in C^p([0, T])$  there exists a constant  $K' > 0$  and an exponent  $p > 0$  such that

$$\|\omega^n - \omega\|_\infty \leq K' |\pi_n|^p. \quad (18)$$

By Assumptions 1 and condition (13) on the partition, we have

$$|R_i^{n,1}| \leq 2K \|\omega^l - \omega\|_\infty^q \leq 2K (K')^q |\pi_l|^{pq} \leq 2K (K')^q \frac{|cT|^{pq}}{m(l)^{pq}}.$$

Summing over the partition  $\pi_n$  and absorbing all constants into a single one denoted  $C$ , this yields:

$$\sum_{\pi_n} |R_i^{n,1}|^2 \leq \frac{Cm(n)}{m(l)^{2pq}}$$

Since  $l = l(n)$  is such that  $m(l) \simeq m(n)^\kappa$ , we obtain:

$$\sum_{\pi_n} |R_i^{n,1}|^2 \leq \frac{C}{m(n)^{2\kappa pq - 1}}.$$

Since  $\omega^l$  is piecewise constant, we can decompose the first term in (17) into horizontal and vertical increments and approximate each horizontal (resp. vertical) increment in terms of the horizontal (resp. vertical) derivative of  $F$ , as in [3]:

$$\begin{aligned}
& F(t_{i+1}^n, \omega_{t_{i+1}^n}^l) - F(t_i^n, \omega_{t_i^n}^l) = \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} \left( F(t_{j+1}^l, \omega_{t_{j+1}^l}^l) - F(t_j^l, \omega_{t_j^l}^l) \right) \\
= & \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} \nabla_{\omega} F(t_j^l, \omega_{t_j^l}^l) (\omega(t_{j+1}^l) - \omega(t_j^l)) + \frac{1}{2} \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} \nabla_{\omega}^2 F(t_j^l, \omega_{t_j^l}^l) (\omega(t_{j+1}^l) - \omega(t_j^l))^2 \\
& + \mathcal{O}(|t_{i+1}^n - t_i^n|) + \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} \varepsilon_{i,j}^l (\omega(t_{j+1}^l) - \omega(t_j^l))^2 \quad \text{with } \varepsilon_l := \max_{i,j} \{\varepsilon_{i,j}^l\} \xrightarrow{n \rightarrow \infty} 0.
\end{aligned}$$

By  $R_i^{n,2}$  we denote the sum of the last three terms in the right hand side of the above equality. We have

$$|R_i^{n,2}|^2 \leq C|\pi^n| |t_{i+1}^n - t_i^n| + C \left( \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} (\omega(t_{j+1}^l) - \omega(t_j^l))^2 \right)^2,$$

Let  $k_i$  denote the number of partition intervals from  $\pi_l$  which are contained in  $[t_i^n, t_{i+1}^n]$ . From the assumptions on the structure of the partitions we deduce that  $k_i \leq C \frac{m(l)}{m(n)}$ . By (18),

$$|\omega(t_{j+1}^l) - \omega(t_j^l)| \leq K' |\pi^l|^p \leq \frac{K' |cT|^p}{m(l)^p}, \quad \text{hence}$$

$$|R_i^{n,2}|^2 \leq C|\pi^n| |t_{i+1}^n - t_i^n| + C \frac{m(l)}{m(n)} \cdot \frac{1}{m(l)^{2p}} \cdot \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} (\omega(t_{j+1}^l) - \omega(t_j^l))^2.$$

Since  $\sum_{i: t_i^n \in \pi^n} \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} (\omega(t_{j+1}^l) - \omega(t_j^l))^2$  converges to  $[\omega](T)$ , it is bounded with respect to  $n$ , so we obtain

$$\sum_{i: t_i^n \in \pi^n} |R_i^{n,2}|^2 \leq CT|\pi^n| + C \frac{m(l)}{m(n)} \cdot \frac{1}{m(l)^{2p}} \leq \frac{CT}{m(n)} + \frac{C}{m(n)^{(2p-1)\kappa+1}}.$$

Next, we write the first term in (19) as follows:

$$\begin{aligned}
& \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} \nabla_{\omega} F(t_j^l, \omega_{t_j^l}^l) (\omega(t_{j+1}^l) - \omega(t_j^l)) = \\
& \quad \nabla_{\omega} F(t_i^n, \omega_{t_i^n}^l) (\omega(t_{i+1}^n) - \omega(t_i^n)) + \\
& \quad \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} \left( \nabla_{\omega} F(t_j^l, \omega_{t_j^l}^l) - \nabla_{\omega} F(t_i^n, \omega_{t_i^n}^l) \right) (\omega(t_{j+1}^l) - \omega(t_j^l)).
\end{aligned} \tag{19}$$

Denote the last sum above by  $R_i^{n,3}$ . Using Abel's summation by parts formula:

$$R_i^{n,3} = \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} \left( \nabla_{\omega} F(t_j^l, \omega_{t_j^l}^l) - \nabla_{\omega} F(t_{j-1}^l, \omega_{t_{j-1}^l}^l) \right) (\omega(t_{i+1}^n) - \omega(t_j^l)) \tag{20}$$

To obtain an estimate for  $R_i^{n,3}$  we control the two increments in the right-hand side separately. We can split the above increment of  $\nabla_\omega F$  into its horizontal and vertical parts:

$$\begin{aligned} \nabla_\omega F(t_j^l, \omega_{t_j^l-}^l) - \nabla_\omega F(t_{j-1}^l, \omega_{t_{j-1}^l-}^l) &= \nabla_\omega F(t_j^l, \omega_{t_j^l-}^l) - \nabla_\omega F(t_{j-1}^l, \omega_{t_{j-1}^l-}^l) + \\ &\quad \nabla_\omega F(t_{j-1}^l, \omega_{t_{j-1}^l-}^l) - \nabla_\omega F(t_{j-1}^l, \omega_{t_{j-1}^l-}^l). \end{aligned}$$

Assumption 2 on  $\nabla_\omega F$  implies the following bound:

$$|\nabla_\omega F(t_j^l, \omega_{t_j^l-}^l) - \nabla_\omega F(t_{j-1}^l, \omega_{t_{j-1}^l-}^l)| \leq C(t_j^l - t_{j-1}^l).$$

The second term is equal to

$$\nabla_\omega F(t_{j-1}^l, \omega_{t_{j-1}^l-}^{l, \omega(t_j^l) - \omega(t_{j-1}^l)}) - \nabla_\omega F(t_{j-1}^l, \omega_{t_{j-1}^l-}^l) = \int_0^{\omega(t_j^l) - \omega(t_{j-1}^l)} \nabla_\omega^2 F(t_{j-1}^l, \omega_{t_{j-1}^l-}^{l, h}) dh.$$

