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Camille Laurent-Gengoux, Mathieu Stiénon, Ping Xu. Kapranov DG-manifolds and Poincaré–Birkhoff–Witt isomorphisms. 2016. hal-01285633

HAL Id: hal-01285633

<https://hal.science/hal-01285633>

Preprint submitted on 9 Mar 2016

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KAPRANOV DG-MANIFOLDS AND POINCARÉ–BIRKHOFF–WITT ISOMORPHISMS

CAMILLE LAURENT-GENGOUX, MATHIEU STIÉNON, AND PING XU

ABSTRACT. We prove that to every pair of Lie algebroids (L, A) corresponds a Kapranov dg-manifold structure on $A[1] \oplus L/A$, canonical up to isomorphism. As a consequence, $\Gamma(\wedge^\bullet A^\vee \otimes L/A)$ carries a canonical $L_\infty[1]$ algebra structure whose binary bracket is a cocycle representative of the Atiyah class of the pair (L, A) . For Lie algebroids over \mathbb{R} , we conjecture that this Kapranov dg-manifold ought to be considered as the derived formal neighborhood of a certain substack in a double-quotient differentiable stack. The second main purpose of the paper is the construction, for Lie algebroid pairs, of Poincaré–Birkhoff–Witt type isomorphisms described by an iteration formula, which allows explicit computations. Such Poincaré–Birkhoff–Witt isomorphisms constitute simultaneous extensions of the classical PBW map of Lie theory and the inverse of the complete symbol map of differential operators.

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Research partially supported by the National Science Foundation and the National Security Agency.

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1. INTRODUCTION

Lately some surprising homotopic algebra structures have arisen in connection with Lie algebroids. This paper aims to solve some puzzles in that direction.

In his work [16] on Rozansky-Witten invariants, Kapranov discovered a natural $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ of an arbitrary Kähler manifold X . Its unary bracket λ_1 is the Dolbeault differential operator $\bar{\partial}$, its binary bracket λ_2 is the composition

$$\Omega^{0,k}(T_X^{1,0}) \otimes \Omega^{0,l}(T_X^{1,0}) \xrightarrow{\wedge} \Omega^{0,k+l}(T_X^{1,0} \otimes T_X^{1,0}) \xrightarrow{\mathcal{R}_2} \Omega^{0,k+l+1}(T_X^{1,0})$$

of the exterior product and the contraction with the Atiyah cocycle

$$\mathcal{R}_2 \in \Omega^{0,1}(\mathrm{Hom}(T_X^{1,0} \otimes T_X^{1,0}, T_X^{1,0})),$$

and the higher brackets are obtained similarly by composing the exterior product and a contraction. This $L_\infty[1]$ algebra plays an important role in derived geometry [13, 27, 31].

The Atiyah class α_E of a holomorphic vector bundle E over a complex manifold X , which is an element of the sheaf cohomology group $H^1(X; \Omega_X^1 \otimes \mathrm{End}(E))$, captures the obstruction to the existence of a holomorphic connection on E [2]. A Dolbeault representative of α_E can be obtained as follows. Considering $T_X^{1,0}$ as a complex Lie algebroid, choose a $T_X^{1,0}$ -connection $\nabla^{1,0}$ on E . Being a holomorphic vector bundle, E carries a canonical flat $T_X^{0,1}$ -connection $\bar{\partial}$. Adding $\nabla^{1,0}$ and $\bar{\partial}$, we obtain a $T_X \otimes \mathbb{C}$ -connection ∇ on E . The element $\mathcal{R}_2 \in \Omega^{1,1}(\mathrm{End} E)$ defined by

$$\mathcal{R}_2(a, b)s = \nabla_a \nabla_b s - \nabla_b \nabla_a s - \nabla_{[a,b]} s, \quad \forall a \in \Gamma(T_X^{0,1}), b \in \Gamma(T_X^{1,0}), s \in \Gamma(E)$$

is a Dolbeault 1-cocycle (called Atiyah cocycle of E) whose cohomology class (which is independent of the choice of $\nabla^{1,0}$) is the Atiyah class

$$\alpha_E \in H^{1,1}(X; \mathrm{End} E) \cong H^1(X; \Omega_X^1 \otimes \mathrm{End} E).$$

Chen and two of the authors discovered that Atiyah classes can be defined in the much wider context of *pairs of Lie algebroids* [9]. We say that (L, A) is a pair of Lie algebroids or *Lie pair* if A is a subalgebroid of a Lie algebroid L over a common base manifold. The Atiyah class $\alpha_{L/A}$ of the pair, which is a simultaneous generalization of both the classical Atiyah class of holomorphic tangent bundles and the Molino

class of foliations [2, 28, 29], captures the obstruction to the existence of an *A-compatible* L -connection on L/A . Furthermore, it was brought to light that the graded vector space $\Gamma(\wedge^\bullet A^\vee \otimes L/A)$ admits a puzzling $Leibniz_\infty[1]$ algebra structure — a natural generalization of Stasheff’s $L_\infty[1]$ algebras [19] first introduced by Loday [24] in which the requirement that the multibrackets be (skew-)symmetric is dropped. Its unary bracket $\lambda_1 : \Gamma(\wedge^\bullet A^\vee \otimes L/A) \rightarrow \Gamma(\wedge^{\bullet+1} A^\vee \otimes L/A)$ is the Chevalley-Eilenberg coboundary operator associated to the Bott representation of the Lie algebroid A on L/A , its binary bracket λ_2 is the composition of the wedge product and the contraction with a Chevalley-Eilenberg Lie algebroid 1-cocycle $\mathcal{R}_2 \in \Gamma(A^\vee \otimes S^2(L/A)^\vee \otimes L/A)$ representative of the Atiyah class

$$\alpha_{L/A} \in H_{\text{CE}}^1(A; S^2(L/A)^\vee \otimes L/A),$$

and all multibrackets $\lambda_k : \otimes^k \Gamma(\wedge^\bullet A^\vee \otimes L/A) \rightarrow \Gamma(\wedge^{\bullet+1} A^\vee \otimes L/A)$ with $k \geq 2$ are $\Gamma(\wedge^\bullet A^\vee)$ -multilinear. The approach in [9] being rather computational, this $Leibniz_\infty[1]$ algebra remained quite enigmatic.

It was natural to wonder whether this result could be improved.

Problem A. *Can the sequence of multibrackets $\{\lambda_k\}_{k \in \mathbb{N}}$ be tweaked so as to make $\Gamma(\wedge^\bullet A^\vee \otimes L/A)$ an $L_\infty[1]$ algebra rather than a mere $Leibniz_\infty[1]$ algebra? If so, how can this $L_\infty[1]$ algebra be interpreted geometrically?*

It is well known that a \mathbb{Z} -graded vector space $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ of finite type (i.e. with each \mathfrak{g}_i finite-dimensional) is an $L_\infty[1]$ algebra if and only if it is a dg-manifold [17, 33]. By a dg-manifold¹, we mean a \mathbb{Z} -graded manifold endowed with a vector field Q of degree +1 satisfying $[Q, Q] = 0$. Such a vector field is said to be homological [34]. Hence we are led to seek a particular type of homological vector field on the \mathbb{Z} -graded manifold $A[1] \oplus L/A$, whose precise characterization is encapsulated in the new notion of Kapranov dg-manifold, the core concept of the present paper.

A *Kapranov dg-manifold* consists of a pair of smooth vector bundles A and E over a common base manifold and a pair of homological vector fields on the \mathbb{Z} -graded manifolds $A[1] \oplus E$ and $A[1]$ such that the inclusion $A[1] \hookrightarrow A[1] \oplus E$ and the projection $A[1] \oplus E \rightarrow E$ are morphisms of dg-manifolds. Isomorphisms of Kapranov dg-manifolds are defined in the obvious way.

It is worth stressing that the addition of a homological vector field on $A[1]$ turns A into a Lie algebroid. Given a representation of a Lie algebroid A on a vector bundle E , the Chevalley-Eilenberg differential provides a pair of homological vector fields on $A[1]$ and $A[1] \oplus E$ compatible with the inclusion and projection maps. The resulting Kapranov dg-manifold is said to be *linear*.

Our main result is the following

Theorem B. *Given a Lie pair (L, A) , there exists a homological vector field*

$$\Gamma(\wedge^\bullet A^\vee \otimes \hat{S}((L/A)^\vee)) \xrightarrow{D} \Gamma(\wedge^{\bullet+1} A^\vee \otimes \hat{S}((L/A)^\vee))$$

1. In the literature, dg-manifolds are sometimes called Q -manifolds [17, 1]. Non-positive grading is assumed for dg-manifolds in derived geometry [10, 11].

on the graded manifold $A[1] \oplus L/A$, compatible with the inclusion $A[1] \hookrightarrow A[1] \oplus E$ and the projection $A[1] \oplus E \twoheadrightarrow A[1]$, of the form

$$D = d_A^\nabla + \sum_{k=2}^{\infty} \mathcal{R}_k,$$

where

- (1) d_A^∇ denotes the Chevalley-Eilenberg differential associated to the Bott representation of the Lie algebroid A on the vector bundle L/A ;
- (2) $\mathcal{R}_2 \in \Gamma(A^\vee \otimes S^2(L/A)^\vee \otimes L/A)$ denotes the symmetric Atiyah 1-cocycle associated to a choice of L -connection ∇ on L/A extending the Bott representation;
- (3) the terms $\mathcal{R}_k \in \Gamma(A^\vee \otimes S^k(L/A)^\vee \otimes L/A)$ with $k \geq 3$ are algebraic functions of \mathcal{R}_2 , the curvature κ^j of the splitting j , the torsion and curvature of ∇ , and their higher order covariant derivatives;

and, abusing notations, we identify the tensor \mathcal{R}_k with the induced contraction operator. This Kapranov dg-manifold structure is natural up to a canonical isomorphism.

We follow a two-step strategy to prove Theorem B, which is also aimed at answering the second question raised in Problem A. First, working within the realm of real Lie algebroids where Lie's third theorem applies (at least locally), we will unveil the geometric meaning of the Kapranov dg-manifold stemming from a Lie pair and translate our geometric intuition into purely algebraic formulas. Later, we will show that the restriction to real Lie algebroids is unnecessary and that Theorem B holds for Lie algebroids over any field \mathbb{K} of characteristic zero as well.

Given a Lie pair (L, A) over \mathbb{R} , Lie's fundamental theorems assert the existence of a pair $(\mathcal{L}, \mathcal{A})$ of local Lie groupoids, which the Lie functor transforms into L and A respectively. Let us assume for the sake of simplicity that both \mathcal{L} and \mathcal{A} are simply s -connected, and \mathcal{A} is a closed Lie subgroupoid of \mathcal{L} . By $\mathfrak{X}_{\mathcal{A}}$, we denote the differentiable stack presented by the Lie groupoid \mathcal{A} [3]. The groupoid \mathcal{A} acts on \mathcal{L} from both the right and the left. Consider the differentiable quotient stack $[\mathcal{A} \backslash \mathcal{L} / \mathcal{A}]$. It contains $\mathfrak{X}_{\mathcal{A}}$ as a smooth substack because the homogeneous space $\mathcal{L} / \mathcal{A}$ obviously contains the unit space M of $(\mathcal{L}, \mathcal{A})$ as a distinguished embedded submanifold [23]. Now the Lie groupoid \mathcal{A} acts from the left on $\mathcal{L} / \mathcal{A}$. Therefore the differentiable quotient stack $[\mathcal{A} \backslash M]$ is a substack of the differentiable quotient stack $[\mathcal{A} \backslash (\mathcal{L} / \mathcal{A})]$, the former being isomorphic to $\mathfrak{X}_{\mathcal{A}}$ and the latter isomorphic to $[\mathcal{A} \backslash \mathcal{L} / \mathcal{A}]$.

Conjecture C. *The Kapranov dg-manifold of Theorem B is the derived formal neighborhood of $\mathfrak{X}_{\mathcal{A}}$ in $[\mathcal{A} \backslash \mathcal{L} / \mathcal{A}]$.*

Although this conjecture remains open, we use it as the guideline of our construction of the Kapranov dg-manifold of Theorem B. Here is a succinct two-step justification. First, the sheaf of functions on the quotient stack $[\mathcal{A} \backslash (\mathcal{L} / \mathcal{A})]$ should be *derived*,

i.e. replaced by a differential graded commutative algebra, namely the Chevalley-Eilenberg cochain complex

$$\cdots \rightarrow \Gamma(\wedge^k A^\vee) \otimes_R C^\infty(\mathcal{L}/\mathcal{A}) \xrightarrow{d_A} \Gamma(\wedge^{k+1} A^\vee) \otimes_R C^\infty(\mathcal{L}/\mathcal{A}) \rightarrow \cdots \quad (1)$$

stemming from the infinitesimal left action of A on \mathcal{L}/\mathcal{A} , whose cohomology plays the role of the submodule of A -invariant functions on \mathcal{L}/\mathcal{A} . Next, we need to replace $C^\infty(\mathcal{L}/\mathcal{A})$ by the functions on the *formal neighborhood* of M in \mathcal{L}/\mathcal{A} . The source map $s : \mathcal{L} \rightarrow M$ factors through the homogeneous space \mathcal{L}/\mathcal{A} and, consequently, turns it into a fiber bundle over M . Thus $s : \mathcal{L}/\mathcal{A} \rightarrow \mathcal{A}$ and $\pi : L/A \rightarrow M$ are two fiber bundles over M . By definition, the functions on the formal neighborhood of M in \mathcal{L}/\mathcal{A} are the s -fiberwise ∞ -jets along the unit space M of smooth functions on \mathcal{L}/\mathcal{A} . Since the vector bundle $\pi : L/A \rightarrow M$ is the restriction to the submanifold M of the tangent bundle to the s -fibers of \mathcal{L}/\mathcal{A} , L/A may be regarded as the normal bundle of M in \mathcal{L}/\mathcal{A} and identified to a tubular neighborhood of M in \mathcal{L}/\mathcal{A} . Therefore, the functions on the formal neighborhood of M in \mathcal{L}/\mathcal{A} may be identified with the functions on the formal neighborhood of the zero section of the vector bundle $\pi : L/A \rightarrow M$, i.e. $\Gamma(\hat{S}(L/A)^\vee)$. Substituting $\Gamma(\hat{S}(L/A)^\vee)$ for $C^\infty(\mathcal{L}/\mathcal{A})$ in (1), we obtain the dgca

$$(\Gamma(\wedge^\bullet A^\vee \otimes \hat{S}(L/A)^\vee), D), \quad (2)$$

whose differential D ought to be the homological vector field on $A[1] \oplus L/A$ of Theorem B.

In order to obtain an explicit formula for the homological vector field, an explicit identification between the fiber bundles L/A and \mathcal{L}/\mathcal{A} is needed. We show that every choice of (i) a splitting j of the short exact sequence of vector bundles $0 \rightarrow A \rightarrow L \rightarrow L/A \rightarrow 0$ and (ii) an L -connection ∇ on L/A extending the Bott representation of A on L/A determines a geodesic flow and an exponential map $\exp^{\nabla, j} : L/A \rightarrow \mathcal{L}/\mathcal{A}$. When the splitting j identifies L/A with a Lie subalgebroid of L , say B , the homogeneous space \mathcal{L}/\mathcal{A} gets identified locally with the corresponding Lie subgroupoid of \mathcal{L} , say \mathcal{B} , L -connections on L/A extending the Bott representation are in one-one correspondence with B -connections on B , and the resulting maps $\exp^\nabla : B \rightarrow \mathcal{B}$ are precisely the exponential maps investigated in [20, 30]. In particular, when the Lie algebroid B is a Lie algebra \mathfrak{g} , a B -connection on B is simply a linear map $\mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{g}$. The classical Lie-theoretic exponential map $\exp : \mathfrak{g} \rightarrow G$ corresponds to the trivial linear map $\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{0} \mathfrak{g}$. This is the content of Section 3.4.

Although an exponential map $\exp^\nabla : L/A \rightarrow \mathcal{L}/\mathcal{A}$ can be defined geometrically on a small neighborhood of the zero section of L/A , what we really need in the present paper is a *formal exponential map* identifying the formal neighborhood of the zero section of L/A to the formal neighborhood of M in \mathcal{L}/\mathcal{A} . We prove this formal exponential map — the ∞ -jet of $\exp^\nabla : L/A \rightarrow \mathcal{L}/\mathcal{A}$ in the direction of the fibers and along the zero section — to admit an alternative and purely algebraic definition, which would be valid for pairs of Lie algebroids over \mathbb{K} , in particular for pairs of complex Lie algebroids as well, despite the loss of the groupoid picture. Indeed, we prove that the algebraic incarnation of the formal exponential map is a *Poincaré-Birkhoff-Witt isomorphism*, which we believe to be of independent interest.

At this point, it is useful to recall the relation between the classical exponential map and PBW isomorphism in Lie theory. Let G be a Lie group and let \mathfrak{g} be its Lie algebra. The algebra $D'_0(\mathfrak{g})$ of distributions on \mathfrak{g} with support $\{0\}$ is canonically identified with the symmetric algebra $S(\mathfrak{g})$ while the algebra $D'_e(G)$ of distributions on G with support $\{e\}$ is canonically identified with the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. The classical Lie-theoretic exponential map $\exp : \mathfrak{g} \rightarrow G$, which is a local diffeomorphism near 0, can be used to push forward the distributions on the Lie algebra to the distributions on the Lie group. The induced isomorphism $D'_0(\mathfrak{g}) \xrightarrow{\cong} D'_e(G)$ is exactly the classical PBW isomorphism given by the symmetrization map

$$X_1 \odot \cdots \odot X_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} X_{\sigma(1)} \cdots X_{\sigma(n)},$$

where S_n denotes the symmetric group over n objects.

The classical exponential map $\mathfrak{g} \rightarrow G$ results from the choice of the zero map $\mathfrak{g} \otimes \mathfrak{g} \xrightarrow{0} \mathfrak{g}$, and therefore the symmetrization map is only one of many different exotic PBW isomorphisms.

Let $\mathcal{N}(L/A)$ denote the space of all functions on L/A which, together with their derivatives of all degrees in the direction of the π -fibers, vanish along the zero section. The space of π -fiberwise differential operators on L/A along the zero section is canonically identified to the symmetric R -algebra $\Gamma(S(L/A))$. Therefore, we have the short exact sequence of R -algebras

$$0 \rightarrow \mathcal{N}(L/A) \rightarrow C^\infty(L/A) \rightarrow \text{Hom}_R(\Gamma(S(L/A)), R) \rightarrow 0. \quad (3)$$

Likewise, let $\mathcal{N}(\mathcal{L}/\mathcal{A})$ denote the space of all functions on \mathcal{L}/\mathcal{A} which, together with their derivatives of all degrees in the direction of the s -fibers, vanish along the unit section. The space of s -fiberwise differential operators on \mathcal{L}/\mathcal{A} along the unit section is canonically identified to the quotient of the enveloping algebra $\mathcal{U}(L)$ by the left ideal generated by $\Gamma(A)$. Therefore, we have the short exact sequence of R -algebras

$$0 \rightarrow \mathcal{N}(\mathcal{L}/\mathcal{A}) \rightarrow C^\infty(\mathcal{L}/\mathcal{A}) \rightarrow \text{Hom}_R\left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R\right) \rightarrow 0. \quad (4)$$

Since the exponential $\exp^{\nabla, j}$ (or more precisely its dual) is an isomorphism from $\mathcal{N}(\mathcal{L}/\mathcal{A})$ to $\mathcal{N}(L/A)$, it induces an isomorphism of R -modules from $\text{Hom}_R(\Gamma(S(L/A)), R)$ to $\text{Hom}_R\left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R\right)$. The latter defines a map $\Gamma(S(L/A)) \rightarrow \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$, which is exactly the *PBW map* we expected. In fact, we prove that such a PBW map can be defined algebraically by an iteration formula for any pair of Lie algebroids over \mathbb{K} :

Theorem D. *Let (L, A) be any pair of Lie algebroids over \mathbb{K} .*

- (1) *Each choice of a splitting $j : L/A \rightarrow L$ of the short exact sequence of vector bundles $0 \rightarrow A \rightarrow L \rightarrow L/A \rightarrow 0$ and of an L -connection ∇ on L/A extending the Bott A -representation determines a unique isomorphism of R -coalgebras $\text{pbw}^{\nabla, j} : \Gamma(S(L/A)) \rightarrow \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$, called the Poincaré-Birkhoff-Witt map, which is also a morphism of filtered R -modules.*

(2) The map $\text{pbw}^{\nabla, j}$ satisfies

$$\begin{aligned} \text{pbw}^{\nabla, j}(1) &= 1 \\ \text{pbw}^{\nabla, j}(b) &= j(b) \\ \text{pbw}^{\nabla, j}(b^{n+1}) &= j(b) \cdot \text{pbw}^{\nabla, j}(b^n) - \text{pbw}^{\nabla, j}(\nabla_{j(b)}(b^n)) \end{aligned}$$

for all $b \in \Gamma(L/A)$, and $n \in \mathbb{N}$. Here b^k stands for the symmetric product $b \odot b \odot \cdots \odot b$ of k copies of b .