Since  $\nabla_\omega^2 F$  is bounded in a neighborhood of  $\omega$ , we get

$$|\nabla_\omega F(t_{j-1}^l, \omega_{t_{j-1}^l-}^{l, \omega(t_j^l) - \omega(t_{j-1}^l)}) - \nabla_\omega F(t_{j-1}^l, \omega_{t_{j-1}^l-}^l)| \leq C|\omega(t_j^l) - \omega(t_{j-1}^l)|.$$

Combining the above estimates we obtain:

$$|\nabla_\omega F(t_j^l, \omega_{t_j^l-}^l) - \nabla_\omega F(t_{j-1}^l, \omega_{t_{j-1}^l-}^l)| \leq C(|t_j^l - t_{j-1}^l| + |\omega(t_j^l) - \omega(t_{j-1}^l)|). \quad (21)$$

thus, since  $\omega$  has finite quadratic variation

$$\begin{aligned} &\sum_{i=0}^{m(n)-1} \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} |\nabla_\omega F(t_j^l, \omega_{t_j^l-}^l) - \nabla_\omega F(t_{j-1}^l, \omega_{t_{j-1}^l-}^l)|^2 \leq \\ &C \sum_{j=0}^{m(l)-1} [|t_j^l - t_{j-1}^l|^2 + |\omega(t_j^l) - \omega(t_{j-1}^l)|^2] \leq M < +\infty, \end{aligned} \quad (22)$$

where the constant  $M$  is independent of  $n, l$ .

Next, note that by (18), we have  $|\omega(t_{i+1}^n) - \omega(t_j^l)| \leq \frac{C}{|\pi_n|^p} \leq \frac{C}{m(n)^p}$ . As before if  $k_i$  denotes the number of partition intervals from  $\pi_l$  which are contained in  $[t_i^n, t_{i+1}^n]$ , we have  $k_i \leq C \frac{m(l)}{m(n)}$ . Hence, using (18)

$$\sum_{t_j^l \in [t_i^n, t_{i+1}^n)} (\omega(t_{i+1}^n) - \omega(t_j^l))^2 \leq k_i |\pi_n|^{2p} \leq \frac{Cm(l)}{m(n)^{2p+1}} \quad (23)$$

Applying the Cauchy-Schwarz inequality to (20) yields:

$$|R_i^{n,3}|^2 \leq \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} |\nabla_\omega F(t_j^l, \omega_{t_j^l-}^l) - \nabla_\omega F(t_{j-1}^l, \omega_{t_{j-1}^l-}^l)|^2 \sum_{t_j^l \in [t_i^n, t_{i+1}^n)} (\omega(t_{i+1}^n) - \omega(t_j^l))^2$$

Inserting (23) here and taking into account (22) we finally obtain:

$$\sum_{i: t_i^n \in \pi^n} |R_i^{n,3}|^2 \leq \frac{CMm(l)}{m(n)^{2p+1}} \leq \frac{CM}{m(n)^{2p+1-\kappa}}.$$

To summarize, we have shown:

$$F(t_{i+1}^n, \omega_{t_{i+1}^n}) - F(t_i^n, \omega_{t_i^n}) = \nabla_\omega F(t_i^n, \omega_{t_i^n}^l)(\omega(t_{i+1}^n) - \omega(t_i^n)) + R_i^{n,1} + R_i^{n,2} + R_i^{n,3}, \quad (24)$$

where

$$\sum_{i: t_i^n \in \pi^n} |R_i^{n,1}|^2 \leq \frac{C}{m(n)^{2\kappa pq - 1}}, \quad \sum_{i: t_i^n \in \pi^n} |R_i^{n,2}|^2 \leq \frac{CT}{m(n)} + \frac{C}{m(n)^{(2p-1)\kappa + 1}}$$

and

$$\sum_{i: t_i^n \in \pi^n} |R_i^{n,3}|^2 \leq \frac{CM}{m(n)^{2p+1-\kappa}}.$$

Now choose  $\kappa > 1$  such that  $2\kappa pq > 1$ ,  $(2p-1)\kappa + 1 > 0$  and  $2p+1-\kappa > 0$  to ensure that  $\sum_{i: t_i^n \in \pi^n} |R_i^{n,\nu}|^2 \rightarrow 0$ ,  $\nu = 1, 2, 3$ . For this we need  $\frac{1}{2pq} < \kappa < \min\{2p+1, \frac{1}{1-2p}\}$ , this is possible for  $p \leq \frac{1}{2}$  and  $q > \frac{1}{2p(2p+1)}$  e.g.  $p = 1/2$  and any  $q > 1/2$  satisfy this condition, and there are solutions with  $p < 1/2$ ,  $q < 1$ .

Denoting  $R_i^n = R_i^{n,1} + R_i^{n,2} + R_i^{n,3}$ , we obtain:

$$\sum_{i: t_i^n \in \pi^n} |R_i^n|^2 \leq 3 \left( \sum_{i: t_i^n \in \pi^n} |R_i^{n,1}|^2 + \sum_{i: t_i^n \in \pi^n} |R_i^{n,2}|^2 + \sum_{i: t_i^n \in \pi^n} |R_i^{n,3}|^2 \right) \xrightarrow{n \rightarrow \infty} 0,$$

therefore

$$\left| \sum_{i: t_i^n \in \pi^n} \left( F(t_{i+1}^n, \omega_{t_{i+1}^n}) - F(t_i^n, \omega_{t_i^n}) \right)^2 - \sum_{i: t_i^n \in \pi^n} (A_i^n)^2 \right| \leq \sum_{i: t_i^n \in \pi^n} |R_i^n|^2 + 2 \sum_{i: t_i^n \in \pi^n} |A_i^n| |R_i^n| \xrightarrow{n \rightarrow \infty} 0,$$

where  $A_i^n = \nabla_\omega F(t_i^n, \omega_{t_i^n}^l)(\omega(t_{i+1}^n) - \omega(t_i^n))$ . Indeed, using the Cauchy-Schwarz inequality, the last term is bounded by

$$2 \sqrt{\sum_{i: t_i^n \in \pi^n} |A_i^n|^2} \sqrt{\sum_{i: t_i^n \in \pi^n} |R_i^n|^2}.$$

It remains to note that

$$\sum_{i: t_i^n \in \pi^n} |A_i^n|^2 = \sum_{i: t_i^n \in \pi^n} \nabla_\omega F(t_i^n, \omega_{t_i^n}^l)^2 (\omega(t_{i+1}^n) - \omega(t_i^n))^2 \rightarrow \int_0^T \nabla_\omega F(t, \omega_t)^2 d[\omega](t).$$

We thus obtain the desired result:

$$\sum_{i: t_i^n \in \pi^n} \left( F(t_{i+1}^n, \omega_{t_{i+1}^n}) - F(t_i^n, \omega_{t_i^n}) \right)^2 \rightarrow \int_0^T \nabla F(t, \omega_t)^2 d[\omega](t).$$

□

**Remark 9.** We note here that the conclusions of Theorem 7 hold if instead of requiring  $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$  we simply require that  $F \in \mathbb{C}_b^{0,2}(\Lambda_T)$  and that  $\nabla_\omega F$  satisfies the horizontal local Lipschitz condition (Assumption 2).

**Remark 10.** Let  $\mathbb{P}$  be the Wiener measure on  $C^0([0, T], \mathbb{R})$ . Then the integral  $\int_0^\cdot \nabla_\omega F(t, \omega) \cdot d^\pi \omega$  is a version of the Ito integral  $\int_0^\cdot \nabla_\omega F(t, W) dW$  and integrating (15) with respect to  $\mathbb{P}$  yields the well-known Ito isometry formula [14]:

$$E \left( \left[ \int_0^\cdot \nabla_\omega F(t, W) dW \right] (T) \right) = E \left( \left| \int_0^T \nabla_\omega F(t, W) dW \right|^2 \right) = E \left( \int_0^T |\nabla_\omega F(t, W)|^2 dt \right).$$

So, Theorem 7 reveals that the Ito isometry is underpinned by a pathwise identity which does not rely on the Wiener measure.

### 3 Continuity property and the rough-smooth decomposition of paths

#### 3.1 Continuity and isometry property of the integral

Theorem 7 is strongly reminiscent of the Ito isometry formula [14] and suggests that the existence of an isometric mapping underlying the pathwise integral. We will now proceed to make this structure explicit.

A consequence of Theorem 7 is that, for a 'rough' path with non-degenerate quadratic variation,  $F(\cdot, \omega)$  has zero quadratic variation along  $\pi$  if and only if  $\nabla_\omega F$  vanishes along  $\omega$ :

**Proposition 11.** Let  $\omega \in Q_\pi([0, T], \mathbb{R}^d) \cap C^p([0, T], \mathbb{R}^d)$  for some  $0 < p \leq 1/2$  such that  $\frac{d[\omega]}{dt} := a(t) > 0$  is a right-continuous, positive definite function and  $F \in \mathbb{C}_b^{0,2}(\Lambda_T)$  be a non-anticipative functional which satisfies Assumption 2 and Assumption 1 for some  $q \leq 1$ .

Then the path  $t \mapsto F(t, \omega)$  has a zero quadratic variation along the partition  $\pi$  if and only if  $\nabla_\omega F(t, \omega) = 0, \forall t \in [0, T]$ .

*Proof.* Indeed, from Theorem 7

$$[F(\cdot, \omega)](T) = \int_0^T {}^t \nabla_\omega F(s, \omega) a(s) \nabla_\omega F(s, \omega) ds.$$

Since  $a(t) \in S_d^+$  the integrand on the right hand side is non-negative and strictly positive unless  $\nabla_\omega F(s, \omega_s) = 0$ . So by right-continuity of the integrand the integral is zero if and only if  $\nabla_\omega F(\cdot, \omega) \equiv 0$ .  $\square$

Thus if we start from an irregular path (meaning, with non-zero quadratic variation), this property is locally preserved by any regular functional transformation  $F$  as long as  $\nabla_\omega F$  does not vanish.

For  $p > 0$ , denote

$$C^{p-}([0, T], \mathbb{R}^d) := \bigcap_{0 < q < p} C^q([0, T], \mathbb{R}^d) \quad (25)$$

the space of functions which are  $p - \epsilon$  Hölder continuous for every  $\epsilon > 0$ .

Let  $F \in C_b^{1,2}(\Lambda_T)$  be a functional satisfying Assumption 2 and Assumption 1 for any  $q < 1$ . We denote  $\mathcal{R}(\Lambda_T)$  the set of such *regular* functionals.

We note that for  $\omega \in C^{\frac{1}{2}-}([0, T], \mathbb{R}^d)$ , we have  $F(\cdot, \omega) \in C^{\frac{1}{2}-}([0, T], \mathbb{R})$ .

Let  $a: [0, T] \rightarrow S_+^d$  be a continuous function taking values in positive-definite symmetric matrices.

**Definition 12** (Harmonic functionals).  $F \in \mathcal{R}(\Lambda_T)$  is called  $a$ -harmonic if

$$\forall (t, \omega) \in \Lambda_T, \quad \mathcal{D}F(t, \omega_t) + \frac{1}{2} \langle \nabla_\omega^2 F(t, \omega_t), a(t) \rangle = 0.$$

We denote  $\mathcal{H}_a(\Lambda_T)$  the space of  $a(\cdot)$ -harmonic functionals.

Note that for any  $F \in \mathcal{R}(\Lambda_T)$  the functional defined by

$$G(t, \omega) = F(t, \omega) - \int_0^t \mathcal{D}F(s, \omega) ds - \frac{1}{2} \int_0^t \langle \nabla_\omega^2 F(s, \omega), a(s) \rangle ds$$

is  $a(\cdot)$ -harmonic.

The class of harmonic functionals plays a key role in probabilistic interpretation of the Functional Ito calculus [13, 5, 2]. We will now see that this class also plays a role in the extension of the pathwise integral.

Let  $\bar{\omega} \in Q_\pi([0, T], \mathbb{R}^d) \cap C^p([0, T], \mathbb{R}^d)$  such that  $d[\bar{\omega}]/dt = a$ . The functional change of variable formula (Theorem 6) then implies that

$$\forall F \in \mathcal{H}_a(\Lambda_T), \quad \forall t \in [0, T], \quad F(t, \bar{\omega}) = F(0, \bar{\omega}) + \int_0^t \nabla_\omega F(u, \bar{\omega}) \cdot d^\pi \bar{\omega}.$$

By Theorem 7, we have

$$[F(\cdot, \bar{\omega})]_\pi(t) = \int_0^t \nabla_\omega F(u, \bar{\omega}) \cdot a(u) \nabla_\omega F(u, \bar{\omega}) du = \|\nabla_\omega F(\cdot, \bar{\omega})\|_{L^2([0, T], a)}^2 < \infty.$$