When $L = T_M$ and A is the trivial Lie subalgebroid of L of rank 0, the $\text{pbw}^{\nabla, j}$ map above is the inverse of the so called ‘‘complete symbol map,’’ which is an isomorphism from the space $\mathcal{U}(T_M)$ of differential operators on M to the space $\Gamma(S(T_M))$ of fiberwisely polynomial functions on T_M^\vee . The complete symbol map was generalized to arbitrary Lie algebroids over \mathbb{R} by Nistor, Weinstein, and one of the authors [30]. It played an important role in quantization theory [15, 30]. However, to the best of our knowledge, even in these classical cases, the iteration formula in Theorem D seems to be new and may have interesting applications.

The pbw-isomorphism allows us to establish an explicit isomorphism of algebras $\text{Hom}_R\left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R\right) \simeq \text{Hom}_R(\Gamma(S(L/A)), R) \cong \Gamma(\hat{S}(L/A)^\vee)$. When (L, A) is a pair of Lie algebroids over \mathbb{R} , $\text{Hom}_R\left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}, R\right)$ is exactly the space of functions on the formal neighborhood of M in \mathcal{L}/\mathcal{A} . Theorem B can thus be proved directly using the iteration formula of pbw-map. This is discussed in Section 4.4.

A Kapranov dg-manifold $A[1] \oplus E$ is said to be *linearizable* if it is isomorphic to a linear Kapranov dg-manifold corresponding to an A -module E .

A natural question is the following

Problem E. *When is the Kapranov dg-manifold associated to a Lie pair as in Theorem B linearizable?*

Our next result completely answers this question.

Theorem F. *Let (L, A) be a Lie pair over a smooth manifold M . The following assertions are equivalent.*

- (1) *The Kapranov dg-manifold stemming from the Lie pair (L, A) (see Theorem 4.23) is linearizable.*
- (2) *The Atiyah class $\alpha_{L/A} \in H_{\text{CE}}^1(A; S^2(L/A)^\vee \otimes L/A)$ of the Lie pair (L, A) vanishes.*
- (3) *For every splitting $j : L/A \rightarrow L$, there exists an extension of the Bott A -connection on L/A to an L -connection ∇ on L/A such that the associated Poincaré-Birkhoff-Witt map $\text{pbw}^{\nabla, j}$ intertwines the infinitesimal A -actions on $\Gamma(S(L/A))$ and $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$.*

In the case of Lie algebroids over \mathbb{R} , the following two assertions may be added to the list above.

- (4) For every splitting $j : L/A \rightarrow L$, there exists an extension of the Bott A -connection on L/A to an L -connection ∇ on L/A such that the fiberwise ∞ -jet of the associated exponential map $\exp^{\nabla, j}$ along the zero section of $L/A \rightarrow M$ intertwines the infinitesimal actions of A on L/A and \mathcal{L}/\mathcal{A} .
- (5) There exists a fiber-preserving diffeomorphism ϕ from a neighborhood of the zero section in L/A to a neighborhood of the identity section of \mathcal{L}/\mathcal{A} , which fixes the submanifold M , and whose fiberwise ∞ -jet along M intertwines the infinitesimal actions of A on L/A and \mathcal{L}/\mathcal{A} .

In fact, a stronger version of the equivalence of (i) and (iv) in Theorem F is proved by Voglaire and one of the authors [21] extending a theorem of Bordemann [4].

As an application, we consider the Lie pair $(L = T_X \otimes \mathbb{C}, A = T_X^{0,1})$ associated to a complex manifold X . We obtain a Kapranov dg-manifold on $T_X^{0,1}[1] \oplus T_X^{1,0}$, whose homological vector field involves the Atiyah cocycle as the second order term. When X is a Kähler manifold, we recover the standard example of Kapranov [16]. As is standard [10], this Kapranov dg manifold describes the formal neighborhood of the diagonal of $X \times X$. As a consequence of Theorem F, we prove that the Kapranov dg-manifold $T_X^{0,1}[1] \oplus T_X^{1,0}$ is linearizable if and only if the Atiyah class vanishes, which is specific to the diagonal embedding.

Some remarks are in order. Kapranov dg-manifolds are closely related to Costello's L_∞ -spaces [12, 14]; the precise relation will be studied somewhere else. However, a Kapranov dg-manifold $A[1] \oplus E$, when considered as a vector bundle over $A[1]$, is not a Q -bundle in the sense of Kotov-Strobl [18] since the dg-structure is not compatible with the linear structure. After [22] appeared, we learned from Bressler [5] that he had also obtained a similar construction. Finally, we would like to point out that this work is related to [6]. However, we warn the reader that 'PBW isomorphism' does not have the same meaning here and in [6]. In the present paper, the PBW map is an explicit R -linear isomorphism which always exists once a connection has been chosen, just as is the case for (the inverse of) the complete symbol map, while the PBW map in [6] is an isomorphism of filtered $\mathcal{U}(A)$ -modules which exists only when the Atiyah class vanishes. Hence existence of the PBW map in [6] is in a certain sense equivalent to linearizability of $A[1] \oplus L/A$ (compare [6, Theorem 5.4] and Theorem F).

Notations. Finally, we list the notation used throughout the paper.

- Given a graded vector space $V = \bigoplus_{k \in \mathbb{Z}} V^{(k)}$, $V[i]$ denotes the graded vector space obtained by shifting the grading on V according to the rule $(V[i])^{(k)} = V^{(i+k)}$. Accordingly, if $E = \bigoplus_{k \in \mathbb{Z}} E^{(k)}$ is a graded vector bundle over M , $E[i]$ denotes the graded vector bundle obtained by shifting the degree in the fibers of E according to the above rule.
- An L_∞ -algebra [19, 33] is a \mathbb{Z} -graded vector space $V = \bigoplus_{n \in \mathbb{Z}} V_n$ endowed with a sequence $(\lambda_k)_{k=1}^\infty$ of multilinear maps $\lambda_k : \wedge^k V \rightarrow V$ of degree $2 - k$ satisfying

the identity

$$\sum_{1 \leq j \leq k \leq n} \sum_{\sigma \in \mathfrak{S}_{k-j}^{j-1}} \operatorname{sgn} \sigma; v_1, \dots, v_{k-1} (-1)^{|v_{\sigma(1)}| + |v_{\sigma(2)}| + \dots + |v_{\sigma(k-j)}|} \lambda_{n-j+1}(v_{\sigma(1)}, \dots, v_{\sigma(k-j)}, \lambda_j(v_{\sigma(k+1-j)}, \dots, v_{\sigma(k-1)}, v_k), v_{k+1}, \dots, v_n) = 0 \quad (5)$$

for each $n \in \mathbb{N}$ and for any homogeneous vectors $v_1, v_2, \dots, v_n \in V$. Here \mathfrak{S}_{k-j}^{j-1} denotes the set of $(k-j, j-1)$ -shuffles², and $\operatorname{sgn} \sigma; v_1, \dots, v_{k-1}$ denotes the Koszul sign³ of the permutation σ of the (homogeneous) vectors v_1, v_2, \dots, v_{k-1} . A \mathbb{Z} -graded vector space V is an $L_\infty[1]$ -algebra if $V[-1]$ is a L_∞ -algebra.

- We use the symbol \mathbb{K} to denote either of the fields \mathbb{R} and \mathbb{C} . A *Lie algebroid* over \mathbb{K} is a \mathbb{K} -vector bundle $L \rightarrow M$ together with a bundle map $\rho : L \rightarrow TM \otimes_{\mathbb{R}} \mathbb{K}$ called the *anchor*, and a bracket $[\cdot, \cdot]$ on sections of L , such that

$$[l_1, fl_2] = f[l_1, l_2] + \rho(l)(f)l_2$$

for all $l_1, l_2 \in \Gamma(L)$ and $f \in C^\infty(M, \mathbb{K})$. In that case ρ seen as a map on sections of L is a morphism of Lie algebras. Unless specified, in this paper, by a Lie algebroid, we always mean a Lie algebroid over \mathbb{K} .

- By a Lie pair (L, A) , we mean an inclusion $A \hookrightarrow L$ of Lie algebroids over a smooth manifold M .
- The symbol R always denotes the algebra of smooth functions on M with values in \mathbb{K} . For any two R -modules P and Q , we write $P \otimes_R Q$ to denote the tensor product of P and Q as R -modules and $P \otimes Q$ to denote the tensor product of P and Q regarded as \mathbb{K} -modules.

Acknowledgments. We would like to thank several institutions for their hospitality while work on this project was being done: Penn State University (Laurent-Gengoux), Université de Lorraine (Stiénon), and Université Paris Diderot (Xu). We also wish to thank many people for useful discussions and comments, including Paul Bressler, Yannick Voglaire, and Jim Stasheff.

2. POINCARÉ-BIRKHOFF-WITT ISOMORPHISMS

2.1. Connections and representations for Lie algebroids. Let M be a smooth manifold, let $L \rightarrow M$ be a Lie \mathbb{K} -algebroid with anchor map $\rho : L \rightarrow T_M \otimes_{\mathbb{R}} \mathbb{K}$, and let $E \xrightarrow{\varpi} M$ be a vector bundle over \mathbb{K} . The algebra of smooth functions on M with values in \mathbb{K} will be denoted R .

The traditional description of a (linear) L -connection on E is in terms of a *covariant derivative*

$$\Gamma(L) \times \Gamma(E) \rightarrow \Gamma(E) : (x, e) \mapsto \nabla_x e$$

characterized by the following two properties:

$$\nabla_{f \cdot x} e = f \cdot \nabla_x e, \quad (6)$$

$$\nabla_x (f \cdot e) = \rho(x)f \cdot e + f \cdot \nabla_x e, \quad (7)$$

2. A $(k-j, j-1)$ -shuffle is a permutation σ of the set $\{1, 2, \dots, k-1\}$ such that $\sigma(1) \leq \sigma(2) \leq \dots \leq \sigma(k-j)$ and $\sigma(k-j+1) \leq \sigma(k-j+2) \leq \dots \leq \sigma(k-1)$.

3. The Koszul sign of a permutation σ of the (homogeneous) vectors v_1, v_2, \dots, v_n is determined by the relation $v_{\sigma(1)} \odot v_{\sigma(2)} \odot \dots \odot v_{\sigma(n)} = \operatorname{sgn} \sigma; v_1, \dots, v_n \cdot v_1 \odot v_2 \odot \dots \odot v_n$.

for all $x \in \Gamma(L)$, $e \in \Gamma(E)$, and $f \in R$.

A *representation of a Lie algebroid* A on a vector bundle $E \rightarrow M$ is a flat A -connection ∇ on E , i.e. a covariant derivative $\nabla : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ satisfying

$$\nabla_{a_1} \nabla_{a_2} e - \nabla_{a_2} \nabla_{a_1} e = \nabla_{[a_1, a_2]} e, \quad (8)$$

for all $a_1, a_2 \in \Gamma(A)$ and $e \in \Gamma(E)$. A vector bundle endowed with a representation of the Lie algebroid A is called an A -*module*. More generally, given a left R -module \mathcal{M} , by an *infinitesimal action* of A on \mathcal{M} , we mean a \mathbb{K} -bilinear map $\nabla : \Gamma(A) \times \mathcal{M} \rightarrow \mathcal{M}$, $(a, e) \mapsto \nabla_a e$ satisfying Eqs. (6), (7), and (8). In other words, ∇ is a representation of the Lie-Rinehart algebra $(\Gamma(A), R)$ [32].

Example 2.1. *Let (L, A) be a Lie pair, i.e. an inclusion $A \hookrightarrow L$ of Lie algebroids. The Bott representation of A on the quotient L/A is the flat connection defined by*

$$\nabla_a^{\text{Bott}} q(l) = q([a, l]), \quad \forall a \in \Gamma(A), l \in \Gamma(L),$$

where q denotes the canonical projection $L \rightarrow L/A$. Thus the quotient L/A of a Lie pair (L, A) is an A -module.

2.2. Universal enveloping algebra of a Lie algebroid. Let L be a Lie \mathbb{K} -algebroid over a smooth manifold M and let R denote the algebra of smooth functions on M taking values in \mathbb{K} . The vector space $\mathfrak{g} := R \oplus \Gamma(L)$ admits a natural Lie algebra structure given by the Lie bracket

$$(f + X) \otimes (g + Y) \mapsto X(g) - Y(f) + [X, Y],$$

where $f, g \in R$ and $X, Y \in \Gamma(L)$. Its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is the quotient of the tensor algebra $T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} (\bigotimes_{\mathbb{K}}^k (R \oplus \Gamma(L)))$ by the ideal generated by the subset of all elements of the form

$$(f + X) \otimes (g + Y) - (g + Y) \otimes (f + X) - (X(g) - Y(f) + [X, Y])$$

with $f, g \in R$ and $X, Y \in \Gamma(L)$. Let i denote the natural inclusion of \mathfrak{g} into $\mathcal{U}(\mathfrak{g})$ and let $\mathcal{V}(\mathfrak{g})$ denote the subalgebra of $\mathcal{U}(\mathfrak{g})$ generated by $i(\mathfrak{g})$. The *universal enveloping algebra $\mathcal{U}(L)$ of the Lie algebroid L* is the quotient of $\mathcal{V}(\mathfrak{g})$ by the two-sided ideal generated by the elements of the form

$$i(f) \otimes i(g + Y) - i(fg + fY)$$

with $f, g \in R$ and $Y \in \Gamma(L)$. Note that we have implicitly used the left R -module structure of \mathfrak{g} . The graduation of $T(\mathfrak{g})$ induces a natural ascending filtration

$$\dots \hookrightarrow \mathcal{U}^{\leq n-1}(L) \hookrightarrow \mathcal{U}^{\leq n}(L) \hookrightarrow \mathcal{U}^{\leq n+1}(L) \hookrightarrow \dots \quad (9)$$

on $\mathcal{U}(L)$.

When the base M of the Lie algebroid L is the one-point space so that the only fiber is a Lie algebra \mathfrak{h} , the universal enveloping algebra of the Lie algebroid is the the universal enveloping algebra of the Lie algebra \mathfrak{h} .

When the Lie algebroid L is the tangent bundle $T_M \rightarrow M$, its universal enveloping algebra $\mathcal{U}(L)$ is the algebra of differential operators on M .

In general, the universal enveloping algebra $\mathcal{U}(L)$ of the Lie algebroid L associated to a Lie groupoid \mathcal{L} is canonically identified with the associative algebra of source-fiberwise differential operators on $C^\infty(\mathcal{L})$ invariant under left translations [7].

The universal enveloping algebra $\mathcal{U}(L)$ of the Lie algebroid $L \rightarrow M$ is a coalgebra over R [37]. Its comultiplication

$$\Delta : \mathcal{U}(L) \rightarrow \mathcal{U}(L) \otimes_R \mathcal{U}(L)$$

is compatible with its filtration (9) and characterized by the identities

$$\begin{aligned} \Delta(1) &= 1 \otimes 1; \\ \Delta(x) &= 1 \otimes x + x \otimes 1, \quad \forall x \in \Gamma(L); \\ \Delta(u \cdot v) &= \Delta(u) \cdot \Delta(v), \quad \forall u, v \in \mathcal{U}(L), \end{aligned}$$

where $1 \in R$ denotes the constant function on M with value 1 while the symbol \cdot denotes the multiplication in $\mathcal{U}(L)$. We refer the reader to [37] for the precise meaning of the last equation above. Explicitly, we have⁴

$$\begin{aligned} \Delta(b_1 \cdot b_2 \cdots b_n) &= 1 \otimes (b_1 \cdot b_2 \cdots b_n) \\ &+ \sum_{\substack{p+q=n \\ p, q \in \mathbb{N}}} \sum_{\sigma \in \mathfrak{S}_p^q} (b_{\sigma(1)} \cdots b_{\sigma(p)}) \otimes (b_{\sigma(p+1)} \cdots b_{\sigma(n)}) \\ &+ (b_1 \cdot b_2 \cdots b_n) \otimes 1, \end{aligned}$$

for all $b_1, \dots, b_n \in \Gamma(L)$.

2.3. The coalgebras $\Gamma(S(L/A))$ and $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$. Let (L, A) be a pair of Lie algebroids over \mathbb{K} .

Writing $\mathcal{U}(L)\Gamma(A)$ for the left ideal of $\mathcal{U}(L)$ generated by $\Gamma(A)$, the quotient $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ is automatically a filtered R -coalgebra since

$$\Delta(\mathcal{U}(L)\Gamma(A)) \subseteq \mathcal{U}(L) \otimes_R (\mathcal{U}(L)\Gamma(A)) + (\mathcal{U}(L)\Gamma(A)) \otimes_R \mathcal{U}(L)$$

and the filtration (9) on $\mathcal{U}(L)$ descends to a filtration

$$\cdots \hookrightarrow \left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)} \right)^{\leq n-1} \hookrightarrow \left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)} \right)^{\leq n} \hookrightarrow \left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)} \right)^{\leq n+1} \hookrightarrow \cdots \quad (10)$$

of $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$.

Likewise, deconcatenation defines a graded R -coalgebra structure on $\Gamma(S(L/A))$. The comultiplication

$$\Delta : \Gamma(S(L/A)) \rightarrow \Gamma(S(L/A)) \otimes_R \Gamma(S(L/A))$$

is given by

$$\begin{aligned} \Delta(b_1 \odot b_2 \odot \cdots \odot b_n) &= 1 \otimes (b_1 \odot b_2 \odot \cdots \odot b_n) \\ &+ \sum_{\substack{p+q=n \\ p, q \in \mathbb{N}}} \sum_{\sigma \in \mathfrak{S}_p^q} (b_{\sigma(1)} \odot \cdots \odot b_{\sigma(p)}) \otimes (b_{\sigma(p+1)} \odot \cdots \odot b_{\sigma(n)}) \\ &+ (b_1 \odot b_2 \odot \cdots \odot b_n) \otimes 1, \quad (11) \end{aligned}$$

for all $b_1, \dots, b_n \in \Gamma(L/A)$. The symbol \odot denotes the symmetric product in $\Gamma(S(L/A))$.

4. The symbol \mathfrak{S}_p^q denotes the set of (p, q) -shuffles. A (p, q) -shuffle is a permutation σ of the set $\{1, 2, \dots, p+q\}$ such that $\sigma(1) \leq \sigma(2) \leq \cdots \leq \sigma(p)$ and $\sigma(p+1) \leq \sigma(p+2) \leq \cdots \leq \sigma(p+q)$.

2.4. Poincaré-Birkhoff-Witt isomorphism. Our first main result is a generalization of the classical Poincaré-Birkhoff-Witt theorem to Lie pairs.