Consider the vector spaces

$$\mathcal{H}_a(\bar{\omega}) := \{ F(\cdot, \bar{\omega}) \mid F \in \mathcal{H}_a(\Lambda_T) \} \subset Q_\pi([0, T], \mathbb{R}), \quad \mathbb{V}_a(\bar{\omega}) := \{ \nabla_\omega F(\cdot, \bar{\omega}) \mid F \in \mathcal{H}_a(\Lambda_T) \} \subset L^2([0, T], a).$$

Proposition 11 then implies that the map

$$\begin{aligned} \mathcal{H}_a(\bar{\omega}) &\rightarrow \mathbb{R}_+ \\ \omega &\rightarrow \|\omega\|_\pi = \sqrt{[\omega]_\pi(T)} \end{aligned}$$

defines a norm on  $\mathcal{H}_a(\bar{\omega})$ . The pathwise integral can then be lifted to a continuous map

$$\begin{aligned} I_{\bar{\omega}}: (\mathbb{V}_a(\bar{\omega}), \|\cdot\|_{L^2([0, T], a)}) &\rightarrow (\mathcal{H}_a(\bar{\omega}), \|\cdot\|_\pi) \\ \phi = \nabla_\omega F(\cdot, \bar{\omega}) &\rightarrow I_{\bar{\omega}}(\phi) := \int_0^\cdot \phi \cdot d^\pi \bar{\omega}. \end{aligned} \quad (26)$$

which is in fact an isometry:

**Proposition 13** (Isometry property).  $I_{\bar{\omega}}: (\mathbb{V}_a(\bar{\omega}), \|\cdot\|_{L^2([0, T], a)}) \rightarrow (\mathcal{H}_a(\bar{\omega}), \|\cdot\|_\pi)$  is an injective isometry.

### 3.2 Rough-smooth decomposition of paths

Consider the space of paths obtained by a regular transformation of the reference path  $\bar{\omega}$ :

$$\mathcal{R}(\bar{\omega}) := \{ F(\cdot, \bar{\omega}) \mid F \in \mathcal{R}(\Lambda_T) \} \subset Q_\pi([0, T], \mathbb{R}).$$

The following result gives a 'signal plus noise' decomposition for such paths, in the sense of Dirichlet processes [12]:

**Proposition 14** (Rough-smooth decomposition of paths). *Any path  $\omega \in \mathcal{R}(\bar{\omega})$  has a unique decomposition*

$$\omega(t) = \omega(0) + \int_0^t \phi \cdot d^\pi \bar{\omega} + s(t) \quad (27)$$

where  $\phi \in \mathbb{V}_a(\bar{\omega})$  and  $[s]_\pi = 0$ .

The term  $\int_0^t \phi \cdot d^\pi \bar{\omega}$  is the 'rough' component of  $\omega$  which inherits the irregularity of  $\bar{\omega}$  while  $s(\cdot)$  represents a 'smooth' component with zero quadratic variation.

Proof: Let  $\omega \in \mathcal{R}(\bar{\omega})$ . Then there exists  $F \in \mathcal{R}(\Lambda_T)$  with  $\omega(t) = F(t, \bar{\omega})$ . The functional change of variable formula (Theorem 6) applied to  $F$  then yields the decomposition with  $\phi = \nabla_\omega F(\bar{\omega}) \in \mathbb{V}_a(\bar{\omega})$ .

Consider now two different decompositions

$$\omega(t) - \omega(0) = \int_0^t \phi_1 \cdot d^\pi \bar{\omega} + s_1(t) = \int_0^t \phi_2 \cdot d^\pi \bar{\omega} + s_2(t).$$

so  $s_1 - s_2 = \int (\phi_1 - \phi_2) \cdot d^\pi \bar{\omega}$ . Since  $[s_1 - s_2]_\pi = 0$ , using Proposition 11 we conclude that  $\phi_1 = \phi_2$  which then yields uniqueness of the decomposition 14.

## 4 Pathwise nature of the integral

The definition (11) of the integral  $\int \nabla_\omega F(\cdot, \omega) d^\pi \omega$  involves values of the integrand  $\nabla_\omega F$  along piecewise constant approximations of  $\omega$ . To be able to interpret  $\int \nabla_\omega F(\cdot, \omega) d^\pi \omega$  as a *pathwise* integral, we must show that it only depends on the values  $\nabla_\omega F(t, \omega), t \in [0, T]$  of the integrand along the path  $\omega$ .

Our goal is to provide conditions under which the following property holds:

**Property 1** (Pathwise nature of the integral). *Let  $F_1, F_2 \in C_b^{1,2}(\Lambda_T)$ , and  $\omega \in Q^\pi([0, T], \mathbb{R}) \cap C^p([0, T])$  for some  $0 < p < 1/2$ . If  $\nabla_\omega F_1(t, \omega_t) = \nabla_\omega F_2(t, \omega_t), \forall t \in [0, T]$  then*

$$\int_0^T \nabla_\omega F_1 \cdot d^\pi \omega = \int_0^T \nabla_\omega F_2 \cdot d^\pi \omega.$$

We will prove several versions of this property under various regularity assumptions on  $F$ .

**Proposition 15.** *Let  $F \in C_b^{1,4}(\Lambda_T)$  be a non-anticipative functional such that  $F, \nabla F, \nabla^2 F, \nabla_\omega^3 F \in C_b^{0,1}(\Lambda_T)$  and satisfy Assumptions 1 and 2 for all  $q < 1$ . Let  $\omega \in Q_\pi([0, T], \mathbb{R}^d) \cap C^{1/2-}([0, T], \mathbb{R}^d)$ . Then*

$$(\forall t \in [0, T], \nabla_\omega F(t, \omega_t) = 0) \quad \Rightarrow \quad \int_0^t \nabla_\omega F(u, \omega) d^\pi \omega = 0.$$

*Proof.* Since  $\nabla_\omega F, \nabla_\omega^2 F$  satisfy the assumptions of Proposition 11 we have first  $\nabla_\omega^2 F(\cdot, \omega) \equiv 0$  and consequently  $\nabla_\omega^3 F(\cdot, \omega) \equiv 0$ . The change of variable formula then yields

$$F(r, \omega_r) = F(0, \omega_0) + \int_0^r \mathcal{D}F(u, \omega_u) du + \int_0^r \nabla_\omega F(u, \omega_u) d\omega(u), \quad \forall r \in [0, T].$$

Thus, we need to prove that the function

$$W(r) := F(r, \omega_r) - \int_0^r \mathcal{D}F(u, \omega_u) du, \quad r \in [0, T]$$

is constant. We will prove only that  $W(T) = W(0)$ . The same proof will work if we take instead of  $T$  any partition point of a partition in  $\pi$ , since  $\pi$  is dense in  $[0, T]$  the result will follow by the continuity of  $W$ .

We will prove by contradiction: assume  $W(T) \neq W(0)$  then there exists  $\delta > 0$  such that  $|W(T) - W(0)| > \delta T$ . For any integer  $n \geq 1$  we can write

$$W(T) - W(0) = \sum_{t_i^n \in \pi^n} (W(t_{i+1}^n) - W(t_i^n))$$

this implies that there exists  $i_n \in \{0, 1, \dots, m(n) - 1\}$  such that

$$|W(t_{i_n+1}^n) - W(t_{i_n}^n)| > \delta(t_{i_n+1}^n - t_{i_n}^n).$$

Our goal is to obtain an estimate on  $W(t_{i_n+1}^n) - W(t_{i_n}^n)$  which will contradict the above inequality.

Fix an integer  $n \geq 1$  and denote  $t := t_{i_n}^n, s := t_{i_n+1}^n$ . Next we choose an integer  $l > n$  and denote by  $\tau_k, k = 0, 1, \dots, N$  the partition points in  $\pi^l$  (in increasing order) which lay in the interval  $[t, s]$ . Since the partitions satisfy the Assumption 3 we have that  $|\pi^l| \simeq \tau_{k+1} - \tau_k \simeq \frac{s-t}{N}$  and  $|\pi^n| \simeq s-t$ . Since  $\omega \in C^p([0, T])$ ,

$$|\omega(\tau_{k+1}) - \omega(\tau_k)| \leq C \frac{|s-t|^p}{N^p} \quad \text{and} \quad |\omega(s) - \omega(t)| \leq C|s-t|^p.$$

where the constant  $C > 0$  does not depend on  $n, l$ . Note also that the number  $N$  is the number of intervals of the form  $[\tau_k, \tau_{k+1}]$  which are contained in  $[t, s]$ , thus  $N \simeq \frac{m(l)}{m(n)}$ .

We introduce the following approximation of the path  $\omega$  on  $[0, s]$  by a path which coincides with  $\omega$  on  $[0, t)$  and is piecewise constant on  $[t, s]$ :

$$\omega^N(r) = \omega(r) \mathbf{1}_{[0, t)}(r) + \sum_{k=0}^{N-1} \omega(\tau_{k+1}) \mathbf{1}_{[\tau_k, \tau_{k+1})}(r) + \omega(s) \mathbf{1}_{\{s\}}.$$

Then, noting that  $\omega_t = \omega_{t-}^N$ , we have

$$F(s, \omega_s) - F(t, \omega_t) = F(s, \omega_s) - F(s, \omega_{s-}^N) + F(s, \omega_{s-}^N) - F(t, \omega_{t-}^N) \quad (28)$$

Next, we write the second term in 28 as a sum:

$$F(s, \omega_{s-}^N) - F(t, \omega_{t-}^N) = \sum_{k=0}^{N-1} (F(\tau_{k+1}, \omega_{\tau_{k+1}-}^N) - F(\tau_k, \omega_{\tau_k-}^N)).$$



Since  $F \in \mathbb{C}_b^{1,2}(\Lambda_T)$ , each summand in the above sum may be expanded using a second-order Taylor expansion, which yields

$$F(\tau_{k+1}, \omega_{\tau_{k+1}-}^N) - F(\tau_k, \omega_{\tau_k-}^N) = \nabla_\omega F(\tau_k, \omega_{\tau_k-}^N) (\omega(\tau_{k+1}) - \omega(\tau_k)) + \frac{1}{2} \nabla_\omega^2 F(\tau_k, \omega_{\tau_k-}^N) (\omega(\tau_{k+1}) - \omega(\tau_k))^2 + \int_{\tau_k}^{\tau_{k+1}} \mathcal{D}F(u, \omega_u^N) du + \mathcal{O}(|\omega(\tau_{k+1}) - \omega(\tau_k)|^2).$$

Hence

$$F(s, \omega_{s-}^N) - F(t, \omega_{t-}^N) = \int_s^t \mathcal{D}F(u, \omega_u^N) du + \sum_{k=0}^{N-1} \nabla_\omega F(\tau_k, \omega_{\tau_k-}^N) (\omega(\tau_{k+1}) - \omega(\tau_k)) + \frac{1}{2} \sum_{k=0}^{N-1} \nabla_\omega^2 F(\tau_k, \omega_{\tau_k-}^N) (\omega(\tau_{k+1}) - \omega(\tau_k))^2 + \mathcal{O}\left(\sum_{k=0}^{N-1} |\omega(\tau_{k+1}) - \omega(\tau_k)|^3\right). \quad (29)$$

We denote  $W^N(r) = F(r, \omega_{r-}^N) - \int_t^r \mathcal{D}F(u, \omega_u^N) du$ ,  $r \in [t, s]$ , then the above equation gives

$$W^N(s) - W^N(t) = F(s, \omega_{s-}^N) - F(t, \omega_{t-}^N) - \int_s^t \mathcal{D}F(u, \omega_u^N) du = \underbrace{\sum_{k=0}^{N-1} \nabla_\omega F(\tau_k, \omega_{\tau_k-}^N) (\omega(\tau_{k+1}) - \omega(\tau_k))}_A + \underbrace{\frac{1}{2} \sum_{k=0}^{N-1} \nabla_\omega^2 F(\tau_k, \omega_{\tau_k-}^N) (\omega(\tau_{k+1}) - \omega(\tau_k))^2}_{R_1} + \underbrace{\mathcal{O}\left(\sum_{k=0}^{N-1} |\omega(\tau_{k+1}) - \omega(\tau_k)|^3\right)}_{R_2}. \quad (30)$$

Since by Proposition 11  $\nabla_\omega^2 F(\tau_0, \omega_{\tau_0-}^N) = \nabla_\omega^2 F(t, \omega_t) = 0$ ,  $\nabla_\omega^2 F$  is locally Lipschitz in time, satisfies Assumption 1, and  $\omega$  is  $p$ -Hölder continuous, we get

$$\begin{aligned} |\nabla_\omega^2 F(\tau_k, \omega_{\tau_k-}^N)| &= |\nabla_\omega^2 F(\tau_k, \omega_{\tau_k-}^N) - \nabla_\omega^2 F(t, \omega_t)| \\ &\leq |\nabla_\omega^2 F(\tau_k, \omega_{\tau_k-}^N) - \nabla_\omega^2 F(\tau_k, \omega_{t, \tau_k-t})| + |\nabla_\omega^2 F(\tau_k, \omega_{t, \tau_k-t}) - \nabla_\omega^2 F(t, \omega_t)| \\ &\leq C \|\omega_{\tau_k-}^N - \omega_{t, \tau_k-t}\|_\infty^q + C(\tau_k - t) \leq C|\tau_k - t|^{pq} + C(\tau_k - t) \leq C|s - t|^{pq}. \end{aligned} \quad (31)$$