Theorem 2.2. *Let (L, A) be a Lie pair. Given a splitting $j : L/A \rightarrow L$ of the short exact sequence $0 \rightarrow A \rightarrow L \rightarrow L/A \rightarrow 0$ and a L -connection ∇ on L/A , there exists a unique isomorphism of filtered R -coalgebras*

$$\text{pbw}^{\nabla, j} : \Gamma(S(L/A)) \rightarrow \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$$

satisfying

$$\text{pbw}^{\nabla, j}(f) = f, \quad \forall f \in R; \quad (12)$$

$$\text{pbw}^{\nabla, j}(b) = j(b), \quad \forall b \in \Gamma(L/A); \quad (13)$$

and, for all $n \in \mathbb{N}$ and $b \in \Gamma(L/A)$,

$$\text{pbw}^{\nabla, j}(b^{n+1}) = j(b) \cdot \text{pbw}^{\nabla, j}(b^n) - \text{pbw}^{\nabla, j}(\nabla_{j(b)}(b^n)). \quad (14)$$

Remark 2.3. *Equation (14) is equivalent to*

$$\begin{aligned} & \text{pbw}^{\nabla, j}(b_0 \odot \cdots \odot b_n) \\ &= \frac{1}{n+1} \sum_{i=0}^n \left(j(b_i) \cdot \text{pbw}^{\nabla, j}(b_0 \odot \cdots \odot \widehat{b}_i \odot \cdots \odot b_n) \right. \\ & \quad \left. - \text{pbw}^{\nabla, j}(\nabla_{j(b_i)}(b_0 \odot \cdots \odot \widehat{b}_i \odot \cdots \odot b_n)) \right) \end{aligned} \quad (15)$$

for all $b_0, \dots, b_n \in \Gamma(L/A)$.

It is immediate that Equations (12), (13), and (15) together define inductively a unique R -linear map $\text{pbw}^{\nabla, j}$. The proof that this Poincaré-Birkhoff-Witt map $\text{pbw}^{\nabla, j}$ is an isomorphism of filtered R -coalgebras will only be sketched here because a more enlightening proof exploiting the close relation between $\text{pbw}^{\nabla, j}$ and exponential maps will be given later.

We introduce the functor Gr which takes a filtered vector space

$$\dots \subset \mathcal{A}^{\leq k-1} \subset \mathcal{A}^{\leq k} \subset \mathcal{A}^{\leq k+1} \subset \dots$$

to the associated graded vector space

$$\text{Gr}(\mathcal{A}) = \bigoplus_k \frac{\mathcal{A}^{\leq k}}{\mathcal{A}^{\leq k-1}}.$$

Rinehart proved that, for every Lie algebroid L , the symmetrization map

$$\text{sym} : \Gamma(S(L)) \rightarrow \text{Gr}(\mathcal{U}(L)),$$

defined by

$$l_1 \odot \cdots \odot l_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} l_{\sigma(1)} \cdots l_{\sigma(n)}, \quad \forall l_1, \dots, l_n \in \Gamma(L),$$

is an isomorphism of graded vector spaces [32].

Lemma 2.4. For all $n \in \mathbb{N}$ and $b_1, \dots, b_n \in \Gamma(L/A)$,

$$\text{pbw}^{\nabla, j}(b_1 \odot \dots \odot b_n) - \frac{1}{n!} \sum_{\sigma \in S_n} j(b_{\sigma(1)}) \cdot j(b_{\sigma(2)}) \cdot \dots \cdot j(b_{\sigma(n)})$$

is an element of $\left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}\right)^{\leq n-1}$.

Proof. It follows from the inductive relation (15) that

$$\begin{aligned} \text{pbw}^{\nabla, j}(b_1 \odot \dots \odot b_n) - \frac{1}{n} \sum_{k=1}^n j(b_k) \cdot \text{pbw}^{\nabla, j}(b_1 \odot \dots \odot \widehat{b}_k \odot \dots \odot b_n) \\ = -\frac{1}{n} \sum_{k=1}^n \text{pbw}^{\nabla, j}(\nabla_{j(b_k)}(b_1 \odot \dots \odot \widehat{b}_k \odot \dots \odot b_n)) \end{aligned}$$

belongs to $\left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}\right)^{\leq n-1}$ as

$$\nabla_{j(b_k)}(b_1 \odot \dots \odot \widehat{b}_k \odot \dots \odot b_n) \in \Gamma(S^{n-1}(L/A))$$

and $\text{pbw}^{\nabla, j}$ respects the filtrations. The result follows by induction on n . \square

Consider the canonical projection

$$p : \mathcal{U}(L) \rightarrow \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$$

and the inclusion

$$j : \Gamma(S(L/A)) \rightarrow \Gamma(S(L))$$

induced by the splitting j .

Lemma 2.4 asserts the commutativity of the diagram

$$\begin{array}{ccc} \Gamma(S(L)) & \xrightarrow{\text{sym}} & \text{Gr}(\mathcal{U}(L)) \\ \uparrow j & & \downarrow \text{Gr}(p) \\ \Gamma(S(L/A)) & \xrightarrow{\text{Gr}(\text{pbw}^{\nabla, j})} & \text{Gr}\left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}\right) \end{array} \quad (16)$$

The functor Gr has the following remarkable property: a morphism ϕ of filtered vector spaces is invertible if and only if the associated morphism of graded vector spaces $\text{Gr}(\phi)$ is invertible. Consequently, in order to prove Theorem 2.2, it suffices to prove that $\text{Gr}(\text{pbw}^{\nabla, j})$ is an isomorphism of graded vector spaces. Since $\text{Gr}(\text{pbw}^{\nabla, j}) = \text{Gr}(p) \circ \text{sym} \circ j$ where j is injective, sym is bijective, and $\text{Gr}(p)$ is surjective, we simply need to check that sym induces an isomorphism between the cokernel of $j : \Gamma(S^n(L/A)) \rightarrow \Gamma(S^n(L))$ and the kernel of $\text{Gr}^n(p)$.

The cokernel of $j : \Gamma(S^n(L/A)) \rightarrow \Gamma(S^n(L))$ is clearly isomorphic to the subspace $\Gamma(S^{n-1}(L) \otimes A)$ of $\Gamma(S^n(L))$. Moreover, $\text{Gr}^n(\mathcal{U}(L)) = \frac{\mathcal{U}^{\leq n}(L)}{\mathcal{U}^{\leq n-1}(L)}$ and

$$\text{Gr}^n\left(\frac{\mathcal{U}(L)}{\mathcal{U}(L) \cdot \Gamma(A)}\right) = \frac{\mathcal{U}^{\leq n}(L)}{\mathcal{U}^{\leq n-1}(L) \cdot \Gamma(A) + \mathcal{U}^{\leq n-1}(L)},$$

imply that

$$\ker(\mathrm{Gr}^n(p)) = \frac{\mathcal{U}^{\leq n-1}(L) \cdot \Gamma(A) + \mathcal{U}^{\leq n-1}(L)}{\mathcal{U}^{\leq n-1}(L)}.$$

Lemma 2.5. *For all $a \in \Gamma(A)$ and $b \in \Gamma(L)$, we have*

$$\mathrm{sym}(\underbrace{b \odot \cdots \odot b}_{n \text{ factors}} \odot a) = \underbrace{b \cdots b}_{n \text{ factors}} \cdot a \quad \text{in } \mathrm{Gr}^{n+1}(\mathcal{U}(L)) = \frac{\mathcal{U}^{\leq n+1}(L)}{\mathcal{U}^{\leq n}(L)}.$$

Proof. First, observe that, for every $k \in \{0, 1, 2, \dots, n\}$,

$$b^k ab^{n-k} - b^n a = b^k [a, b] b^{n-k-1} + b^{k+1} [a, b] b^{n-k-2} + \cdots + b^{n-1} [a, b] \in \mathcal{U}^{\leq n}(L).$$

Then, in $\mathrm{Gr}^{n+1}(\mathcal{U}(L))$, we have

$$\begin{aligned} \mathrm{sym}(b^n \odot a) - b^n \cdot a &= \frac{1}{n+1} (ab^n + bab^{n-1} + \cdots + b^{n-1}ab + b^n a) - b^n a \\ &= \frac{1}{n+1} \sum_{k=0}^n (b^k ab^{n-k} - b^n a) = 0. \quad \square \end{aligned}$$

Proposition 2.6. *For every $n \in \mathbb{N}$, the graded linear isomorphism*

$$\mathrm{sym} : \Gamma(S(L)) \rightarrow \mathrm{Gr}(\mathcal{U}(L))$$

maps the subspace $\Gamma(S^{n-1}(L) \odot A)$ of $\Gamma(S^n(L))$ isomorphically onto the subspace $\frac{\mathcal{U}^{\leq n-1}(L) \cdot \Gamma(A) + \mathcal{U}^{\leq n-1}(L)}{\mathcal{U}^{\leq n-1}(L)}$ of $\mathrm{Gr}^n(\mathcal{U}(L))$.

Proof. Since $\mathrm{sym} : \Gamma(S(L)) \rightarrow \mathrm{Gr}(\mathcal{U}(L))$ is an isomorphism of graded R -modules, $\mathcal{U}^{\leq n-1}(L)$ is spanned by its elements of the form b^k with $b \in \Gamma(L)$ and $k \in \{0, 1, 2, \dots, n-1\}$. Consequently, $\frac{\mathcal{U}^{\leq n-1}(L) \cdot \Gamma(A) + \mathcal{U}^{\leq n-1}(L)}{\mathcal{U}^{\leq n-1}(L)}$ is spanned by its elements of the form $b^{n-1} \cdot a$ with $b \in \Gamma(L)$ and $a \in \Gamma(A)$. By Lemma 2.5, sym induces an isomorphism of R -modules

$$\Gamma(S^{n-1}(L) \odot A) \xrightarrow{\mathrm{sym}} \frac{\mathcal{U}^{\leq n-1}(L) \cdot \Gamma(A) + \mathcal{U}^{\leq n-1}(L)}{\mathcal{U}^{\leq n-1}(L)}$$

mapping $b^{n-1} \odot a$ to $b^{n-1} \cdot a$. □

3. EXPONENTIAL MAPS

Troughout this section, we work over the field \mathbb{R} of real numbers exclusively. Thus we restrict ourselves to real Lie algebroids.

For any Lie group, there is a canonical exponential map from the associated Lie algebra to the Lie group itself. The construction of the exponential map generalizes to Lie groupoids though in a noncanonical way as a choice of connection is needed (see Landsman [20]). In the particular case of a Lie group \mathcal{G} , one of these connection-induced exponential maps stands out: the one associated to the ‘zero’ connection defined by $\nabla_X Y = 0$ for every left-invariant vector fields X and Y on \mathcal{G} .

Here, given an inclusion of Lie groupoids $\mathcal{A} \hookrightarrow \mathcal{L}$ (and the corresponding inclusion of Lie algebroids $A \hookrightarrow L$), we define an exponential map which takes (a neighborhood of the zero section of) L/A to (a neighborhood of the “identity” section of) \mathcal{L}/\mathcal{A} . The definition requires two choices: (1) a splitting of the short exact sequence of vector bundles

$$0 \rightarrow A \rightarrow L \rightarrow L/A \rightarrow 0$$

and (2) an L -connection on L/A . When the Lie subalgebroid A is a vector bundle of rank zero, we recover Landsman’s exponential map [20].

3.1. s - and ρ -paths. Let \mathcal{L} be a Lie groupoid over a manifold M with source $s : \mathcal{L} \rightarrow M$, target $t : \mathcal{L} \rightarrow M$, and identity $1 : M \rightarrow \mathcal{L}$. The subbundle $L = \{X \in 1^*(T_{\mathcal{L}}) \mid s_*X = 0\}$ of the pullback of $T_{\mathcal{L}}$ via $1 : M \rightarrow \mathcal{L}$ carries a Lie algebroid structure whose anchor $\rho : L \rightarrow T_M$ is the (restriction of) the differential $t_* : T_{\mathcal{L}} \rightarrow T_M$ of the target [25]. We write π to denote the bundle projection $L \rightarrow M$.

In what follows, the symbol I always denotes some open interval of the real line containing 0.

Definition 3.1. An s -path is a smooth curve $\gamma : I \rightarrow \mathcal{L}$ originating from a point of the unit submanifold of \mathcal{L} (i.e. $\gamma(0) = 1_m$ for some $m \in M$) and fully contained in one of the s -fibers (i.e. $s \circ \gamma(t) = m$ for all $t \in I$).

Definition 3.2. A ρ -path is a smooth curve $\beta : I \rightarrow L$ satisfying

$$\pi_*(\beta'(t)) = \rho(\beta(t)), \quad \forall t \in I.$$

Lemma 3.3. Every s -path γ determines a unique ρ -path β through the relation

$$\beta(t) = \left. \frac{d}{d\tau} (\gamma(t))^{-1} \gamma(\tau) \right|_t.$$

Conversely, every ρ -path β determines a unique s -path γ . The origins of the two paths satisfy the relation $\gamma(0) = 1_{\pi(\beta(0))}$.

3.2. Connections as horizontal liftings. Let M be a smooth manifold, let $L \rightarrow M$ be a Lie algebroid with anchor map ρ , and let $E \xrightarrow{\varpi} M$ be a vector bundle.

In Section 2.1, we recalled the traditional definition of L -connections on E in terms of covariant derivatives $\Gamma(L) \times \Gamma(E) \xrightarrow{\nabla} \Gamma(E)$. However, L -connections on E admit the following alternate description:

Definition 3.4. A (linear) L -connection on E is a map $L \times_M E \xrightarrow{h} T_E$, called horizontal lifting, such that the diagram

$$\begin{array}{ccccc}
 & & L & \xrightarrow{\rho} & T_M \\
 & \nearrow & \searrow & \searrow \varpi_* & \nearrow \\
 L \times_M E & \xrightarrow{h} & T_E & & M \\
 & \searrow & \nearrow \varpi & & \\
 & & E & &
 \end{array} \tag{17}$$

commutes and its faces

$$\begin{array}{ccc} L \times_M E & \xrightarrow{h} & T_E \\ \downarrow & & \downarrow \\ E & \xrightarrow{\text{id}} & E \end{array} \quad \text{and} \quad \begin{array}{ccc} L \times_M E & \xrightarrow{h} & T_E \\ \downarrow & & \downarrow \varpi_* \\ L & \xrightarrow{\rho} & T_M \end{array}$$

are vector bundle maps.

The covariant derivative and horizontal lift describing a given connection are related to one another by the identity

$$e_*(\rho(l_m)) - h(l_m, e_m) = \tau_{e_m}((\nabla l e)_m), \quad (18)$$

which holds for all $m \in M$, $l \in \Gamma(L)$, and $e \in \Gamma(E)$. Here, τ_{e_m} denotes the canonical linear isomorphism between the fiber E_m and its tangent space at the point e_m .

3.3. Geodesic vector field. Suppose a splitting $j : L/A \rightarrow L$ of the short exact sequence $0 \rightarrow A \rightarrow L \rightarrow L/A \rightarrow 0$ and an L -connection ∇ on L/A have been chosen. Together, the connection and the splitting determine a ‘geodesic’ vector field $\Xi^{\nabla, j}$ on L/A : the value of Ξ at a point $x \in L/A$ is the horizontal lift of the vector $j(x) \in L$. More precisely,

$$\Xi_x = h(j(x), x), \quad \forall x \in L/A,$$

where $h : L \times_M (L/A) \rightarrow T_{L/A}$ denotes the horizontal lifting associated to the linear L -connection on L/A . This vector field Ξ is called geodesic because its integral curves $t \mapsto b(t)$ satisfy the equation

$$b'(t) = h(j(b(t)), b(t)), \quad (19)$$

which is similar to the usual geodesic equation.

Lemma 3.5. *The flow $t \mapsto \Phi_t^{\Xi}$ of the geodesic vector field Ξ satisfies the following two properties:*

- (1) $\Phi_t^{\Xi}(rx) = r\Phi_{rt}^{\Xi}(x)$, for all $x \in L/A$ and all real numbers t, r in a sufficiently small open interval around 0;
- (2) $\Phi_{t_1+t_2}^{\Xi} = \Phi_{t_2}^{\Xi} \circ \Phi_{t_1}^{\Xi}$, for all real numbers t_1, t_2 in a sufficiently small open interval around 0.

Given a section σ of the vector bundle $L/A \xrightarrow{\pi} M$, consider the map

$$\Psi^{\Xi, \sigma} : \mathbb{R} \times M \rightarrow \mathbb{R} \times M$$

defined by

$$\Psi^{\Xi, \sigma}(t, m) = (t, \pi \circ \Phi_t^{\Xi} \circ \sigma(m)).$$

Since $\pi \circ \Phi_0^{\Xi} \circ \sigma = \text{id}_M$ and

$$\left. \frac{d}{d\tau} \pi \circ \Phi_{\tau}^{\Xi} \circ \sigma(m) \right|_0 = \pi_* \Xi_{\sigma(m)} = \pi_* h(j(\sigma(m)), \sigma(m)) = \rho(j(\sigma(m))), \quad (20)$$

the differential of $\Psi^{\Xi, \sigma}$ at the point $(0, m)$ is represented by the Jacobian matrix

$$J_{\Psi^{\Xi, \sigma}}|_{(0, m)} = \begin{pmatrix} 1 & 0 \\ \rho(j(\sigma(m))) & \text{id}_{(T_M)_m} \end{pmatrix},$$

which is invertible. Hence $\Psi^{\Xi, \sigma}$ is locally diffeomorphic in a small neighborhood of $\{0\} \times M$.

Therefore, for t sufficiently close to 0, the map

$$\psi_t^{\Xi, \sigma} := \pi \circ \Phi_t^{\Xi} \circ \sigma$$

is locally diffeomorphic and we can define a one-parameter family of sections σ_t of $L/A \xrightarrow{\pi} M$ by setting

$$\sigma_t = \Phi_t^{\Xi} \circ \sigma \circ (\psi_t^{\Xi, \sigma})^{-1}. \quad (21)$$

Obviously, we have $\sigma_0 = \sigma$.

The following Lemma will be used later in the proof of Lemma 3.18.

Lemma 3.6. *For every section σ of the vector bundle $L/A \xrightarrow{\pi} M$, we have*

$$\left. \frac{d\sigma_t}{dt} \right|_0 = -\nabla_{j(\sigma)}\sigma.$$

Proof. Differentiating $\Phi_t^{\Xi} \circ \sigma(m) = \sigma_t \circ \psi_t^{\Xi, \sigma}(m)$ with respect to the parameter t , we obtain

$$\begin{aligned} \left. \frac{d}{dt} \Phi_t^{\Xi}(\sigma(m)) \right|_0 &= \left. \frac{d}{dt} \sigma_t \circ \psi_0^{\Xi, \sigma}(m) \right|_0 + \left. \frac{d}{dt} \sigma_0 \circ \psi_t^{\Xi, \sigma}(m) \right|_0 \\ &= \left. \frac{d}{dt} \sigma_t(m) \right|_0 + \left. \frac{d}{dt} \sigma \circ \pi \circ \Phi_t^{\Xi} \circ \sigma(m) \right|_0 \end{aligned}$$

and hence, making use of Equations (20) and (18),

$$\begin{aligned} \left. \frac{d}{dt} \sigma_t(m) \right|_0 &= \Xi_{\sigma(m)} - \sigma_* \pi_* \Xi_{\sigma(m)} \\ &= h(j(\sigma(m)), \sigma(m)) - \sigma_* \rho(j(\sigma(m))) \\ &= -\tau_{\sigma(m)}((\nabla_{j(\sigma)}\sigma)(m)). \end{aligned}$$

Therefore, we have established the equality $\left. \frac{d\sigma_t}{dt} \right|_0 = -\nabla_{j(\sigma)}\sigma$ in $\Gamma(L/A)$. \square

3.4. Exponential map. We can now outline the definition of the exponential map $\exp^{\nabla, j}$ from a neighborhood of the zero section of L/A to a neighborhood of the identity section of \mathcal{L}/\mathcal{A} . This map depends on the splitting j and the connection ∇ we have chosen; different choices yielding different exponential maps.