Hence

$$|R_1| \leq C|s - t|^{pq} \sum_{k=0}^{N-1} (\omega(\tau_{k+1}) - \omega(\tau_k))^2 \leq CN|s - t|^{pq} \frac{|s - t|^{2p}}{N^{2p}} = CN^{1-2p}|s - t|^{p(q+2)} \quad (32)$$

Note also that since  $\omega$  is  $p$ -Hölder continuous

$$|R_2| \leq CN \frac{|s - t|^{3p}}{N^{3p}} = CN^{1-3p}|s - t|^{3p}. \quad (33)$$

Using the condition  $\nabla_\omega F(\tau_0, \omega_{\tau_0-}^N) = \nabla_\omega F(t, \omega_t) = 0$ , for the first term on the right hand

side of 30, we have

$$A = \sum_{k=0}^{N-1} \nabla_{\omega} F(\tau_k, \omega_{\tau_k}^N) (\omega(\tau_{k+1}) - \omega(\tau_k)) = \sum_{k=0}^{N-1} \sum_{j=0}^{k-1} \left( \nabla_{\omega} F(\tau_{j+1}, \omega_{\tau_{j+1}}^N) - \nabla_{\omega} F(\tau_j, \omega_{\tau_j}^N) \right) (\omega(\tau_{k+1}) - \omega(\tau_k)) \quad (34)$$

We decompose the increments of  $\nabla_{\omega} F$  in the above formula into vertical and horizontal components:

$$\nabla_{\omega} F(\tau_{j+1}, \omega_{\tau_{j+1}}^N) - \nabla_{\omega} F(\tau_j, \omega_{\tau_j}^N) = \underbrace{\nabla_{\omega} F(\tau_{j+1}, \omega_{\tau_{j+1}}^N) - \nabla_{\omega} F(\tau_j, \omega_{\tau_j}^N)}_{r_{1,j}} + \nabla_{\omega} F(\tau_j, \omega_{\tau_j}^N) - \nabla_{\omega} F(\tau_j, \omega_{\tau_j}^N).$$

The first term on the right hand side is bounded by the horizontal Lipschitz Assumption 2 on  $\nabla_{\omega} F$ :

$$|r_{1,j}| = |\nabla_{\omega} F(\tau_{j+1}, \omega_{\tau_j, \tau_{j+1} - \tau_j}^N) - \nabla_{\omega} F(\tau_j, \omega_{\tau_j}^N)| \leq C(\tau_{j+1} - \tau_j) = C \frac{|s - t|}{N}.$$

Using Taylor expansion for the function  $h \mapsto \nabla_{\omega} F(\tau_j, \omega_{\tau_j}^{N,h})$ , the second term can be written as

$$\nabla_{\omega} F(\tau_j, \omega_{\tau_j}^{N, \omega(\tau_{j+1}) - \omega(\tau_j)}) - \nabla_{\omega} F(\tau_j, \omega_{\tau_j}^N) = \nabla_{\omega}^2 F(\tau_j, \omega_{\tau_j}^N) (\omega(\tau_{j+1}) - \omega(\tau_j)) + \frac{1}{2} \nabla_{\omega}^3 F(\tau_j, \omega_{\tau_j}^N) (\omega(\tau_{j+1}) - \omega(\tau_j))^2 + \underbrace{\mathcal{O}(|\omega(\tau_{j+1}) - \omega(\tau_j)|^3)}_{r_{2,j}} \quad (35)$$

$$A = \underbrace{\sum_{k=0}^{N-1} \sum_{j=0}^{k-1} \nabla_{\omega}^2 F(\tau_j, \omega_{\tau_j}^N) (\omega(\tau_{j+1}) - \omega(\tau_j)) (\omega(\tau_{k+1}) - \omega(\tau_k))}_{R_1} + \frac{1}{2} \underbrace{\sum_{k=0}^{N-1} \sum_{j=0}^{k-1} \nabla_{\omega}^3 F(\tau_j, \omega_{\tau_j}^N) (\omega(\tau_{j+1}) - \omega(\tau_j))^2 (\omega(\tau_{k+1}) - \omega(\tau_k))}_{R_2} + \underbrace{\sum_{k=0}^{N-1} \sum_{j=0}^{k-1} r_{2,j} (\omega(\tau_{j+1}) - \omega(\tau_j))^2 (\omega(\tau_{k+1}) - \omega(\tau_k))}_{R_3}$$

To estimate  $R_3$  we note that using the same argument as in 31, since by we get

$$|\nabla_{\omega}^3 F(\tau_j, \omega_{\tau_j}^N)| \leq C|s - t|^{pq},$$

consequently

$$|R_3| \leq CN^2 |s - t|^{pq} \frac{|s - t|^{2p}}{N^{2p}} \frac{|s - t|^p}{N^p} = CN^{2-3p} |s - t|^{p(q+3)}. \quad (36)$$

Since  $|r_{2,j}| \leq C \frac{|s-t|^{3-p}}{N^{3p}}$ , we have

$$|R_4| \leq CN^2 \frac{|s-t|^{3p}}{N^{3p}} \frac{|s-t|^{2p}}{N^{2p}} \frac{|s-t|^p}{N^p} = CN^{2-6p} |s-t|^{6p} \quad (37)$$

To estimate  $B$  we do similar computations as for  $A$ , using  $\nabla_\omega^3 F(\tau_0, \omega_{\tau_0-}^N) = \nabla_\omega^3 F(t, \omega_t) = 0$ :

$$B = \sum_{k=0}^{N-1} \sum_{j=0}^{k-1} \sum_{i=0}^{j-1} \left( \nabla_\omega^2 F(\tau_{i+1}, \omega_{\tau_{i+1}-}^N) - \nabla_\omega^2 F(\tau_i, \omega_{\tau_i-}^N) \right) (\omega(\tau_{j+1}) - \omega(\tau_j)) (\omega(\tau_{k+1}) - \omega(\tau_k))$$

changing the order of summation

$$B = \sum_{i=0}^{N-1} \left( \nabla_\omega^2 F(\tau_{i+1}, \omega_{\tau_{i+1}-}^N) - \nabla_\omega^2 F(\tau_i, \omega_{\tau_i-}^N) \right) \sum_{k>j>i} (\omega(\tau_{j+1}) - \omega(\tau_j)) (\omega(\tau_{k+1}) - \omega(\tau_k)).$$

$$\left| \sum_{k>j>i} (\omega(\tau_{j+1}) - \omega(\tau_j)) (\omega(\tau_{k+1}) - \omega(\tau_k)) \right| = \left| \frac{(\omega(\tau_N) - \omega(\tau_{i+1}))^2 - \sum_{j>i} (\omega(\tau_{j+1}) - \omega(\tau_j))^2}{2} \right|$$

$$\leq C|s-t|^{2p} + NC \frac{|s-t|^{2p}}{N^{2p}} \leq CN^{1-2p} |s-t|^{2p}. \quad (38)$$

Next, we decompose the increments of  $\nabla_\omega^2 F$  in the above formula into vertical and horizontal components:

$$\nabla_\omega^2 F(\tau_{j+1}, \omega_{\tau_{j+1}-}^N) - \nabla_\omega^2 F(\tau_j, \omega_{\tau_j-}^N) = \nabla_\omega^2 F(\tau_{j+1}, \omega_{\tau_{j+1}-}^N) - \nabla_\omega^2 F(\tau_j, \omega_{\tau_j}^N) + \nabla_\omega^2 F(\tau_j, \omega_{\tau_j}^N) - \nabla_\omega^2 F(\tau_j, \omega_{\tau_j-}^N).$$

The first term on the right hand side is bounded by the horizontal local Lipschitz continuity of  $\nabla_\omega F$  (Assumption 2):

$$|\nabla_\omega F(\tau_{j+1}, \omega_{\tau_{j+1}-}^N) - \nabla_\omega F(\tau_j, \omega_{\tau_j}^N)| = |\nabla_\omega F(\tau_{j+1}, \omega_{\tau_j, \tau_{j+1}-\tau_j}^N) - \nabla_\omega F(\tau_j, \omega_{\tau_j}^N)| \leq C(\tau_{j+1} - \tau_j).$$

For the second term using Taylor expansion we have

$$\nabla_\omega^2 F(\tau_j, \omega_{\tau_j}^N) - \nabla_\omega^2 F(\tau_j, \omega_{\tau_j-}^N) = \nabla_\omega^3 F(\tau_j, \omega_{\tau_j-}^N) (\omega(\tau_{j+1}) - \omega(\tau_j)) + \mathcal{O} \left( (\omega(\tau_{j+1}) - \omega(\tau_j))^2 \right)$$

since as was mentioned above  $|\nabla_\omega^3 F(\tau_j, \omega_{\tau_j-}^N)| \leq C|s-t|^{pq}$ , we obtain

$$|\nabla_\omega^2 F(\tau_j, \omega_{\tau_j}^N) - \nabla_\omega^2 F(\tau_j, \omega_{\tau_j-}^N)| \leq C \frac{|s-t|^{p(q+1)}}{N^p} + C \frac{|s-t|^{2p}}{N^{2p}} \leq C \frac{|s-t|^{p(q+1)}}{N^p}. \quad (39)$$

Combining the estimates (38) and (39) yields:

$$|B| \leq NC \frac{|s-t|^{p(q+1)}}{N^p} CN^{1-2p} |s-t|^{2p} = CN^{2-3p} |s-t|^{p(q+3)}. \quad (40)$$

From (30) and (35) we have

$$W^N(s) - W^N(t) = A + R_1 + R_2 = B + R_1 + R_2 + R_3 + R_4$$

Combining (32),(33), (36), (37) and (40) we obtain:

$$\begin{aligned} |W^N(s) - W^N(t)| &\leq C[N^{2-3p}|s-t|^{p(q+3)} + N^{1-2p}|s-t|^{p(q+2)} + N^{1-3p}|s-t|^{3p} \\ &\quad + N^{2-6p}|s-t|^{6p}] \leq C[N^{2-3p}|s-t|^{p(q+3)} + N^{1-2p}|s-t|^{p(q+2)}]. \end{aligned}$$

Recall that

$$W(r) := F(r, \omega_r) - \int_0^r \mathcal{D}F(u, \omega_u) du, \text{ and } W^N(r) := F(r, \omega_{r-}^N) - \int_0^r \mathcal{D}F(u, \omega_u^N) du,$$

since  $F(t, \omega_{t-}) = F(t, \omega_{t-}^N)$

$$W(s) - W(t) = W^N(s) - W^N(t) + F(s, \omega_s) - F(s, \omega_{s-}^N) + \int_t^s (\mathcal{D}F(u, \omega_u) - \mathcal{D}F(u, \omega_u^N)) du$$

On the other hand by Assumption 1 and using  $\omega \in C^p([0, T])$ , we have

$$|F(s, \omega_s) - F(s, \omega_{s-}^N)| \leq K \|\omega_s - \omega_{s-}^N\|_\infty^q \leq C \frac{|s-t|^{pq}}{N^{pq}}.$$

Thus

$$\begin{aligned} |W(s) - W(t)| &\leq |W^N(s) - W^N(t)| + C \frac{|s-t|^{pq}}{N^{pq}} + \sup_u |\mathcal{D}F(u, \omega_u) - \mathcal{D}F(u, \omega_u^N)| |s-t| \\ &\leq C[N^{2-3p}|s-t|^{p(q+3)} + N^{1-2p}|s-t|^{p(q+2)}] + C \frac{|s-t|^{pq}}{N^{pq}} + |s-t| \sup_u |\mathcal{D}F(u, \omega_u) - \mathcal{D}F(u, \omega_u^N)|. \end{aligned}$$

Next we are going to arrange  $N$  to be such that the sum of the first three terms on the right-hand side in the above inequality is optimal. For that we need the two dominant terms of that sum to be roughly equal:  $N^{2-3p}|s-t|^{p(q+3)} \simeq \frac{|s-t|^{pq}}{N^{pq}}$ , i.e.,  $N \simeq |s-t|^{-\frac{3p}{2-3p+pq}}$ . Recall that  $N \simeq \frac{m(l)}{m(n)}$  and  $|s-t| \simeq m(n)^{-1}$ , thus, we need to choose  $l = l(n)$  such that  $m(l) \simeq m(n)^{\frac{2+pq}{2-3p+pq}}$ . Plugging  $N \simeq |s-t|^{-\frac{3p}{2-3p+pq}}$  in the inequality, after routine computations, we get

$$|W(s) - W(t)| \leq C|s-t|^{\alpha_{p,q}} + C|s-t|^{\alpha'_{p,q}} + \sup_u |\mathcal{D}F(u, \omega_u) - \mathcal{D}F(u, \omega_u^N)| |s-t|.$$

where

$$\alpha_{p,q} := p \left( q + \frac{3pq}{2-3p+pq} \right) < \alpha'_{p,q} := p \left( q + \frac{1+2pq}{2-3p+pq} \right).$$