- (1) Given an element $x \in L/A$, consider the integral curve $t \mapsto b_x(t)$ in L/A of the geodesic vector field $\Xi^{\nabla, j}$ originating from $b_x(0) = x$. Note that, provided x lies sufficiently close to the zero section of L/A , this curve is defined up to time $t = 1$.
- (2) The lifted curve $t \mapsto j(b_x(t))$ is a ρ -path in the Lie algebroid L , for the ‘image’ of the geodesic equation $b'_x(t) = h(j(b_x(t)), b_x(t))$ under the projection $\pi_* : T_{L/A} \rightarrow T_M$ is precisely the ρ -path equation $\pi_*((j \circ b_x)'(t)) = \rho(j \circ b_x(t))$ as $\pi_*(h(l, e)) = \rho(l)$ per Diagram (17).
- (3) The ρ -path $t \mapsto j \circ b_x(t)$ determines a unique s -path $t \mapsto g^x(t)$ in the Lie groupoid \mathcal{L} . (See Lemma 3.3.)

- (4) Composing the path g^x with the canonical projection $\mathcal{L} \rightarrow \mathcal{L}/\mathcal{A}$, we obtain a path in \mathcal{L}/\mathcal{A} whose value at $t = 1$ is taken to be the image of x by the exponential map $\exp^{\nabla, j}$.

Definition 3.7. *The exponential*

$$\exp^{\nabla, j} : L/A \rightarrow \mathcal{L}/\mathcal{A}$$

associated to an L -connection ∇ on L/A and a splitting $j : L/A \rightarrow L$ is the map which takes a point x of L/A to the projection of $g^x(1)$ in \mathcal{L}/\mathcal{A} , where $t \mapsto g^x(t)$ is the unique path in \mathcal{L} satisfying

$$\left. \frac{d}{d\tau} (g^x(t))^{-1} g^x(\tau) \right|_t = j(\Phi_t^{\Xi, \nabla, j}(x)) \quad \text{and} \quad g^x(0) = 1_{\pi(x)}. \quad (22)$$

We reformulate Lemma 3.5.

Lemma 3.8. (1) *For any $x \in L/A$ and any real numbers r, t in a sufficiently small open interval around 0, we have*

$$g^{rx}(t) = g^x(rt).$$

- (2) *For any $x \in L/A$ and any real numbers t_1, t_2 in a sufficiently small open interval around 0, we have*

$$g^x(t_1 + t_2) = g^x(t_1) \cdot g^{\Phi_{t_1}^{\Xi}(x)}(t_2).$$

Proposition 3.9. *The exponential map $\exp^{\nabla, j} : L/A \rightarrow \mathcal{L}/\mathcal{A}$ is a fiber bundle map over $\text{id} : M \rightarrow M$ which maps the zero section of L/A to the identity section of \mathcal{L}/\mathcal{A} and is locally diffeomorphic around these sections.*

Proof. The result follows from the following two observations: **(1)** $\exp^{\nabla, j}(0_m) = 1_m$ for every $m \in M$ since the geodesic vector field Ξ vanishes along the zero section of L/A ; **(2)** the differential of $\exp^{\nabla, j}$ in the direction of the fibers and along the zero section is the identity map since, by Lemma 3.8,

$$\begin{aligned} \left. \frac{d}{dr} \exp(rx) \right|_0 &= \left. \frac{d}{dr} \mathbf{q}(g^{rx}(1)) \right|_0 = \mathbf{q}_* \left(\left. \frac{d}{dr} g^x(r) \right|_0 \right) \\ &= \mathbf{q}_* \left(\left. \frac{d}{dr} (g^x(0))^{-1} \cdot g^x(r) \right|_0 \right) = \mathbf{q}(j \circ \Phi_0^{\Xi}(x)) = \Phi_0^{\Xi}(x) = x, \end{aligned}$$

where $\mathbf{q} : \mathcal{L} \rightarrow \mathcal{L}/\mathcal{A}$ and $q : L \rightarrow L/A$ are the canonical projections. \square

Example 3.10. *Consider the special case where L is the tangent bundle to a smooth manifold M and A is its trivial subbundle of rank 0. Here j is necessarily the identity map on T_M while ∇ is simply a connection on T_M . If $t \mapsto \gamma(t)$ is the geodesic curve on M with tangent vector x at time 0, then $t \mapsto \dot{\gamma}(t)$ is the integral curve b_x of $\Xi^{\nabla, j}$ starting from the point x at time 0 and $t \mapsto (\gamma(0), \gamma(t))$ is the corresponding s -path g^x in the pair groupoid $M \times M \rightrightarrows M$. Thus we recover the usual exponential map of classical differential geometry: $\exp^{\nabla, j}(x) = (x, \gamma(1))$.*

Example 3.11. *When A is the trivial subbundle of rank 0 of a Lie algebroid L , j is necessarily the identity map on L and $\exp^{\nabla, j}$ is the Landsman exponential map corresponding to the L -connection ∇ on L [20, 30]. In particular, given a Lie algebra $\mathfrak{g} = \text{Lie}(G)$, consider the Lie pair (L, A) with $L = \mathfrak{g}$ and $A = \{0\}$. The*

exponential map $\exp^{\nabla, j}$ determined by the trivial connection $\nabla_\xi \eta = 0, \forall \xi, \eta \in \mathfrak{g}$ is the usual exponential map $\exp : \mathfrak{g} \rightarrow G$ of Lie theory.

Example 3.12. When the image of the splitting $j : L/A \rightarrow L$ is a Lie subalgebroid B of L , the L -connection ∇ on L/A is the sum of the Bott A -connection on B and some B -connection on B , and $\exp^{\nabla, j}$ is the Landsman exponential map corresponding to this B -connection on B [20].

3.5. Fiberwise jets and differential operators along a section. Given a surjective submersion $\pi : P \rightarrow M$ together with a section $\epsilon : M \rightarrow P$, we define an equivalence relation \sim on $C^\infty(P)$. Two functions f and g are equivalent ($f \sim g$) if and only if $f - g$ vanishes together with all its derivatives on $\epsilon(M)$, or equivalently, if and only if, for every π -fiberwise differential operator D , the function $D(f - g)$ vanishes along $\epsilon(M)$. Thus the equivalence class of a function $f \in C^\infty(P)$ is its π -fiberwise infinite jet along $\epsilon(M)$.

The quotient of $C^\infty(P)$ by the equivalence class $\mathcal{N}(P, M)$ of the zero function is, by construction, the algebra of all such jets. It will be denoted $C^\infty_\sim(P, M)$.

Now, consider the space $\mathcal{D}(P, M)$ of all maps $C^\infty(P) \rightarrow C^\infty(M)$ obtained by composition of a π -fiberwise differential operator on P (seen as an endomorphism of $C^\infty(P)$) with the restriction $\epsilon^* : C^\infty(P) \rightarrow C^\infty(M)$. By construction, $\mathcal{D}(P, M)$ is a coalgebra.

Proposition 3.13. In the category $R\text{-mod}$ of left modules over $R = C^\infty(M)$, the algebra $C^\infty_\sim(P, M)$ is the dual of the coalgebra $\mathcal{D}(P, M)$:

$$\text{Hom}_{R\text{-mod}}(\mathcal{D}(P, M), R) \cong C^\infty_\sim(P, M).$$

Denoting by $F^k \mathcal{D}(P, M)$ the π -fiberwise differential operators evaluated along $\epsilon(M)$ of order at most k and by $F^k C^\infty_\sim(P, M)$ the π -fiberwise jets (along $\epsilon(M)$) of functions on P which vanish up to order k along $\epsilon(M)$, we obtain dual filtrations

$$\begin{aligned} \dots &\hookrightarrow F^{k-1} \mathcal{D}(P, M) \hookrightarrow F^k \mathcal{D}(P, M) \hookrightarrow F^{k+1} \mathcal{D}(P, M) \hookrightarrow \dots \\ \dots &\hookrightarrow F^{k+1} C^\infty_\sim(P, M) \hookrightarrow F^k C^\infty_\sim(P, M) \hookrightarrow F^{k-1} C^\infty_\sim(P, M) \hookrightarrow \dots \end{aligned}$$

of the coalgebra $\mathcal{D}(P, M)$ of π -fiberwise differential operators on P evaluated along $\epsilon(M)$ and the algebra $C^\infty_\sim(P, M)$ of π -fiberwise infinite jets of functions on P along $\epsilon(M)$.

3.6. Induced exponential map on jets and differential operators. Given a pair of surjective submersions $\pi_1 : P_1 \rightarrow M_1$ and $\pi_2 : P_2 \rightarrow M_2$, a pair of sections $\epsilon_1 : M_1 \rightarrow P_1$ and $\epsilon_2 : M_2 \rightarrow P_2$, and a pair of diffeomorphisms $\Psi : P_1 \rightarrow P_2$ and $\psi : M_1 \rightarrow M_2$ intertwining the submersions π_1 and π_2 and the sections ϵ_1 and ϵ_2 as well, the algebra isomorphism $\Psi^* : C^\infty(P_2) \rightarrow C^\infty(P_1)$ maps $\mathcal{N}(P_1, M_1)$ onto $\mathcal{N}(P_2, M_2)$ and, therefore, determines a coalgebra isomorphism

$$\Psi_* : \mathcal{D}(P_1, M_1) \rightarrow \mathcal{D}(P_2, M_2).$$

For instance, given an inclusion of Lie groupoids $\mathcal{A} \hookrightarrow \mathcal{L}$ over a manifold M and the corresponding inclusion of Lie algebroids $A \hookrightarrow L$, the exponential map $\exp^{\nabla, j} : L/A \rightarrow \mathcal{L}/\mathcal{A}$ of Definition 3.7 induces an isomorphism of coalgebras

$$\exp_*^{\nabla, j} : \mathcal{D}(L/A, M) \rightarrow \mathcal{D}(\mathcal{L}/\mathcal{A}, M). \quad (23)$$

Indeed, in view of Proposition 3.9, it suffices to take

$$\begin{aligned} \pi_1 &= (L/A \rightarrow M) & \pi_2 &= (\mathcal{L}/\mathcal{A} \xrightarrow{s} M) \\ \epsilon_1 &= (M \xrightarrow{0} L/A) & \epsilon_2 &= (M \xrightarrow{1} \mathcal{L}/\mathcal{A}) \\ \Psi &= (L/A \xrightarrow{\exp^{\nabla, j}} \mathcal{L}/\mathcal{A}) & \psi &= (M \xrightarrow{\text{id}} M) \end{aligned}$$

in the construction above.

Lemma 3.14. *Let $E \rightarrow M$ be a vector bundle. The filtered coalgebras $\mathcal{D}(E, M)$ and $\Gamma(S(E))$ are canonically isomorphic.*

Proof. To the symmetric product $x_1 \odot x_2 \odot \cdots \odot x_n$ in $\Gamma(S(E))$ of any sections x_1, x_2, \dots, x_n of $E \rightarrow M$ we associate the fiberwise differential operator in $\mathcal{D}(E, M)$ which takes a function $f \in C^\infty(E)$ to the function

$$m \mapsto \frac{d}{dt_1} \Big|_0 \frac{d}{dt_2} \Big|_0 \cdots \frac{d}{dt_n} \Big|_0 f(t_1 x_1(m) + t_2 x_2(m) + \cdots + t_n x_n(m))$$

of $C^\infty(M)$. \square

Lemma 3.15. *Let $\mathcal{A} \hookrightarrow \mathcal{L}$ be an inclusion of Lie groupoids and let $A \hookrightarrow L$ denote the corresponding inclusion of Lie algebroids. The filtered coalgebras $\mathcal{D}(\mathcal{L}/\mathcal{A}, M)$ and $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ are canonically isomorphic.*

Proof. To the image in $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ of the product $x_1 \cdot x_2 \cdots x_n \in \mathcal{U}(L)$ of any sections x_1, x_2, \dots, x_n of L we associate the fiberwise differential operator in $\mathcal{D}(\mathcal{L}/\mathcal{A}, M)$ which takes a function $f \in C^\infty(\mathcal{L}/\mathcal{A})$ to the function

$$1^*(\vec{x}_1 \circ \vec{x}_2 \circ \cdots \circ \vec{x}_n(\mathbf{q}^* f)) \in C^\infty(M).$$

Here \vec{x} stands for the left-invariant vector field on \mathcal{L} corresponding to the section x of L . As previously, \mathbf{q} denotes the canonical projection $\mathbf{q} : \mathcal{L} \rightarrow \mathcal{L}/\mathcal{A}$ and 1 denotes the identity section $1 : M \rightarrow \mathcal{L}$. \square

It turns out that the isomorphism (23) coincides with the pbw map defined in Section 2.4. This is our second core result.

Theorem 3.16. *Let $\mathcal{A} \hookrightarrow \mathcal{L}$ be an inclusion of Lie groupoids and let $A \hookrightarrow L$ denote the corresponding inclusion of Lie algebroids. Given a splitting $j : L/A \rightarrow L$ of the short exact sequence of vector bundles $0 \rightarrow A \rightarrow L \rightarrow L/A \rightarrow 0$ and an L -connection ∇ on L/A , the filtered coalgebra isomorphism $\exp_*^{\nabla, j} : \mathcal{D}(L/A, M) \rightarrow \mathcal{D}(\mathcal{L}/\mathcal{A}, M)$ induced by the exponential map coincides with the Poincaré-Birkhoff-Witt isomorphism $\text{pbw}^{\nabla, j} : \Gamma(S(L/A)) \rightarrow \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$.*

We will make use of the bundle map

$$\begin{array}{ccc} L/A & \xrightarrow{E} & \mathcal{L} \\ \pi \downarrow & & \downarrow s \\ M & \xrightarrow{\text{id}} & M \end{array}$$

defined by $\mathbf{E}(x) = g^x(1)$ for all $x \in L/A$, and the associated map

$$\mathcal{E} : \Gamma(L/A) \rightarrow \Gamma(\mathcal{L}).$$

We remind the reader that the exponential map $\exp^{\nabla, j}$ is simply the composition of $\mathbf{E} : L/A \rightarrow \mathcal{L}$ with the canonical projection $\mathbf{q} : \mathcal{L} \rightarrow \mathcal{L}/\mathcal{A}$.

Lemma 3.17. (1) *For any $l \in \mathcal{L}$ and $x \in \Gamma(L/A)$, we have*

$$\left. \frac{d}{d\tau} l \cdot \mathcal{E}(\tau x) \right|_0 = \overrightarrow{j(x)}|_l,$$

where $\overrightarrow{j(x)}$ denotes the left-invariant vector field on the Lie groupoid \mathcal{L} (tangent to the s -fibers) associated to the section $j(x)$ of the Lie algebroid L .

(2) *For any $x \in \Gamma(L/A)$ and any real numbers t_1, t_2 in a sufficiently small open interval around 0, we have*

$$\mathcal{E}((t_1 + t_2)x) = \mathcal{E}(t_1x) \cdot \mathcal{E}(t_2x_{t_1}),$$

where the r.h.s. is a product of s -sections of \mathcal{L} and

$$x_{t_1} = \Phi_{t_1}^{\Xi} \circ x \circ (\psi_{t_1}^{\Xi, x})^{-1}$$

as in Eq. (21).

Proof. The first assertion follows from

$$\left. \frac{d}{d\tau} f(l \cdot \mathcal{E}(\tau x)) \right|_0 = \left. \frac{d}{d\tau} f(l \cdot g^{\tau x}(1)) \right|_0 = \left. \frac{d}{d\tau} f(l \cdot g^x(\tau)) \right|_0 = \overrightarrow{j(x)}|_l(f),$$

in which we have made use of the first part of Lemma 3.8. The second part of Lemma 3.8 can be rewritten as

$$\mathbf{E}((t_1 + t_2)x) = \mathbf{E}(t_1x) \cdot \mathbf{E}(t_2\Phi_{t_1}^{\Xi}(x)).$$

Passing to sections of $L/A \rightarrow M$, we obtain the second assertion above. \square

Recall that the universal enveloping algebra $\mathcal{U}(L)$ of the Lie algebroid L can be regarded as the associative algebra of s -fiberwise differential operators on $C^\infty(\mathcal{L})$ invariant under left translations. The map $\mathcal{E} : \Gamma(L/A) \rightarrow \Gamma(\mathcal{L})$ induces a map

$$\mathcal{E} : \Gamma(S(L/A)) \rightarrow \mathcal{U}(L)$$

as follows. The element $\mathcal{E}[x_1 \odot x_2 \odot \cdots \odot x_n]$ of $\mathcal{U}(L)$ corresponding to $x_1 \odot x_2 \odot \cdots \odot x_n \in \Gamma(S^n(L/A))$ is the left invariant differential operator which takes a function $f \in C^\infty(\mathcal{L})$ to the function

$$\mathcal{L} \ni l \mapsto \left. \frac{d}{dt_1} \right|_0 \left. \frac{d}{dt_2} \right|_0 \cdots \left. \frac{d}{dt_n} \right|_0 f(l \cdot \mathcal{E}(t_1x_1 + t_2x_2 + \cdots + t_nx_n)) \in \mathbb{R}.$$

In particular, for any $x \in \Gamma(L/A)$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} (\mathcal{E}[x^n]f)(l) &= \left. \frac{d}{dt_1} \right|_0 \left. \frac{d}{dt_2} \right|_0 \cdots \left. \frac{d}{dt_n} \right|_0 f(l \cdot \mathcal{E}((t_1 + t_2 + \cdots + t_n)x)) \\ &= \left. \frac{d^n}{dt^n} f(l \cdot \mathcal{E}(tx)) \right|_0. \end{aligned}$$

(Note that, for every $r \in R = C^\infty(M)$ seen as an element of degree 0 in $\Gamma(S(L/A))$, we have $\mathcal{E}[r] = r$.)

It is easy to check that \mathcal{E} is a morphism of left R -modules.

Lemma 3.18. *For all $x \in \Gamma(L/A)$ and $n \in \mathbb{N}$, we have $\mathcal{E}[x] = \overrightarrow{j(x)}$ and*

$$\mathcal{E}[x^{n+1}] = \overrightarrow{j(x)} \circ \mathcal{E}[x^n] - \mathcal{E}[\nabla_{j(x)}(x^n)].$$

Proof. Indeed, it follows from Lemma 3.17 that

$$(\mathcal{E}[x]f)(l) = \overrightarrow{j(x)}|_l(f)$$

and

$$\begin{aligned} (\mathcal{E}[x^{n+1}]f)(l) &= \frac{d}{dt_0} \Big|_0 \frac{d}{dt_1} \Big|_0 \cdots \frac{d}{dt_n} \Big|_0 f \left(l \cdot \mathcal{E}((t_0 + t_1 + \cdots + t_n)x) \right) \\ &= \frac{d}{dt_0} \Big|_0 \frac{d}{dt_1} \Big|_0 \cdots \frac{d}{dt_n} \Big|_0 f \left(l \cdot \mathcal{E}(t_0x) \cdot \mathcal{E}((t_1 + t_2 + \cdots + t_n)x_{t_0}) \right) \\ &= \frac{d}{dt_0} \Big|_0 \left(\mathcal{E}[(x_{t_0})^n]f \right) (l \cdot \mathcal{E}(t_0x)) \\ &= \frac{d}{dt_0} \Big|_0 \left(\mathcal{E}[x^n]f \right) (l \cdot \mathcal{E}(t_0x)) + \frac{d}{dt_0} \Big|_0 \left(\mathcal{E}[(x_{t_0})^n]f \right) (l) \\ &= \left(\overrightarrow{j(x)}(\mathcal{E}[x^n]f) \right) (l) + \left(\mathcal{E} \left[\frac{d}{dt_0} (x_{t_0})^n \Big|_0 \right] f \right) (l) \\ &= \left(\overrightarrow{j(x)}(\mathcal{E}[x^n]f) \right) (l) - \mathcal{E}[\nabla_{j(x)}(x^n)]f(l). \end{aligned}$$

The last equality above is a consequence of Lemma 3.6. □

Proof of Proposition 3.16. One easily observes that

$$\Gamma(S(L/A)) \cong \mathcal{D}(L/A, M) \xrightarrow{\exp_*^{\nabla, j}} \mathcal{D}(\mathcal{L}/\mathcal{A}, M) \cong \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$$

is the composition of \mathcal{E} with the projection $\mathcal{U}(L) \rightarrow \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ the same way $\exp^{\nabla, j}$ is the composition of \mathbf{E} with the projection $\mathbf{q} : \mathcal{L} \rightarrow \mathcal{L}/\mathcal{A}$. Then Lemma 3.18 asserts that

$$\exp_*^{\nabla, j} : \Gamma(S(L/A)) \rightarrow \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$$

is precisely the map $\text{pbw}^{\nabla, j}$ inductively defined in Theorem 2.2. □

Remark 3.19. *When $L = T_M$ and A is the trivial Lie subalgebroid of L of rank 0, the $\text{pbw}^{\nabla, j}$ map of Theorem 3.16 is the inverse of the so called “complete symbol map,” which is an isomorphism from the space $\mathcal{U}(T_M)$ of differential operators on M to the space $\Gamma(S(T_M))$ of fiberwisely polynomial functions on T_M^\vee . The complete symbol map was generalized to arbitrary Lie algebroids over \mathbb{R} by Nistor, Weinstein, and one of the authors [30]. It played an important role in quantization theory [15, 30].*

4. KAPRANOV DG-MANIFOLDS

4.1. Kapranov dg-manifolds. Let A be a Lie algebroid over a smooth manifold M , and R be the algebra of smooth functions on M valued in \mathbb{K} . The Chevalley-Eilenberg differential

$$d_A : \Gamma(\wedge^k A^\vee) \rightarrow \Gamma(\wedge^{k+1} A^\vee)$$

defined by

$$(d_A \alpha)(a_0, a_1, \dots, a_k) = \sum_{i=0}^n \rho(a_i)(\alpha(a_0, \dots, \widehat{a}_i, \dots, a_k)) \\ + \sum_{i < j} \alpha([a_i, a_j], a_0, \dots, \widehat{a}_i, \dots, \widehat{a}_j, \dots, a_k)$$

and the exterior product make $\bigoplus_{k \geq 0} \Gamma(\wedge^k A^\vee)$ into a differential graded commutative R -algebra.

The Chevalley-Eilenberg covariant differential associated to a representation ∇ of a Lie algebroid $A \rightarrow M$ of rank n on a vector bundle $E \rightarrow M$ is the operator

$$d_A^\nabla : \Gamma(\wedge^k A^\vee \otimes E) \rightarrow \Gamma(\wedge^{k+1} A^\vee \otimes E)$$

that takes a section $\alpha \otimes e$ of $\wedge^k A^\vee \otimes E$ to

$$d_A^\nabla(\alpha \otimes e) = (d_A \alpha) \otimes e + (-1)^k \sum_{j=1}^n (\alpha \wedge \beta_j) \otimes \nabla_{b_j} e, \quad (24)$$

where b_1, b_2, \dots, b_n and $\beta_1, \beta_2, \dots, \beta_n$ are any pair of dual local frames for the vector bundles A and A^\vee . Because the connection ∇ is flat, d_A^∇ is a coboundary operator: $d_A^\nabla \circ d_A^\nabla = 0$.

Recall that a *dg-manifold* is a \mathbb{Z} -graded manifold endowed with a vector field Q of degree $+1$ satisfying $[Q, Q] = 0$. Such a vector field Q is said to be homological [34]. In the literature, dg-manifolds are also called Q -manifolds [1, 17]. We refer the reader to [18, 35] for details. Below are several standard examples of dg-manifolds.

Example 4.1. *If A is a Lie algebroid, then $A[1]$ is a sg-manifold with the Chevalley-Eilenberg differential d_A serving as the homological vector field Q . In fact, according to Vaintrob [34], for any vector bundle $A \rightarrow M$, $A[1]$ is a dg-manifold if and only if A is a Lie algebroid over M .*

Example 4.2. *If $E \rightarrow M$ is a vector bundle and $s \in \Gamma(E)$, then $E[-1]$ is a dg-manifold with the contraction operator i_s serving as the homological vector field Q .*

Example 4.3. *Suppose $\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}_i$ is a \mathbb{Z} -graded vector space of finite type, i.e. each \mathfrak{g}_i is a finite dimensional vector space. Then $\mathfrak{g}[1]$ is a dg-manifold if and only if \mathfrak{g} is an L_∞ -algebra.*

We are now ready to introduce the main object of the paper.

Definition 4.4. *A Kapranov dg-manifold is a pair of vector bundles A and E over a common base manifold together with a homological vector field on each of the graded manifolds $A[1]$ and $A[1] \oplus E$ such that the inclusion $A[1] \hookrightarrow A[1] \oplus E$ and the projection $A[1] \oplus E \twoheadrightarrow A[1]$ are morphisms of dg-manifolds.*

Note that the requirement of $A[1]$ being a dg-manifold implies that A must be a Lie algebroid. Kapranov dg-manifolds are related to Costello's L_∞ algebras over the differential graded algebra $(\Gamma(\wedge^\bullet A^\vee), d_A)$ (see [12]).

Example 4.5. *Given a representation of a Lie algebroid A over a vector bundle E , the Chevalley-Eilenberg differentials provide a pair of homological vector fields on $A[1]$ and $A[1] \oplus E$ compatible with the inclusion and projection maps. The resulting Kapranov dg-manifold $A[1] \oplus E$ is said to be linear.*

Definition 4.6. *A morphism of Kapranov dg-manifolds from $A[1] \oplus E$ to $A'[1] \oplus E'$ consists of a pair (Φ, ϕ) of morphisms of dg-manifolds $\Phi : A[1] \oplus E \rightarrow A'[1] \oplus E'$ and $\phi : A[1] \rightarrow A'[1]$ such that the following two diagrams commute:*

$$\begin{array}{ccc}
A[1] \oplus E & \xrightarrow{\Phi} & A'[1] \oplus E' \\
\uparrow i & & \uparrow i \\
A[1] & \xrightarrow{\phi} & A'[1]
\end{array}
\quad
\begin{array}{ccc}
A[1] \oplus E & \xrightarrow{\Phi} & A'[1] \oplus E' \\
q \downarrow & & \downarrow q \\
A[1] & \xrightarrow{\phi} & A'[1].
\end{array}
\tag{25}$$

Such a morphism is said to fix the dg-submanifold $A[1]$ if $A = A'$ and $\phi = \text{id}_{A[1]}$, in which case it is simply denoted by Φ .

An isomorphism of Kapranov dg-manifolds is a morphism (Φ, ϕ) of Kapranov dg-manifolds where both Φ and ϕ are dg-manifold isomorphisms.

Definition 4.7. *A Kapranov dg-manifold $(A[1] \oplus E, D)$ is said to be linearizable if there exists a representation ∇ of the Lie algebroid A on the vector bundle E and an isomorphism of Kapranov dg-manifolds from $(A[1] \oplus E, D)$ to $(A[1] \oplus E, d_A^\nabla)$ fixing the dg-submanifold $(A[1], d_A)$.*

4.2. Equivalent definitions of Kapranov dg-manifolds. Given a smooth vector bundle E over the manifold M , recall that deconcatenation (see Equation (11)) defines a comultiplication on $\Gamma(S(E))$ while concatenation defines a multiplication on $\Gamma(S(E^\vee))$. Let ϵ denote the ideal of the graded algebra $\Gamma(S(E^\vee))$ generated by $\Gamma(E^\vee)$, and R the algebra of smooth functions on M . The algebra $\text{Hom}_R(\Gamma(S(E)), R)$ dual to the coalgebra $\Gamma(S(E))$ is the ϵ -adic completion of $\Gamma(S(E^\vee))$. It will be denoted by $\Gamma(\hat{S}(E^\vee))$. Equivalently, one can think of the completion $\hat{S}(E^\vee)$ of $S(E^\vee)$ as a bundle of algebras over M . Therefore we have a pairing $\langle \varepsilon | e \rangle \in R$ for any $\varepsilon \in \Gamma(\hat{S}(E^\vee))$ and $e \in \Gamma(S(E))$, which can be obtained as follows. Consider e as a fiberwise polynomial on E^\vee and ε a fiberwise differential operator on E^\vee with fiberwisely constant coefficients. Then $\langle \varepsilon | e \rangle$ is the function on M obtained by applying ε on e fiberwisely. The following lemma is immediate.

Lemma 4.8. (1) *Let ε be a section of $\hat{S}(E^\vee)$. Then $\varepsilon \in \Gamma(\hat{S}^{\geq n}(E^\vee))$ if and only if $\langle \varepsilon | e \rangle = 0$ for all $e \in \Gamma(\hat{S}^{> n}(E^\vee))$.*

(2) *Let e be a section of $S(E)$. Then $e \in \Gamma(S^{\leq n}(E))$ if and only if $\langle \varepsilon | e \rangle = 0$ for all $\varepsilon \in \Gamma(\hat{S}^{> n}(E^\vee))$.*

Let $\nabla : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ be a representation of A on E . Consider the space $\mathfrak{X}^{\text{fv}}(E)$ of formal vertical vector fields on E along the zero section, which are by definition R -linear derivations of the algebra of formal functions on E along the zero section. Since the latter can be identified naturally with $\Gamma(\hat{S}(E^\vee))$, the space

$\mathfrak{X}^{\text{fv}}(E)$ is naturally identified with $\Gamma(\hat{S}(E^\vee) \otimes E)$. The Lie algebra structure on $\Gamma(\hat{S}(E^\vee) \otimes E)$ inherited from $\mathfrak{X}^{\text{fv}}(E)$ extends to a graded Lie algebra structure on $\bigoplus_{n \geq 0} \Gamma(\wedge^n(A^\vee) \otimes \hat{S}(E^\vee) \otimes E)$:

$$\begin{aligned} & [\alpha_1 \otimes \varepsilon_1^{n_1} \otimes e_1, \alpha_2 \otimes \varepsilon_2^{n_2} \otimes e_2] \\ &= n_2 \langle \varepsilon_2, e_1 \rangle \cdot (\alpha_1 \wedge \alpha_2) \otimes (\varepsilon_1^{n_1} \odot \varepsilon_2^{n_2-1}) \otimes e_2 \\ &\quad - n_1 \langle \varepsilon_1, e_2 \rangle \cdot (\alpha_1 \wedge \alpha_2) \otimes (\varepsilon_1^{n_1-1} \odot \varepsilon_2^{n_2}) \otimes e_1, \end{aligned} \quad (26)$$

for all $\alpha_1, \alpha_2 \in \Gamma(\wedge A^\vee)$, $\varepsilon_1, \varepsilon_2 \in \Gamma(E^\vee)$, $e_1, e_2 \in \Gamma(E)$, and $n_1, n_2 \in \mathbb{N}$. Since E is an A -module, so is $\hat{S}(E^\vee) \otimes E$ and we can consider the corresponding Chevalley-Eilenberg differential

$$d_A^\nabla : \Gamma(\wedge(A^\vee) \otimes \hat{S}(E^\vee) \otimes E) \rightarrow \Gamma(\wedge(A^\vee) \otimes \hat{S}(E^\vee) \otimes E), \quad (27)$$

The following result can be easily proved by a direct verification.

Lemma 4.9. *Given a representation $\nabla : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ of A on E , the differential (27) and the Lie bracket (26) make*

$$\bigoplus_{n \geq 0} \Gamma(\wedge^n(A^\vee) \otimes \hat{S}(E^\vee) \otimes E)$$

a differential graded Lie algebra.

Theorem 4.10. *Given a Lie algebroid A over a smooth manifold M and a smooth vector bundle E over M , the following data are equivalent.*

- (1) A homological vector field D on $A[1] \oplus E$ making it a Kapranov dg-manifold.
- (2) An infinitesimal action $\delta : \Gamma(A) \times \Gamma(S(E)) \rightarrow \Gamma(S(E))$ of A on the symmetric coalgebra $\Gamma(S(E))$ by coderivations such that $\delta_a(1) = 0$, for all $a \in \Gamma(A)$.
- (3) An infinitesimal action $\delta^* : \Gamma(A) \times \Gamma(\hat{S}(E^\vee)) \rightarrow \Gamma(\hat{S}(E^\vee))$ of A on the completed symmetric algebra $\Gamma(\hat{S}(E^\vee))$ by derivations such that $\delta_a^*(\varepsilon) \in \hat{S}^{\geq 1}(E^\vee)$, for all $a \in \Gamma(A)$ and $\varepsilon \in \Gamma(E^\vee)$.
- (4) A degree +1 derivation D of the graded algebra $\bigoplus_{n \geq 0} \Gamma(\wedge^n A^\vee \otimes \hat{S}(E^\vee))$, which preserves the decreasing filtration

$$\dots \hookrightarrow \Gamma(\wedge A^\vee \otimes \hat{S}^{\geq n+1}(E^\vee)) \hookrightarrow \Gamma(\wedge A^\vee \otimes \hat{S}^{\geq n}(E^\vee)) \hookrightarrow \dots \quad (28)$$

and satisfies $D^2 = 0$ and $D(\alpha \otimes 1) = d_A(\alpha) \otimes 1$, for all $\alpha \in \Gamma(\wedge A^\vee)$.

- (5) A representation $\nabla : \Gamma(A) \times \Gamma(E) \rightarrow \Gamma(E)$ of A on E and a solution $\mathcal{R} \in \Gamma(A^\vee \otimes \hat{S}^{\geq 2}(E^\vee) \otimes E)$ of the Maurer-Cartan equation

$$d_A^\nabla \mathcal{R} + \frac{1}{2}[\mathcal{R}, \mathcal{R}] = 0$$

in the differential graded Lie algebra $\bigoplus_{n \geq 0} \Gamma(\wedge^n A^\vee \otimes \hat{S}(E^\vee) \otimes E)$ of Lemma 4.9.

- (6) An $L_\infty[1]$ algebra structure on the graded vector space

$$\bigoplus_{n \geq 0} \Gamma(\wedge^n A^\vee \otimes E)$$

defined by a sequence $(\lambda_k)_{k \in \mathbb{N}}$ of multibrackets

$$\lambda_k : S^k(\Gamma(\wedge A^\vee \otimes E)) \rightarrow \Gamma(\wedge A^\vee \otimes E)[1]$$

such that each λ_k with $k \geq 2$ is $\Gamma(\wedge A^\vee)$ -multilinear and λ_1 is the Chevalley-Eilenberg covariant differential d_A^∇ associated to a representation ∇ of the Lie algebroid A on the vector bundle E .

Actually, the infinitesimal A -actions δ and δ^* are related to the homological vector field D through the identities

$$\begin{aligned}\delta_a^* \sigma &= i_a D(1 \otimes \sigma) \\ \rho(a) \langle \sigma | s \rangle &= \langle \delta_a^* \sigma | s \rangle + \langle \sigma | \delta_a s \rangle\end{aligned}$$

where $a \in \Gamma(A)$, $\sigma \in \Gamma(\hat{S}(E^\vee))$, $s \in \Gamma(S(E))$. Moreover, we have

$$D = d_A^\nabla + \sum_{k \geq 2} \mathcal{R}_k, \quad (29)$$

where the representation ∇ of A on E and the bundle maps $\mathcal{R}_k : A \otimes S^k(E) \rightarrow E$ (with $k \geq 2$) are related to the coderivation δ through the identities

$$\nabla_a e = -\text{pr}_1(\delta_a e) \quad (30)$$

$$\mathcal{R}_k(a; s) = -\text{pr}_1(\delta_a s) \quad (31)$$

where $a \in \Gamma(A)$, $e \in \Gamma(E)$, $s \in \Gamma(S^k(E))$, and pr_1 denotes the canonical projection of $\bigoplus_{n=0}^\infty S^n(E)$ onto $S^1(E)$. Finally, for all $e_1, \dots, e_k \in \Gamma(E)$ and all homogeneous elements ξ_1, \dots, ξ_k of $\Gamma(\wedge A^\vee)$, we have

$$\lambda_k(\xi_1 \otimes e_1, \dots, \xi_k \otimes e_k) = (-1)^{|\xi_1| + \dots + |\xi_k|} \xi_1 \wedge \dots \wedge \xi_k \wedge \mathcal{R}_k(e_1, \dots, e_k).$$

Proof. (iii) \Leftrightarrow (vi) Given an infinitesimal A -action $\delta^* : \Gamma(A) \times \Gamma(\hat{S}(E^\vee)) \rightarrow \Gamma(\hat{S}(E^\vee))$, set

$$D(\alpha \otimes \sigma) = (d_A \alpha) \otimes \sigma + (-1)^k \sum_j (\alpha \wedge \nu_j) \otimes (\delta_{v_j}^* \sigma)$$

for all $\alpha \in \Gamma(\wedge^k A^\vee)$, $k \in \{0, 1, 2, \dots\}$, and $\sigma \in \Gamma(\hat{S}(E^\vee))$. Here v_1, \dots, v_l and ν_1, \dots, ν_l are any pair of dual local frames of the vector bundles A and A^\vee . The flatness of the infinitesimal action δ^* implies that $D^2 = 0$ while the other properties of D follow immediately from those of δ^* . Conversely, given a degree +1 derivation D as in (ii), one can recover δ^* by the relation

$$\delta_a^*(\sigma) = i_a D(1 \otimes \sigma), \forall a \in \Gamma(A), \sigma \in \Gamma(\hat{S}(E^\vee)).$$

(i) \Leftrightarrow (vi) A homological vector field on $A[1] \oplus E$ is, by definition, equivalent to a degree +1 derivation D of the graded algebra $\bigoplus_{n \geq 0} \Gamma(\wedge^n A^\vee \otimes \hat{S}(E^\vee))$. The projection $A[1] \oplus E \rightarrow A[1]$ being a morphism of dg-manifolds is equivalent to the condition

$$D(\alpha \otimes 1) = d_A \alpha \otimes 1, \quad \forall \alpha \in \Gamma(\wedge A^\vee), \quad (32)$$

while the inclusion $A[1] \hookrightarrow A[1] \oplus E$ being a dg submanifold is equivalent to

$$\text{Pr}(D(\alpha \otimes \sigma)) = d_A(\alpha \sigma_0). \quad (33)$$

Here $\text{Pr} : \Gamma(\wedge A^\vee \otimes \hat{S}(E^\vee)) \rightarrow \Gamma(\wedge A^\vee)$ is the natural projection map: $\text{Pr}(\alpha \otimes \sigma) = \alpha \sigma_0$, $\forall \alpha \in \Gamma(\wedge A^\vee)$ and $\sigma \in \Gamma(\hat{S}(E^\vee))$, where $\sigma_0 \in R$ denotes the $\Gamma(\hat{S}^0(E^\vee))$ -component of σ . From Eq. (32), it follows that Eq. (33) is indeed equivalent to $\text{Pr}(D(1 \otimes \sigma)) = d_A(\sigma_0)$, which essentially means that $\text{Pr}(D(1 \otimes \sigma)) = 0$ for all $\sigma \in \Gamma(\hat{S}^{\geq 1}(E^\vee))$. Since D is a derivation, the latter is equivalent to $D(1 \otimes$

$\Gamma(\hat{S}^{\geq 1}(E^\vee)) \subseteq (1 \otimes \Gamma(\hat{S}^{\geq 1}(E^\vee)))$, which in turn is equivalent to that D preserves the decreasing filtration (28).