One can check that  $\alpha_{p,q} \rightarrow 5/4 > 1$  as  $p \rightarrow 1/2-$ ,  $q \rightarrow 1-$ , thus we can choose  $p$  and  $q$  close to  $1/2$  and  $1$  respectively so that  $\alpha_{p,q} > 1$ . As  $n \rightarrow +\infty$  we have  $N \simeq m(n)^{\frac{3p}{2-3p+pq}} \rightarrow +\infty$  and  $|s-t| \simeq m(n)^{-1} \rightarrow 0$ . Hence, by continuity of  $\mathcal{D}F$  and since  $\|\omega_s - \omega_{s-}^N\|_\infty \rightarrow 0$ , we have  $\sup_{u \in [t, s]} |\mathcal{D}F(u, \omega_u) - \mathcal{D}F(u, \omega_u^N)| \rightarrow 0$ , this with the above inequality implies that

$$\frac{|W(s) - W(t)|}{s-t} \leq C|s-t|^{\alpha_{p,q}-1} + \sup_{u \in [t, s]} |\mathcal{D}F(u, \omega_u) - \mathcal{D}F(u, \omega_u^N)| \rightarrow 0,$$

which is a contradiction, since by the choice of  $s = t_{i_n+1}^n$ ,  $t = t_{i_n}^n$  we have  $\frac{|W(s) - W(t)|}{s-t} > \delta$ .  $\square$

We are now ready to prove the main result of this section, which gives sufficient conditions on the functional  $F$  under which the pathwise integral  $\int \nabla_\omega F(t, \omega) \cdot d^\pi \omega$  is a (pathwise) limit of Riemann sums computed along  $\omega$  itself:

**Theorem 16.** *Let  $F \in C_b^{1,4}(\Lambda_T)$  be a non-anticipative functional such that  $F, \nabla F, \nabla^2 F, \nabla_\omega^3 F \in C_b^{0,1}(\Lambda_T)$  satisfy Assumptions 1, 2 for all  $q < 1$ . Let  $\omega \in Q_\pi([0, T], \mathbb{R}^d) \cap C^{1/2-}([0, T], \mathbb{R}^d)$ . Then, the path-wise integral  $\int \nabla_\omega F(t, \omega) \cdot d^\pi \omega$  is a limit sum of Riemann sums computed along  $\omega$ :*

$$\int_0^T \nabla_\omega F(u, \omega) d^\pi \omega = \lim_{n \rightarrow +\infty} \sum_{i=0}^{m(n)-1} \nabla_\omega F(t_i^n, \omega_{t_i^n}) \cdot (\omega_{t_{i+1}^n} - \omega_{t_i^n}).$$

In particular, if  $F(t, \omega) = 0$  for all  $t \in \pi$  then  $\int_0^T \nabla_\omega F(u, \omega) d^\pi \omega = 0$ .

This is a corollary of the following Lemma:

**Lemma 17.** *Let  $F \in C_b^{1,4}(\Lambda_T)$  be a non-anticipative functional such that  $F, \nabla F, \nabla^2 F, \nabla_\omega^3 F$  are  $C_b^{0,1}(\Lambda_T)$  non-anticipative functionals satisfying the Assumptions 1, 2 for all  $q < 1$ . Let  $\omega \in Q_\pi([0, T], \mathbb{R}^d) \cap C^{1/2-}([0, T], \mathbb{R}^d)$ . For  $t < s \in \pi^n$  we have*

$$\begin{aligned} F(s, \omega_s) - F(t, \omega_t) &= \int_t^s \mathcal{D}_t F(u, \omega_u) du + \nabla_\omega F(t, \omega_t) (\omega(s) - \omega(t)) \\ &+ \frac{1}{2} \nabla_\omega^2 F(t, \omega_t) (\omega(s) - \omega(t))^2 + \frac{1}{3!} \nabla_\omega^3 F(t, \omega_t) (\omega(s) - \omega(t))^3 + o(|s - t|). \end{aligned}$$

*Proof.* Let us first consider the case where  $\nabla_\omega F(t, \omega_t) = 0$ ,  $\nabla_\omega^2 F(t, \omega_t) = 0$ ,  $\nabla_\omega^3 F(t, \omega_t) = 0$ . Then, if we set

$$W(r) := F(r, \omega_r) - \int_0^r \mathcal{D}F(u, \omega_u) du, \quad r \in [0, T]$$

Now, note that in the proof of Proposition 15, when estimating the difference  $W(s) - W(t)$  we only used the conditions  $\nabla_\omega F(t, \omega_t) = 0$ ,  $\nabla_\omega^2 F(t, \omega_t) = 0$ ,  $\nabla_\omega^3 F(t, \omega_t) = 0$ . Therefore, using the same estimates as in the proof of Proposition 15 we obtain that there exists  $\alpha > 1$  such that

$$|W(s) - W(t)| \leq C|s - t|^\alpha + \sup_u |\mathcal{D}F(u, \omega_u) - \mathcal{D}F(u, \omega_u^N)| |s - t|.$$

Thus

$$F(s, \omega_s) - F(t, \omega_t) - \int_t^s \mathcal{D}F(u, \omega_u) du = W(s) - W(t) = o(|s - t|).$$

Let us now consider the general case. For the given path  $\omega \in Q_\pi([0, T], \mathbb{R}^d) \cap C^{1/2-}([0, T], \mathbb{R}^d)$ , we consider an auxiliary functional  $\tilde{F}$ , defined for a path  $x \in D([0, T], \mathbb{R}^d)$  by

$$\begin{aligned} \tilde{F}(u, x_u) &= F(u, x_u) - \nabla_\omega F(t, \omega_t) (x(u) - \omega(t)) \\ &- \frac{1}{2} \nabla_\omega^2 F(t, \omega_t) (x(u) - \omega(t))^2 - \frac{1}{3!} \nabla_\omega^3 F(t, \omega_t) (x(u) - \omega(t))^3. \end{aligned}$$

It is easy to check that  $\tilde{F}$  satisfies the assumptions of Proposition 15, with  $\nabla_\omega \tilde{F}(t, \omega_t) = 0$ ,  $\nabla_\omega^2 \tilde{F}(t, \omega_t) = 0$ ,  $\nabla_\omega^3 \tilde{F}(t, \omega_t) = 0$  and  $\mathcal{D}\tilde{F}(t, \omega_t) = \mathcal{D}F(t, \omega_t)$ . Thus, as proved above

$$\tilde{F}(s, \omega_s) - \tilde{F}(t, \omega_t) = \int_t^s \mathcal{D}F(u, \omega_u) du + o(|s - t|),$$

and the result follows.  $\square$

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