(vi) \Leftrightarrow (v) Let D be a derivation as in (iii). By the equivalence of (iv) and (iii), there is an infinitesimal action $\delta^* : \Gamma(A) \times \Gamma(\hat{S}(E^\vee)) \rightarrow \Gamma(\hat{S}(E^\vee))$ on the symmetric algebra $\Gamma(\hat{S}(E^\vee))$ by derivations such that $\delta_a^*(\varepsilon) \in \Gamma(\hat{S}^{\geq 1}(E^\vee))$ for any $a \in \Gamma(A)$ and $\varepsilon \in \Gamma(E^\vee)$. Write $\gamma_a^*(\varepsilon) = \text{Pr}_1(\delta_a^*(\varepsilon))$, where $\text{Pr}_1 : \Gamma(\hat{S}^{\geq 1}(E^\vee)) \rightarrow \Gamma(E^\vee)$ is the natural projection. It is simple to see that $\gamma^* : \Gamma(A) \times \Gamma(E^\vee) \rightarrow \Gamma(E^\vee)$ defines a flat A -connection on the vector bundle E^\vee , and therefore a representation ∇ of A on the vector bundle E . That is,

$$\nabla_a(\varepsilon) = \text{Pr}_1(\delta_a^*(\varepsilon)). \quad (34)$$

Thus d_A^∇ defines a degree +1 homological vector field on $A[1] \oplus E$, which makes it into a linear Kapranov dg-manifold. Hence the difference $D - d_A^\nabla$ is a $\Gamma(\wedge A^\vee)$ -linear derivation on $\Gamma(\wedge A^\vee \otimes \hat{S}(E^\vee))$. Moreover since both D and d_A^∇ preserve the decreasing filtration (28) and $D(\alpha \otimes \sigma) = d_A^\nabla(\alpha \otimes \sigma)$ for all $\alpha \in \Gamma(\wedge A^\vee)$ and $\sigma \in \Gamma(S^{\leq 1}(E^\vee))$, it follows that there must exist $\mathcal{R} \in \Gamma(A^\vee \otimes \hat{S}^{\geq 2}(E^\vee) \otimes E)$ such that for all $k \in \{0, 1, 2, \dots\}$, $\alpha \in \Gamma(\wedge^k A^\vee)$, and $\sigma \in \Gamma(\hat{S}(E^\vee))$,

$$(D - d_A^\nabla)(\alpha \otimes \sigma) = (-1)^k \sum_j (\alpha \wedge \beta_j) \otimes i_{b_j} \mathcal{R}(\sigma), \quad (35)$$

where $i_{b_j} \mathcal{R} \in \Gamma(\hat{S}^{\geq 2}(E^\vee) \otimes E)$ is seen as a derivation of the algebra $\Gamma(\hat{S}(E^\vee))$. Since $D^2 = 0$, and $(d_A^\nabla)^2 = 0$, it follows that \mathcal{R} must be a solution of the Maurer-Cartan equation $d_A^\nabla \mathcal{R} + \frac{1}{2}[\mathcal{R}, \mathcal{R}] = 0$.

Conversely, given a representation ∇ of A on a vector bundle E and a solution of the Maurer-Cartan equation $\mathcal{R} \in \Gamma(A^\vee \otimes \hat{S}^{\geq 2}(E^\vee) \otimes E)$, the operator $D = d_A^\nabla + \mathcal{R}$, where \mathcal{R} acts on $\Gamma(\wedge A^\vee \otimes \hat{S}(E^\vee))$ by the right hand side of Eq. (35), clearly satisfies all the desired properties.

(iv) \Leftrightarrow (vi) Assume that $\mathcal{R} \in \Gamma(A^\vee \otimes \hat{S}^{\geq 2}(E^\vee) \otimes E)$ is a solution of the Maurer-Cartan equation. Then $D = d_A^\nabla + \mathcal{R}$ is a derivation satisfying all the properties as in (iv). It is standard [17] that D induces an $L_\infty[1]$ -algebra whose multibrackets λ_k are the coefficients of the Talyer expansion of the homological vector field D . For completeness, we sketch a proof using the so called higher derived brackets of Voronov [36] Let \mathfrak{h} be the graded Lie algebra $\bigoplus_{n \geq 0} \Gamma(\wedge^n A^\vee \otimes \hat{S}(E^\vee) \otimes E)$ equipped with the Lie bracket as given by Eq. (26), and $\mathfrak{a} \subset \mathfrak{h}$ the abelian Lie subalgebra $\bigoplus_{n \geq 0} \Gamma(\wedge^n A^\vee \otimes E)$. Let $\Pi_{\mathfrak{a}} : \mathfrak{h} \rightarrow \mathfrak{a}$ be the natural projection. It is simple to check that

$$\Pi_{\mathfrak{a}}[x, y] = \Pi_{\mathfrak{a}}[\Pi_{\mathfrak{a}}x, y] + \Pi_{\mathfrak{a}}[x, \Pi_{\mathfrak{a}}y]$$

$\forall x, y \in \mathfrak{h}$. Hence the triple $(\mathfrak{h}, \mathfrak{a}, \Pi_{\mathfrak{a}})$ is a V-algebra in the sense of Cattaneo-Schätz [8, Definition 2.3]. Moreover, $D = d_A^\nabla + \mathcal{R}$ is a degree +1 derivation of the graded Lie algebra \mathfrak{h} , where \mathcal{R} is considered as a vertical vector field on $A[1] \oplus E$, i.e. an element in $\Gamma(\wedge A^\vee) \otimes \mathfrak{X}^{\text{fv}}(E)$ and acts on \mathfrak{h} by the commutator $[\mathcal{R}, -]$. It is simple to check that D is an *adapted derivation* in the sense of [8, Definition 2.4]. Consider the higher derived brackets:

$$\lambda_k : S^k(\mathfrak{a}) \rightarrow \mathfrak{a}[1] \quad (36)$$

$$\lambda_k(x_1, \dots, x_k) = \Pi_{\mathfrak{a}}[[\dots [D(x_1), x_2] \dots] \dots x_k] \quad (37)$$

for every $k > 0$, and

$$\lambda_0 = \Pi_a \mathcal{R} = 0.$$

According to [8, Theorem 2.5], the sequence of multibrackets $(\lambda_k)_{k \in \mathbb{N}}$ equips $\mathfrak{a} = \Gamma(\wedge A^\vee \otimes E)$ with the structure of an $L_\infty[1]$ -algebra. It follows from a direct computation that $\lambda_1 = d_A^\nabla$, and $(\lambda_k)_{k \geq 2}$ are indeed $\Gamma(\wedge(A^\vee))$ -multilinear. In fact, for any $k \geq 2$, the k -th multibracket λ_k is related to the k -th component $\mathcal{R}_k \in \Gamma(A^\vee \otimes S^k(E^\vee) \otimes E)$ of the Maurer-Cartan element \mathcal{R} through the equation

$$\lambda_k(\xi_1 \otimes e_1, \dots, \xi_k \otimes e_k) = (-1)^{|\xi_1| + \dots + |\xi_k|} \xi_1 \wedge \dots \wedge \xi_k \wedge \mathcal{R}_k(e_1, \dots, e_k), \quad (38)$$

$\forall e_1, \dots, e_k \in \Gamma(E)$ and any homogeneous elements ξ_1, \dots, ξ_k of $\Gamma(\wedge A^\vee)$.

Conversely, given a sequence of multibrackets $(\lambda_k)_{k \geq 2}$ as in (vi), since each λ_k is $\Gamma(\wedge A^\vee)$ -multilinear, it follows that λ_k must be of the form (38) for a section $\mathcal{R}_k \in \Gamma(A^\vee \otimes S^k(E^\vee) \otimes E)$. Let $\mathcal{R} = \sum_{n \geq 2} \mathcal{R}_n \in \Gamma(A^\vee \otimes \hat{S}^{\geq 2}(E^\vee) \otimes E)$ and consider the operator $D = d_A^\nabla + \mathcal{R}$ which acts on \mathfrak{h} as a degree +1-derivation of the graded Lie algebra. It is simple to check, by going backward, that the compatibility conditions of the multibrackets $(\lambda_k)_{k \geq 2}$ imply that

$$\Pi_a([[[D, D], e_1], \dots, e_n]) = 0, \forall e_1, \dots, e_n \in \Gamma(E). \quad (39)$$

Since $[D, D] = 2(d_A^\nabla \mathcal{R} + \frac{1}{2}[\mathcal{R}, \mathcal{R}])$ is an element in $\Gamma(\wedge^2 A^\vee \otimes \hat{S}^{\geq 2}(E^\vee) \otimes E)$, it follows that $D^2 = 0$.

(ii) \Leftrightarrow (iii) The infinitesimal A -actions δ and δ^* are related to each other by the identity

$$\rho(a)\langle \varepsilon | e \rangle = \langle \delta_a^*(\varepsilon) | e \rangle + \langle \varepsilon | \delta_a(e) \rangle, \quad (40)$$

where $a \in \Gamma(A)$, $\varepsilon \in \Gamma(\hat{S}(E^\vee))$, and $e \in \Gamma(S(E))$. Since $\Gamma(\hat{S}^{>n}(E^\vee))$ is the annihilator of $\Gamma(S^{\leq n}(E))$ according to Lemma 4.8, $\Gamma(\hat{S}^{>n}(E^\vee))$ is δ^* -stable if and only if $\Gamma(S^{\leq n}(E))$ is δ -stable. It follows from Theorem 5.3 that $\delta_a(1) = 0$ for all $a \in \Gamma(A)$ if and only if $\delta_a^*(\varepsilon) \in \Gamma(\hat{S}^{\geq 1}(E^\vee))$ for all $a \in \Gamma(A)$ and $\varepsilon \in \Gamma(E^\vee)$. \square

Remark 4.11. *According to (the proof of) Theorem 4.10, the homological vector field on a Kapranov dg-manifold $A[1] \oplus E$ is the sum $D = d_A^\nabla + \mathcal{R}$ of the Chevalley-Eilenberg differential relative to a representation ∇ of the Lie algebroid A on the vector bundle E and a solution $\mathcal{R} = \sum_{k \geq 2} \mathcal{R}_k$, with $\mathcal{R}_k \in \Gamma(A^\vee \otimes S^k(E^\vee) \otimes E)$, of the Maurer-Cartan equation $d_A^\nabla \mathcal{R} + [\mathcal{R}, \mathcal{R}] = 0$. Projecting the Maurer-Cartan equation on $\Gamma(\wedge A^\vee \otimes S^2(E^\vee) \otimes E)$, we obtain $d_A^\nabla \mathcal{R}_2 = 0$. Hence $\mathcal{R}_2 \in \Gamma(A^\vee \otimes S^2(E^\vee) \otimes E)$ is a Chevalley-Eilenberg 1-cocycle with respect to the representation of A on $S^2(E^\vee) \otimes E$ induced by ∇ . We have shown that a Kapranov dg-manifold $A[1] \oplus E$ determines a cohomology class $[\mathcal{R}_2]$ in $H_{\text{CE}}^1(A; S^2(E^\vee) \otimes E)$.*

Lemma 4.12. *There is a bijection between the following objects.*

- (1) an isomorphism of Kapranov dg-manifolds $A[1] \oplus E$ over the identity of A ;
- (2) an isomorphism of R -filtered coalgebras $\Psi : \Gamma(S(E)) \rightarrow \Gamma(S(E))$ with respect to the increasing filtration:

$$\Gamma(S^{\leq 0}(E)) \subset \Gamma(S^{\leq 1}(E)) \subset \Gamma(S^{\leq 2}(E)) \subset \Gamma(S^{\leq 3}(E)) \subset \dots \quad (41)$$

which is equivariant under the infinitesimal A -action;

(3) an isomorphism of R -filtered algebras $\Phi : \Gamma(\hat{S}(E^\vee)) \rightarrow \Gamma(\hat{S}(E^\vee))$ with respect to the decreasing filtration:

$$\dots \hookrightarrow \Gamma(\hat{S}^{\geq n+1}(E^\vee)) \hookrightarrow \Gamma(\hat{S}^{\geq n}(E^\vee)) \hookrightarrow \dots, \quad (42)$$

which is equivariant under the infinitesimal A -action.

It is clear that any isomorphism of R -filtered algebras $\Phi : \Gamma(\hat{S}(E^\vee)) \rightarrow \Gamma(\hat{S}(E^\vee))$ is equivalent to a family $(\phi_k)_{k \geq 1}$ with $\phi_k \in \Gamma(E \otimes S^k(E^\vee))$ such that $\phi_1 \in \Gamma(E \otimes E^\vee)$ is invertible when being considered as a bundle map $E \rightarrow E$. The relation between Φ and $(\phi_k)_{k \geq 1}$ is as follows:

$$\Phi(\varepsilon) = \sum_{k=1}^{\infty} \phi_k(\varepsilon), \forall \varepsilon \in \Gamma(E^\vee).$$

By abuse of notation, we write $\Phi = (\phi_k)_{k \geq 1}$.

Proposition 4.13. *Assume that $\Phi = (\phi_k)_{k \geq 1}$, where $\phi_k \in \Gamma(E \otimes S^k(E^\vee))$, corresponds to an isomorphism of Kapranov dg-manifolds over the identity of A from $(A[1] \oplus E, D)$ to $(A[1] \oplus E, D')$, where $D = d_A^\nabla + \sum_{k \geq 2} \mathcal{R}_k$ and $D' = d_A^{\nabla'} + \sum_{k \geq 2} \mathcal{R}'_k$. Then*

- (1) $\phi_1 : E \rightarrow E$ is an isomorphism of A -modules;
- (2) under the induced isomorphism

$$\phi_1^* : H_{\text{CE}}^1(A; S^2(E^\vee) \otimes E) \rightarrow H_{\text{CE}}^1(A; S^2(E^\vee) \otimes E),$$

the class of $[\mathcal{R}'_2]$ maps to the class of $[\mathcal{R}_2]$.

Proof. (i) follows immediately from (34).

To prove (ii), without loss of generality, we may assume that $\phi_1 = \text{id}$ and $\nabla' = \nabla$. By assumption, we have

$$(1 \otimes \Phi) \circ (d_A^\nabla + \mathcal{R}_2 + \sum_{k \geq 3} \mathcal{R}_k) = (d_A^\nabla + \mathcal{R}'_2 + \sum_{k \geq 3} \mathcal{R}'_k) \circ \Phi.$$

Applying the equation above to $\varepsilon \in \Gamma(E^\vee)$ and projecting onto the components in $\Gamma(A^\vee \otimes S^2(E^\vee))$, we obtain $\phi_2(d_A^\nabla(\varepsilon)) + \mathcal{R}_2(\varepsilon) = d_A^\nabla(\phi_2(\varepsilon)) + \mathcal{R}'_2(\varepsilon)$. It thus follows that $\mathcal{R}_2 - \mathcal{R}'_2 = d_A^\nabla \phi_2$. \square

Corollary 4.14. *If a Kapranov dg-manifold $A[1] \oplus E$ is linearizable, then $[\mathcal{R}_2]$ vanishes in $H_{\text{CE}}^1(A; S^2(E^\vee) \otimes E)$.*

4.3. Examples. In this section, we provide a standard construction of Kapranov dg-manifolds, which in a certain sense is a geometrical interpretation of a Kapranov dg-manifold in light of Theorem 4.10 (iii).

We say that a Lie algebroid A (over M) acts infinitesimally on a surjective submersion $\pi : X \rightarrow M$, if there is a Lie algebra morphism from $\Gamma(A)$ to vector fields on X , denoted by $a \mapsto \underline{a}$, which satisfies the properties: $\underline{f}a = \pi^* f \underline{a}$, and $\pi_* \underline{a} = \rho(a)$, $\forall a \in \Gamma(A)$ and $f \in R$.

Let $\pi : E \rightarrow M$ be a vector bundle, on which Lie algebroid A acts infinitesimally preserving the zero section. That is, \underline{a} is tangent to the zero section, $\forall a \in \Gamma(A)$. Now for any $s \in \Gamma(\hat{S}(E^\vee))$ and $a \in \Gamma(A)$, define $\delta_a^* s \in \Gamma(\hat{S}(E^\vee))$ to be the infinit jets

of $\underline{a}(s)$ along the zero section, where s is considered as a fiberwise formal function on E . Since \underline{a} is tangent to the zero section, it follows that $\delta_a^*(\varepsilon) \in \hat{S}^{\geq 1}(E^\vee)$, for all $a \in \Gamma(A)$ and $\varepsilon \in \Gamma(E^\vee)$. Hence, according to Theorem 4.10, $A[1] \oplus E$ naturally admits a Kapranov dg-manifold structure. We thus have the following

Proposition 4.15. *Let A be a Lie algebroid over M , and $\pi : E \rightarrow M$ a vector bundle. Assume that E admits an infinitesimal A -action which preserves the zero section. Then $A[1] \oplus E$ is naturally a Kapranov dg-manifold.*

Remark 4.16. *From the discussions above, one easily sees that the same result is valid provided that the infinitesimal A -action is defined in a neighborhood of the zero section of E .*

Note that when E is an A -module, that is, when the infinitesimal A -action preserves the vector bundle structure on E (or \underline{a} , for all $a \in \Gamma(A)$, is a linear vector field on E [26]), we reduce to the situation of linear Kapranov dg-manifold in Example 4.5.

Alternatively, we may obtain the homological vector field on the Kapranov dg-manifold in Proposition 4.15 as the Chevalley-Eilenberg differential of a Lie algebroid as follows.

Consider the transformation Lie algebroid $A \ltimes E \rightarrow E$ [25], where $A \ltimes E = A \times_M E$ is the pullback vector bundle of $A \rightarrow M$ over the projection $\pi : E \rightarrow M$. The anchor map is essentially the action map $(a, e) \rightarrow \underline{a}|_e$, $\forall (a, e) \in A \times_M E$, while the Lie bracket on the sections of $A \ltimes E \rightarrow E$ is induced by the Lie bracket on $\Gamma(A)$ in the sense that $[\pi^*a_1, \pi^*a_2] = \pi^*[a_1, a_2]$, $\forall a_1, a_2 \in \Gamma(A)$. It is clear that $\Gamma(\wedge^k(A \ltimes E)^\vee) \cong \Gamma(\wedge^k A^\vee) \otimes_R C^\infty(E)$.

Proposition 4.17. *Under the same hypothesis as in Proposition 4.15, the formal Chevalley-Eilenberg differential*

$$\Gamma(\wedge^k A^\vee) \otimes_R \Gamma(\hat{S}E^\vee) \xrightarrow{d_A^{(\infty)}} \Gamma(\wedge^{k+1} A^\vee) \otimes_R \Gamma(\hat{S}E^\vee) \quad (43)$$

of the transformation Lie algebroid $A \ltimes E \rightarrow E$ is precisely the homological vector field on the Kapranov dg-manifold $A[1] \oplus E$ as in Proposition 4.15.

As an immediate consequence, we have the following

Corollary 4.18. *Let A be a Lie algebroid over M , and $\pi : X \rightarrow M$ a surjective submersion. Assume that $\pi : X \rightarrow M$ admits an infinitesimal A -action, and $\sigma : M \rightarrow X$ is a section of π which is stable under the infinitesimal A -action. Then $A[1] \oplus N_M$ admits a Kapranov dg-manifold structure, canonical up to an isomorphism, where N_M denotes the normal bundle of $\sigma(M)$ in X .*

Proof. Choose a Riemannian metric on X . The corresponding exponential map establishes a diffeomorphism between a tubular neighborhood of $\sigma(M)$ in X and a tubular neighborhood of the zero section of the normal bundle N_M . Thus it follows from Proposition 4.15 and Remark 4.16 that $A[1] \oplus N_M$ admits a Kapranov dg-manifold structure. According to Lemma 4.12, the Kapranov dg-manifold structures resulting from different choices of Riemannian metric on X are indeed isomorphic. \square

From the proof of Corollary 4.18, we immediately obtain the following

Corollary 4.19. *Under the same hypothesis as in Corollary 4.18, if there exists a local diffeomorphism ϕ from a tubular neighborhood of the zero section of the normal bundle N_M to a tubular neighborhood of $\sigma(M)$ in X such that the fiberwise ∞ -jets of ϕ along the zero section intertwines the infinitesimal A -actions on N_M and on X , the resulting Kapranov dg-manifold $A[1] \oplus N_M$ is linearizable.*

4.4. Kapranov dg-manifolds stemming from Lie pairs. In this section, we will describe the main results of the paper. More precisely, given a Lie algebroid pair (L, A) , we construct a Kapranov dg-manifold on $A[1] \oplus L/A$ canonical up to isomorphism. To get a rough idea behind, let us for the moment that consider real Lie algebroid pair (L, A) , i.e. $\mathbb{K} = \mathbb{R}$. In this case, Lie's fundamental theorems assert the existence of a pair $(\mathcal{L}, \mathcal{A})$ of local Lie groupoids, which the Lie functor transforms into L and A respectively. The Lie groupoid \mathcal{A} acts from the right on the groupoid \mathcal{L} and the resulting homogeneous space \mathcal{L}/\mathcal{A} obviously contains the unit space M of $(\mathcal{L}, \mathcal{A})$ as a distinguished embedded submanifold [23]. Now the Lie groupoid \mathcal{A} acts from the left on \mathcal{L}/\mathcal{A} . Therefore we obtain an infinitesimal A -action on \mathcal{L}/\mathcal{A} , under which M is stable. Thus we may apply Corollary 4.18. Note furthermore that the normal bundle N_M in this situation is naturally isomorphic to L/A . Therefore we obtain a Kapranov dg-manifold on $A[1] \oplus L/A$ provided we choose a tubular neighborhood identification of M . The later can be achieved exactly by using the exponential map developed in Section 3. This is essentially the underlying geometry behind our construction. However, for a general Lie algebroid pair (L, A) over a field \mathbb{K} of characteristic zero, the Lie groupoid picture unfortunately is not valid any more. Nevertheless, we can still obtain the same result by working on the algebraic counterparts by thinking of them as ‘‘formal geometric objects’’ in a certain sense.

Consider the short exact sequence of vector bundles

$$0 \longrightarrow A \xrightarrow{i} L \xrightarrow{q} L/A \longrightarrow 0 \quad (44)$$

and choose an L -connection ∇ on L/A extending the Bott A -connection on L/A (see Example 2.1). Its *torsion* is the bundle map $T^\nabla : \wedge^2 L \rightarrow L/A$ defined by

$$T^\nabla(l_1, l_2) = \nabla_{l_1} q(l_2) - \nabla_{l_2} q(l_1) - q([l_1, l_2]), \quad \forall l_1, l_2 \in \Gamma(L).$$

Since $T^\nabla(a, l) = 0$ if $a \in \Gamma(A)$ and $l \in \Gamma(L)$, there exists a unique bundle map $\beta : \wedge^2(L/A) \rightarrow L/A$ making the diagram

$$\begin{array}{ccc} \wedge^2 L & \xrightarrow{T^\nabla} & L/A \\ q \downarrow & \nearrow \beta & \\ \wedge^2(L/A) & & \end{array}$$

commute.

Lemma 4.20. *Given an L -connection ∇ on L/A , setting*

$$\nabla'_{l_1} q(l_2) = \nabla_{l_1} q(l_2) - \frac{1}{2} T^\nabla(l_1, l_2), \quad \forall l_1, l_2 \in \Gamma(L)$$

defines a torsion-free L -connection ∇' on L/A .

The connection ∇ extends by derivation to an L -connection on $S(L/A)$, which we also denote by ∇ by abuse of notation. Its *curvature* is the bundle map $R^\nabla : \wedge^2 L \rightarrow \text{Der}(S(L/A))$ defined by

$$R^\nabla(l_1, l_2) = [\nabla_{l_1}, \nabla_{l_2}] - \nabla_{[l_1, l_2]}, \quad \forall l_1, l_2 \in \Gamma(L).$$

The sections of $\text{Der}(S(L/A))$ are the derivations of the R -algebra $\Gamma(S(L/A))$.

Now choose a splitting $j : L/A \rightarrow L$ of the short exact sequence (44) and consider the bundle map $\mathcal{Z}^\nabla : A \otimes S^2(L/A) \rightarrow L/A$ defined by

$$\mathcal{Z}^\nabla(a; b_1 \odot b_2) = \frac{1}{2} \{ R^\nabla(a, j(b_1))b_2 + R^\nabla(a, j(b_2))b_1 \}$$

for all $a \in \Gamma(A)$ and $b_1, b_2 \in \Gamma(L/A)$. In particular, we have

$$\mathcal{Z}^\nabla(a; b^2) = R^\nabla(a, j(b))b, \quad \forall a \in \Gamma(A), b \in \Gamma(L/A).$$

Since the Bott A -connection on L/A is flat, \mathcal{Z}^∇ does not actually depend on the chosen splitting.

Lemma 4.21. (1) *The element $\mathcal{Z}^\nabla \in \Gamma(A^\vee \otimes S^2(L/A)^\vee \otimes L/A)$ is a Chevalley-Eilenberg 1-cocycle with respect to the representation of A on $S^2(L/A)^\vee \otimes L/A$ induced by the Bott connection.*

(2) *Given two L -connections $\check{\nabla}$ and $\check{\nabla}'$ extending the Bott representation of A on L/A , the difference $\mathcal{Z}^{\check{\nabla}} - \mathcal{Z}^{\check{\nabla}'}$ is a Chevalley-Eilenberg 1-coboundary with respect to the representation of A on $S^2(L/A)^\vee \otimes L/A$ induced by the Bott connection.*

The cohomology class $[\mathcal{Z}^\nabla] \in H_{\text{CE}}^1(A; S^2(L/A)^\vee \otimes L/A)$, which is independent of the chosen connection ∇ , is called the *Atiyah class* of the Lie pair (L, A) and is denoted $\alpha_{L/A}$.

Remark 4.22. *The definition of the Atiyah class given above agrees with the definition of [9] even though the Atiyah cocycle \mathcal{Z}^∇ defined here is slightly different from (but cohomologous to) the Atiyah cocycle defined in [9]. Indeed, a straightforward computation (involving the Jacobi identity in the Lie algebra $\Gamma(L)$) yields*

$$(\nabla_a \beta)(b_1, b_2) = R^\nabla(a, j(b_1))b_2 - R^\nabla(a, j(b_2))b_1,$$

which implies that

$$\mathcal{Z}^\nabla(a; b_1 \odot b_2) = R^\nabla(a, j(b_1))b_2 - \frac{1}{2}(\nabla_a \beta)(b_1, b_2)$$

and

$$\mathcal{Z}^{\nabla'}(a; b_1 \odot b_2) = R^{\nabla'}(a, j(b_1))b_2,$$

where ∇' stands for the torsion-free connection obtained from ∇ as in Lemma 4.20.

Theorem 4.23. *Let (L, A) be a Lie pair. Given a splitting $j : L/A \rightarrow L$ of the short exact sequence of vector bundles $0 \rightarrow A \rightarrow L \rightarrow L/A \rightarrow 0$ and an L -connection ∇ on L/A extending the Bott representation of A on L/A , there exists a Kapranov dg-manifold structure on $A[1] \oplus L/A$ whose homological vector field $D \in \mathfrak{X}(A[1] \oplus L/A)$ is a derivation*

$$\Gamma(\wedge^\bullet A^\vee \otimes \hat{S}((L/A)^\vee)) \xrightarrow{D} \Gamma(\wedge^{\bullet+1} A^\vee \otimes \hat{S}((L/A)^\vee))$$

of degree +1 of the algebra of functions on $A[1] \oplus L/A$ of the form

$$D = d_A^{\nabla \text{Bott}} + \sum_{k=2}^{\infty} \mathcal{R}_k,$$

where

- (1) $d_A^{\nabla \text{Bott}}$ denotes the Chevalley-Eilenberg differential associated to the Bott representation of the Lie algebroid A on the vector bundle L/A ;
- (2) $\mathcal{R}_2 \in \Gamma(A^\vee \otimes S^2(L/A)^\vee \otimes L/A)$ is the Atiyah 1-cocycle \mathcal{Z}^∇ associated to the connection ∇ ;
- (3) the terms $\mathcal{R}_k \in \Gamma(A^\vee \otimes S^k(L/A)^\vee \otimes L/A)$ with $k \geq 3$ are algebraic functions of \mathcal{R}_2 , α , β , R^∇ , and their higher order covariant derivatives.

Moreover, the various Kapranov dg-manifold structures on $A[1] \oplus L/A$ resulting from all possible choices of splitting and connection are canonically isomorphic.

We have already observed that the Bott representation of A on L/A extends to the symmetric algebra $S(L/A)$, resulting in an infinitesimal A -action by coderivations on the R -coalgebra $\Gamma(S(L/A))$.

Lemma 4.24. *Multiplication from the left by elements of $\Gamma(A)$ defines an infinitesimal A -action by coderivations on the R -coalgebra $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$, which preserves the filtration (10).*

Proof. It is clear that the multiplication from the left by elements $a \in \Gamma(A)$ defines an infinitesimal A -action on the left R -module $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ by coderivations. For all $a \in \Gamma(A)$ and $l_1, l_2, \dots, l_n \in \Gamma(L)$, we have

$$al_1 \cdots l_n = \sum_{i=1}^n (l_1 \cdots l_{i-1} [a, l_i] l_{i+1} \cdots l_n) + l_1 \cdots l_n a,$$

where every term of the right hand side belongs to $\mathcal{U}^{\leq n}(L)$ except for the last which belongs to $\mathcal{U}(L)\Gamma(A)$. \square

Theorem 4.25. *Let (L, A) be a Lie pair. The following assertions are equivalent.*

- (1) *The Kapranov dg-manifold stemming from the Lie pair (L, A) (see Theorem 4.23) is linearizable.*
- (2) *The Atiyah class $\alpha_{L/A} \in H_{\text{CE}}^1(A; S^2(L/A)^\vee \otimes L/A)$ of the Lie pair (L, A) vanishes.*
- (3) *For every splitting $j : L/A \rightarrow L$, there exists an extension of the Bott A -connection on L/A to an L -connection ∇ on L/A such that the associated Poincaré-Birkhoff-Witt map $\text{pbw}^{\nabla, j}$ intertwines the infinitesimal A -actions on $\Gamma(S(L/A))$ and $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$.*

In the case of Lie algebroids over \mathbb{R} , the following two assertions may be added to the list above.

- (4) *For every splitting $j : L/A \rightarrow L$, there exists an extension of the Bott A -connection on L/A to an L -connection ∇ on L/A such that the fiberwise ∞ -jet of the associated exponential map $\exp^{\nabla, j}$ along the zero section of $L/A \rightarrow M$ intertwines the infinitesimal actions of A on L/A and \mathcal{L}/\mathcal{A} .*

- (5) *There exists a fiber-preserving diffeomorphism ϕ from a neighborhood of the zero section in L/A to a neighborhood of the identity section of \mathcal{L}/\mathcal{A} , which fixes the submanifold M , and whose fiberwise ∞ -jet along M intertwines the infinitesimal actions of A on L/A and \mathcal{L}/\mathcal{A} .*

4.5. Proofs of Theorems 4.23 and 4.25. Let (L, A) be a Lie pair. Given a splitting

$$0 \longrightarrow A \begin{array}{c} \xrightarrow{i} \\ \xleftarrow{p} \end{array} L \begin{array}{c} \xrightarrow{q} \\ \xleftarrow{j} \end{array} L/A \longrightarrow 0$$

of the short exact sequence (44), i.e. a pair of bundle maps $j : L/A \rightarrow L$ and $p : L \rightarrow A$ satisfying $p \circ i = \text{id}_A$, $q \circ j = \text{id}_{L/A}$, and $i \circ p + j \circ q = \text{id}_L$, we can define a bundle map $\alpha : \wedge^2(L/A) \rightarrow A$ by setting

$$\alpha(b_1, b_2) = p([j(b_1), j(b_2)]), \quad \forall b_1, b_2 \in \Gamma(L/A).$$

Every choice of a splitting $j : L/A \rightarrow L$ of the short exact sequence (44) and an L -connection ∇ on L/A extending the Bott representation of A on L/A determines a \mathbb{K} -bilinear map

$$\Delta : \Gamma(L/A) \times \Gamma(L) \rightarrow \Gamma(L)$$

defined by

$$\Delta_b l = j(\nabla_l b) - [l, j(b)], \quad \forall b \in \Gamma(L/A), l \in \Gamma(L).$$

Lemma 4.26. *The map Δ is R -linear in its first argument and satisfies*

$$\Delta_b(j(c)) = \alpha(b, c) + j\{\nabla_{j(b)}c - \beta(b, c)\}, \quad \forall b, c \in \Gamma(L/A). \quad (45)$$

Likewise, every choice of a splitting $j : L/A \rightarrow L$ of the short exact sequence of vector bundles (44) and an L -connection ∇ on L/A extending the Bott representation of A on L/A determines a Poincaré-Birkhoff-Witt map

$$\text{pbw}^{\nabla, j} : \Gamma(S(L/A)) \rightarrow \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)},$$

which is an isomorphism of filtered R -coalgebras according to Theorem 2.2. In what follows, we will use the simplified symbol pbw to denote the Poincaré-Birkhoff-Witt isomorphism $\text{pbw}^{\nabla, j}$.

Being a quotient of the universal enveloping algebra $\mathcal{U}(L)$, the R -coalgebra $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ is naturally a left $\mathcal{U}(L)$ -module. In other words, $\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ is endowed with a canonical infinitesimal L -action. Pulling back this infinitesimal action through pbw , we obtain an infinitesimal L -action δ on $\Gamma(S(L/A))$ (by coderivations):

$$\delta_l(s) = \text{pbw}^{-1}(l \cdot \text{pbw}(s)), \quad (46)$$

for all $l \in \Gamma(L)$ and $s \in \Gamma(S(L/A))$.

At this stage, it is useful to introduce the bundle map

$$\mathbf{P} : L \otimes S(L/A) \rightarrow S(L/A)$$

defined by

$$\mathbf{P}(l; s) = q(l) \odot s + \nabla_l s - \delta_l s, \quad \forall l \in \Gamma(L), s \in \Gamma(S(L/A)).$$

Lemma 4.27. *For all $l \in \Gamma(L)$, the map $s \mapsto \mathbf{P}(l; s)$ is a coderivation of the R -coalgebra $\Gamma(S(L/A))$ which preserves the filtration*

$$\dots \hookrightarrow S^{\leq n-1}(L/A) \hookrightarrow S^{\leq n}(L/A) \hookrightarrow S^{\leq n+1}(L/A) \hookrightarrow \dots \quad (47)$$

Sketch of proof. Using (15) and reasoning by induction on n , it is easy to check that

$$\text{pbw}(q(l) \odot b^n + \nabla_l(b^n)) - l \cdot \text{pbw}(b^n) \in \left(\frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)} \right)^{\leq n}$$

for all $l \in \Gamma(L)$, $b \in \Gamma(L/A)$, and $n \in \mathbb{N}$. \square

For all $l \in \Gamma(L)$ and $s \in \Gamma(S(L/A))$, we have

$$l \cdot \text{pbw}(s) = \text{pbw}(\nabla_l s + q(l) \odot s - \mathbf{P}(l; s)). \quad (48)$$

Lemma 4.28. *For all $b, c \in \Gamma(L/A)$ and $n \in \mathbb{N}$, we have*

$$\mathbf{P}(j(c); b^n) + n\mathbf{P}(j(b); c \odot b^{n-1}) = 0. \quad (49)$$

Sketch of proof. In (15), take $b_0 = c$ and $b_1 = b_2 = \dots = b_n = b$. Then make use of (48) to write each term as $\text{pbw}(\dots)$. \square

In particular, taking $c = \nabla_l b$ in (49), we obtain

$$\mathbf{P}(j(\nabla_l b); b^n) + \mathbf{P}(j(b); \nabla_l(b^n)) = 0, \quad \forall b, c \in \Gamma(L/A), n \in \mathbb{N}. \quad (50)$$

Proposition 4.29. *For all $l \in \Gamma(L)$, $b \in \Gamma(L/A)$, and $n \in \mathbb{N}$, we have*

$$\begin{aligned} & \mathbf{P}(l; b^{n+1}) - \mathbf{P}(j(b); q(l) \odot b^n) \\ &= \beta(q(l), b) \odot b^n + R^\nabla(l, j(b))(b^n) + b \odot \mathbf{P}(l; b^n) - \mathbf{P}(j(b); \mathbf{P}(l; b^n)) \\ & \quad + \nabla_{j(b)}(\mathbf{P}(l; b^n)) - \mathbf{P}(\Delta_b l; b^n) - \mathbf{P}(l; \nabla_{j(b)}(b^n)). \end{aligned} \quad (51)$$

Sketch of proof. Multiply (14) from the left by l . Then rewrite each term as $\text{pbw}(\dots)$ by making use of (48) repeatedly if necessary. Finally, simplify using the definitions of β , R^∇ , and Δ . \square

Consider the bundle maps

$$\mathbf{R} : A \otimes S(L/A) \rightarrow S(L/A) \quad \text{and} \quad \mathbf{H} : L/A \otimes S(L/A) \rightarrow S(L/A)$$

defined by

$$\begin{aligned} \mathbf{R}(a; s) &= \mathbf{P}(i(a); s), \quad \forall a \in \Gamma(A), s \in \Gamma(S(L/A)); \\ \mathbf{H}(b; s) &= \mathbf{P}(j(b); s), \quad \forall b \in \Gamma(L/A), s \in \Gamma(S(L/A)). \end{aligned}$$

It follows from Lemma 4.27 that, for all $a \in \Gamma(A)$ and $b \in \Gamma(L/A)$, the maps $\mathbf{R}(a; -)$ and $\mathbf{H}(b; -)$ are R -linear coderivations of $\Gamma(S(L/A))$ which preserve the filtration (47).

Proposition 4.30. *For all $l \in \Gamma(L)$ and $s \in \Gamma(S(L/A))$, we have*

$$\mathbf{P}(l; s) = \mathbf{R}(p(l); s) + \mathbf{H}(q(l); s). \quad (52)$$

For all $a \in \Gamma(A)$, $b \in \Gamma(L/A)$, and $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathbf{R}(a; b^{n+1}) &= R^\nabla(a, j(b))(b^n) + b \odot \mathbf{R}(a; b^n) - \mathbf{H}(b; \mathbf{R}(a; b^n)) \\ &\quad + \nabla_{j(b)}(\mathbf{R}(a; b^n)) - \mathbf{R}(\Delta_b a; b^n) - \mathbf{R}(a; \nabla_{j(b)}(b^n)). \end{aligned} \quad (53)$$

For all $b, c \in \Gamma(L/A)$ and $n \in \mathbb{N}$, we have

$$\begin{aligned} \mathbf{H}(c; b^{n+1}) &= \frac{n+1}{n+2} \left\{ R^\nabla(j(c), j(b))(b^n) + b \odot \mathbf{H}(c; b^n) - \mathbf{H}(b; \mathbf{H}(c; b^n)) \right. \\ &\quad + \nabla_{j(b)}(\mathbf{H}(c; b^n)) - \mathbf{H}(\nabla_{j(b)}c; b^n) - \mathbf{H}(c; \nabla_{j(b)}(b^n)) \\ &\quad \left. + \beta(c, b) \odot b^n - \mathbf{R}(\alpha(b, c); b^n) + \mathbf{H}(\beta(b, c); b^n) \right\}. \end{aligned} \quad (54)$$

Sketch of proof. Equation (52) is a trivial consequence of $i \circ p + j \circ q = \text{id}_L$. Taking $l = i(a)$ in (51) yields (53). Finally, to establish (54), take $l = j(c)$ in (51), use (49) to rewrite the l.h.s. as a single term, and use (45) to eliminate Δ from the r.h.s. \square

Lemma 4.31. *For all $l \in \Gamma(L)$, we have $\mathbf{P}(l; 1) = 0$.*

Proof. Since $\text{pbw}(1) = 1$ and $\text{pbw}(b) = j(b)$ for all $b \in \Gamma(L/A)$, straightforward computation gives $\mathbf{R}(a; 1) = 0$ and $\mathbf{H}(c; 1) = 0$ for all $a \in \Gamma(A)$ and $c \in \Gamma(L/A)$. The result now follows from (52). \square

Remark 4.32. *Since the coderivations \mathbf{R} and \mathbf{H} preserve the filtration (47) of $S(L/A)$, Proposition 4.30 and Lemma 4.31 allow us to compute all expressions of the form $\mathbf{P}(l; b^n)$ by induction on n . Indeed, we have devised an algebraic algorithm for computing \mathbf{P} from the connection ∇ , its torsion β , its curvature R^∇ , the splitting j , and the associated bundle map α .*

We note that if $j(L/A)$ is a Lie subalgebroid of L (i.e. $\alpha = 0$), \mathbf{H} may be computed inductively using (54) independently of \mathbf{R} .

We leave the proofs of the next three corollaries to the reader, who will have noted that

$$R^\nabla(a, j(b))(b^n) = nb^{n-1} \odot R^\nabla(a, j(b))b.$$

Corollary 4.33. *For all $a \in \Gamma(A)$ and $b, c \in \Gamma(L/A)$, we have*

$$\mathbf{R}(a; b) = 0$$

$$\mathbf{H}(c; b) = \frac{1}{2}\beta(c, b)$$

$$\mathbf{R}(a; b^2) = R^\nabla(a, j(b))b = \mathcal{Z}^\nabla(a; b^2)$$

$$\mathbf{H}(c; b^2) = \beta(c, b) \odot b + \frac{1}{2}\beta(\beta(c, b), b) + \frac{1}{3}(\nabla_{j(b)}\beta)(c, b) + \frac{2}{3}R^\nabla(j(b), j(c))b.$$

Corollary 4.34. *If $\alpha = 0$, $\beta = 0$, and $R^\nabla(j(c), j(b)) = 0$ for all $b, c \in \Gamma(L/A)$, then $\mathbf{H} = 0$ and*

$$\begin{aligned} \mathbf{R}(a; b^{n+1}) &= nb^{n-1} \odot \mathcal{Z}^\nabla(a; b^2) + b \odot \mathbf{R}(a; b^n) \\ &\quad + \nabla_{j(b)}(\mathbf{R}(a; b^n)) - \mathbf{R}(\Delta_b a; b^n) - \mathbf{R}(a; \nabla_{j(b)}(b^n)), \end{aligned}$$

for all $a \in \Gamma(A)$, $b \in \Gamma(L/A)$, and $n \in \mathbb{N}$.

Corollary 4.35. *The following assertions are equivalent:*

- (1) $\mathcal{Z}^\nabla = 0$;
- (2) $\mathbf{R} = 0$;
- (3) $a \cdot \text{pbw}(s) = \text{pbw}(\nabla_a s)$ for all $a \in \Gamma(A)$ and $s \in \Gamma(S(L/A))$;
- (4) $\text{pbw} : \Gamma(S(L/A)) \rightarrow \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$ is a morphism of left $\mathcal{U}(A)$ -modules.

We now return to the central issue of Theorem 4.23: how to make the graded manifold $A[1] \oplus L/A$ a Kapranov dg-manifold.

Proposition 4.36. (1) *Every isomorphism of filtered R -coalgebras*

$$\psi : \Gamma(S(L/A)) \rightarrow \frac{\mathcal{U}(L)}{\mathcal{U}(L)\Gamma(A)}$$

gives rise to a homological vector field D_ψ on $A[1] \oplus L/A$ making it a Kapranov dg-manifold with the Chevalley-Eilenberg differential d_A as homological vector field on the dg-submanifold $A[1]$.

- (2) *Given any two such isomorphisms of filtered R -coalgebras ϕ and ψ , there exists a canonical isomorphism of Kapranov dg-manifolds between $(A[1] \oplus L/A, D_\phi)$ and $(A[1] \oplus L/A, D_\psi)$ fixing the dg-submanifold $(A[1], d_A)$.*

Proof. (i) Since ψ respects the filtrations, it follows from Lemma 4.24 that the map $\delta : \Gamma(A) \times \Gamma(S(L/A)) \rightarrow \Gamma(S(L/A))$ defined by

$$\delta_a(s) = \psi^{-1}(a \cdot \psi(s)), \quad \forall a \in \Gamma(A), s \in \Gamma(S(L/A)) \quad (55)$$

is an infinitesimal A -action by coderivations on $\Gamma(S(L/A))$, which preserves the filtration. Hence, we have $\delta_a(1) = 0$ for all $a \in \Gamma(A)$ according to Theorem 5.3. The conclusion follows from Theorem 4.10.

(ii) For every $a \in \Gamma(A)$, the automorphism $\phi^{-1} \circ \psi$ of the filtered R -coalgebra $\Gamma(S(L/A))$ intertwines the coderivations $\phi^{-1}(a \cdot \phi(-))$ and $\psi^{-1}(a \cdot \psi(-))$. Therefore, by Lemma 4.12, there exists a diffeomorphism of the graded manifold $A[1] \oplus L/A$ that fixes the submanifold $A[1]$ and maps the homological vector field D_ϕ onto D_ψ . \square

Forearmed with Proposition 4.36, we can now prove Theorem 4.23.

Proof of Theorem 4.23. Taking $\psi = \text{pbw}^{\nabla, j}$ in Proposition 4.36, we obtain the Kapranov dg-manifold $(A[1] \oplus L/A, D_{\text{pbw}^{\nabla, j}})$ associated to the infinitesimal A -action by coderivations

$$\delta_a(s) = \text{pbw}^{-1}(a \cdot \text{pbw}(s))$$

on $\Gamma(S(L/A))$.

From Corollary 4.33 and the definitions of \mathbf{P} and \mathbf{R} , it follows that

$$\begin{aligned} -\text{pr}_1(\delta_a b) &= \text{pr}_1(\mathbf{R}(a; b) - \nabla_a b) = -\nabla_a^{\text{Bott}} b \\ -\text{pr}_1(\delta_a(b^2)) &= \text{pr}_1(\mathbf{R}(a; b^2) - \nabla_a(b^2)) = \mathcal{Z}^\nabla(a; b^2) \end{aligned}$$

for all $a \in \Gamma(A)$ and $b \in \Gamma(L/A)$.

Therefore, according to Theorem 4.10, we have

$$D_{\text{pbw}^{\nabla, j}} = d_A^{-\nabla^{\text{Bott}}} + \sum_{k \geq 2} \mathcal{R}_k,$$

where $\mathcal{R}_2 = \mathcal{Z}^\nabla$ and the bundles maps $\mathcal{R}_k : A \otimes S^k(L/A) \rightarrow L/A$ with $k \geq 3$ satisfy

$$\mathcal{R}_k(a; b^k) = -\text{pr}_1(\delta_a(b^k)) = \text{pr}_1(\mathbf{R}(a; b^k) - \nabla_a(b^k)) = \text{pr}_1(\mathbf{R}(a; b^k)) \quad (56)$$

for all $a \in \Gamma(A)$ and $b \in \Gamma(L/A)$.

Remark 4.32 suffices to explain why each \mathcal{R}_k is an algebraic function of \mathcal{R}_2 , α , β , R^∇ , and their higher order covariant derivatives.

Finally, given any other choice of connection $\check{\nabla}$ and splitting \check{j} , Proposition 4.36 warrants the existence of a canonical isomorphism of Kapranov dg-manifolds between $(A[1] \oplus L/A, D_{\text{pbw}^\nabla, j})$ and $(A[1] \oplus L/A, D_{\text{pbw}^{\check{\nabla}}, \check{j}})$ fixing the dg-submanifold $(A[1], d_A)$. \square

Proof of Theorem 4.25. It follows from Corollary 4.14 and Theorem 4.23 that (i) implies (ii). If the Atiyah class $\alpha_{L/A}$ vanishes, there exists an L -connection ∇ on L/A extending the Bott representation such that the corresponding 1-cocycle \mathcal{Z}^∇ vanishes. So the equivalence between (ii) and (iii) essentially follows from Corollary 4.35. The fact that (iii) implies (i) is a consequence of applying Proposition 4.36 to the isomorphism pbw. The equivalence between (iii) and (iv) follows from Theorem 3.16. It is obvious that (iv) implies (v). Finally, the fact that (v) implies (i) follows from Corollary 4.19. This concludes the proof. \square

Proposition 4.37. *Given a Lie pair (L, A) , suppose there exists a Lie subalgebroid B of L such that $L = A \oplus B$ and a torsion-free flat B -connection ∇ on B . Then there exists a Kapranov dg-manifold structure on $A[1] \oplus B$ with homological vector field $D = d_A^{\nabla^{\text{Bott}}} + \sum_{k \geq 2} \mathcal{R}_k$ where $\mathcal{R}_2 = \mathcal{Z}^{\nabla^{\text{Bott}} + \nabla}$ and $\mathcal{R}_{n+1} = d_B^\nabla \mathcal{R}_n$ for $n \geq 2$.*

Proof. The obvious identification of L/A with B is a splitting of the short exact sequence $0 \rightarrow A \rightarrow L \rightarrow L/A \rightarrow 0$ and the sum $\nabla^{\text{Bott}} + \nabla$ is an L -connection on L/A extending the Bott representation of A on L/A . Therefore, we may apply Theorem 4.23 to obtain a Kapranov dg-manifold structure on $A[1] \oplus L/A$. The conclusion follows from Corollary 4.34 and Eq. (56). \square

4.6. Examples from complex geometry. A complex manifold X provides a Lie pair $(L = T_X \otimes \mathbb{C}, A = T_X^{0,1})$. In this case, $L/A \cong T_X^{1,0}$. As a consequence of Theorems 4.23 and 4.25, we have

Theorem 4.38. *Let X be a complex manifold.*

- (1) *Every $T_X^{1,0}$ -connection $\nabla^{1,0}$ on $T_X^{1,0}$ determines a Kapranov dg-manifold structure on $T_X^{0,1}[1] \oplus T_X^{1,0}$ with homological vector field*

$$D = \bar{\partial} + \sum_{k \geq 2} \mathcal{R}_k,$$

where $\mathcal{R}_2 = \mathcal{Z}^{\nabla^{\text{Bott}} + \nabla^{1,0}}$ is the Atiyah cocycle associated to the $T_X \otimes \mathbb{C}$ -connection $\nabla^{\text{Bott}} + \nabla^{1,0}$ on $T_X^{1,0}$, and all

$$\mathcal{R}_k \in \Omega^{0,1}(S^k(T_X^{1,0})^\vee \otimes (T_X^{1,0})^\vee)$$

with $k \geq 3$ are algebraic functions of \mathcal{R}_2 , the curvature of $\nabla^{1,0}$ and their higher covariant derivatives.

- (2) The Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ is thus an $L_\infty[1]$ algebra. For $n \geq 2$, the n -th multibracket λ_n is the composition of the wedge product

$$\Omega^{0,j_1}(T_X^{1,0}) \otimes \cdots \otimes \Omega^{0,j_n}(T_X^{1,0}) \rightarrow \Omega^{0,j_1+\cdots+j_n}(\otimes^n T_X^{1,0})$$

with the map

$$\Omega^{0,j_1+\cdots+j_n}(\otimes^n T_X^{1,0}) \rightarrow \Omega^{0,j_1+\cdots+j_n+1}(T_X^{1,0})$$

induced by $\mathcal{R}_n \in \Omega^{0,1}(\text{Hom}(\otimes^n T_X^{1,0}, T_X^{1,0}))$, while λ_1 is the Dolbeault operator $\bar{\partial} : \Omega^{0,j}(T_X^{1,0}) \rightarrow \Omega^{0,j+1}(T_X^{1,0})$.

- (3) For different choices of connections $\nabla^{1,0}$, the resulting Kapranov dg-manifold structures on $T_X^{0,1}[1] \oplus T_X^{1,0}$, and thus the corresponding $L_\infty[1]$ algebra structure on the Dolbeault complex $\Omega^{0,\bullet}(T_X^{1,0})$ are canonically isomorphic.
- (4) This Kapranov dg-manifold is linearizable if the Atiyah class of the holomorphic tangent bundle T_X vanishes.

When X is a Kähler manifold, the Levi-Civita connection ∇^{LC} induces a $T_X^{1,0}$ -connection $\nabla^{1,0}$ on $T_X^{1,0}$ as follows. First, extend the Levi-Civita connection \mathbb{C} -linearly to a $T_X \otimes \mathbb{C}$ -connection $\nabla^{\mathbb{C}}$ on $T_X \otimes \mathbb{C}$. Since X is Kähler, $\nabla^{\text{LC}}J = 0$ and $\nabla^{\mathbb{C}}$ restricts to a $T_X \otimes \mathbb{C}$ -connection on $T_X^{1,0}$. It is easy to check that the induced $T_X^{0,1}$ -connection on $T_X^{1,0}$ is the canonical infinitesimal $T_X^{0,1}$ -action on $T_X^{1,0}$ — a section of $T_X^{1,0}$ is $T_X^{0,1}$ -horizontal iff it is holomorphic — while the induced $T_X^{1,0}$ -connection $\nabla^{1,0}$ on $T_X^{1,0}$ is flat and torsion free. Thus according to Proposition 4.37, we have a simpler formula for \mathcal{R}_n :

$$\mathcal{R}_{n+1} = d^{\nabla^{1,0}} \mathcal{R}_n, \quad \forall n \geq 2.$$

Thus we recover a classical theorem of Kapranov [16].

5. APPENDIX: CODERIVATIONS OF THE SYMMETRIC COALGEBRA

Given a module V over a commutative ring R with identity 1, deconcatenation (see Equation (11)) defines a comultiplication Δ on $S(V)$. For every $v \in V$ and every non-negative integer n , we have

$$\Delta(v^n) = \sum_{k=0}^n \binom{n}{k} v^k \otimes v^{n-k},$$

where v^k denotes the symmetric product of k copies of v while v^0 denotes the unit element 1 of the ring R . Thus $(S(V), \Delta)$ is a coalgebra.

The following result is immediate.

Lemma 5.1. $\ker(\Delta - (1 \otimes \text{id} + \text{id} \otimes 1)) = S^1(V)$

Lemma 5.2. If $\delta \in \text{End}(S(V))$ is a coderivation of the symmetric coalgebra $(S(V), \Delta)$, then $\delta(1) \in S^1(V)$.

Proof. We have $\delta(1) = \varepsilon + v + H$ for some $\varepsilon \in R$, $v \in V$, and $H \in S^{\geq 2}(V)$. Since δ is a coderivation, we have

$$\begin{aligned} \Delta \circ \delta(1) &= (\delta \otimes \text{id} + \text{id} \otimes \delta) \circ \Delta(1) \\ \Delta(\varepsilon + v + H) &= (\delta \otimes \text{id} + \text{id} \otimes \delta)(1 \otimes 1). \end{aligned}$$

Since $\Delta(v) = v \otimes 1 + 1 \otimes v$ and $\Delta(1) = 1 \otimes 1$, it follows that $\varepsilon \cdot 1 \otimes 1 + \Delta(H) = \varepsilon \cdot 1 \otimes 1 + H \otimes 1 + \varepsilon \cdot 1 \otimes 1 + 1 \otimes H$, which implies that

$$\Delta(H) - (1 \otimes H + H \otimes 1) = \varepsilon \cdot 1 \otimes 1.$$

In this last equation, the right hand side belongs to $S^0(V) \otimes S^0(V)$ while the left hand side belongs to $S^{\geq 1}(V) \otimes S^{\geq 1}(V)$. Therefore, we obtain the system of equations

$$\begin{cases} \Delta(H) - (1 \otimes H + H \otimes 1) = 0 \\ \varepsilon \cdot 1 \otimes 1 = 0 \end{cases}.$$

By Lemma 5.1, the only solution is $\varepsilon = 0$, $H = 0$. Hence $\delta(1) = v \in S^1(V)$. \square

We are now ready to state the main result.

Theorem 5.3. *A coderivation δ of the symmetric coalgebra $(S(V), \Delta)$ preserves the increasing filtration*

$$S^{\leq 0}(V) \subset S^{\leq 1}(V) \subset S^{\leq 2}(V) \subset S^{\leq 3}(V) \subset \dots$$

if and only if $\delta(1) = 0$.

Proof. Suppose $\delta(S^{\leq n}(V)) \subset S^{\leq n}(V)$ for all non-negative integers n . Then $\delta(1) \in S^0(V)$ as $1 \in S^0(V)$. Since δ is a coderivation, we must have $\delta(1) \in S^1(V)$ by Lemma 5.2. Therefore $\delta(1) = 0$.

Conversely, assume that $\delta(1) = 0$. Then, obviously, $\delta(S^0(V)) = \{0\} \subset S^0(V)$. Given $v \in V$, we have $\delta(v) = \varepsilon + w + H$ for some $\varepsilon \in R$, $w \in V$, and $H \in S^{\geq 2}(V)$. Since δ is a coderivation, we have $\Delta \circ \delta(v) = (\delta \otimes \text{id} + \text{id} \otimes \delta) \circ \Delta(v)$, which implies that $\varepsilon \cdot 1 \otimes 1 + \Delta(H) = 1 \otimes \varepsilon + 1 \otimes H + \varepsilon \otimes 1 + H \otimes 1$. Therefore

$$\Delta(H) - (1 \otimes H + H \otimes 1) = \varepsilon \cdot 1 \otimes 1.$$

In this last equation, the right hand side belongs to $S^0(V) \otimes S^0(V)$ while the left hand side belongs to $S^{\geq 1}(V) \otimes S^{\geq 1}(V)$. Therefore, we obtain the system of equations

$$\begin{cases} \Delta(H) - (1 \otimes H + H \otimes 1) = 0 \\ \varepsilon \cdot 1 \otimes 1 = 0 \end{cases}.$$

By Lemma 5.1, the only solution is $\varepsilon = 0$, $H = 0$. Hence $\delta(v) = w \in V$, which shows that $\delta(S^1(V)) \subset S^1(V)$. For $n \geq 2$, we argue by induction. Suppose $\delta(v^k) \in S^{\leq k}(V)$ for $0 \leq k \leq n-1$. Given any $v \in V$, we have $\delta(v^n) = x + y$ for some $x \in S^{\leq n}(V)$ and $y \in S^{> n}(V)$. Since δ is a coderivation, we have

$$\begin{aligned} \Delta \circ \delta(v^n) &= (\delta \otimes \text{id} + \text{id} \otimes \delta) \circ \Delta(v^n) \\ \Delta(x + y) &= (\delta \otimes \text{id} + \text{id} \otimes \delta) \left(1 \otimes v^n + \sum_{k=1}^{n-1} \binom{n}{k} v^k \otimes v^{n-k} + v^n \otimes 1 \right) \end{aligned}$$

which implies that

$$\begin{aligned} \Delta(y) - (1 \otimes y + y \otimes 1) &= 1 \otimes x + x \otimes 1 - \Delta(x) \\ &\quad + \sum_{k=1}^{n-1} \binom{n}{k} (\delta(v^k) \otimes v^{n-k} + v^k \otimes \delta(v^{n-k})). \end{aligned}$$

In this last equation, while the left hand side belongs to $\bigoplus_{p+q>n} (S^p(V) \otimes S^q(V))$, the right hand side belongs to $\bigoplus_{p+q\leq n} (S^p(V) \otimes S^q(V))$ by the induction hypothesis. Therefore, $y \in S^{>n}(V) \cap \ker(\Delta - (1 \otimes \text{id} + \text{id} \otimes 1))$ and it follows from Lemma 5.1 that $y = 0$. Hence $\delta(v^n) = x \in S^{\leq n}(V)$. \square

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