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Concise formulae for the cumulant matrices of a random vector

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ABSTRACT

Concise formulae are given for the cumulant matrices of a real-valued (zero-mean) random vector up to order 6. In addition to usual matrix operations, they involve only the Kronecker product, the vec operator, and the commutation matrix. Orders 5 and 6 are provided here for the first time; the same method as provided in the paper can be applied to compute higher orders. An immediate consequence of these formulae is to return 1) upper bounds on the rank of the cumulant matrices and 2) the expression of the sixth-order moment matrix of a Gaussian vector. Due to their conciseness, the proposed formulae also have a computational advantage as compared to the repeated use of Leonov and Shiryaev formula.

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1. Introduction

Cumulants are useful quantities for characterizing some important statistical properties, such as independence, gaussianity, or possible (a)symmetry of probability distributions. They were first introduced in 1899 by T.N. Thiele under the name *half-invariants* (see Hald [1] where Thiele's original paper is translated into English). Later, in Leonov and Shiryaev [2] (translated into English by James R. Brown), they were called *semi-invariants*, whereas Lukacs already called them *cumulants* in his 1955 paper [3].

Higher-order cumulants make it possible to solve problems that are otherwise intractable on the second order. This is typically the case in Independent Component Analysis (ICA, as introduced in Comon [4]), where fourth-order cumulants are used to blindly separate (over-)determined (i.e. as many as or less sources than sensors) mixtures of independent sources (e.g. Cardoso and Souloumiac [5], Albera et al. [6]). Recently, several researchers have investigated the use of sixth-order cumulants (Albera et al. [7]) and, more generally, cumulants of order $2k$, $k > 2$ (Albera et al. [8]), for the separation of under-determined mixtures of sources (i.e. more sources than sensors).

This paper is concerned with the matrix expressions of the cumulants of a random vector. Loosely speaking, the cumulant matrix of a random vector is a collection of the various cross-cumulants between its elements. It is often convenient to arrange such a collection into a matrix (a square symmetric matrix in the case of even order cumulants) for both notational and practical reasons. This makes it easier to handle cumulants for algorithmic perspectives and to exploit some of their statistical properties, such as "matrix redundancy" (see e.g. Albera et al. [7,6,8]). An important example is ICA where the blind separation of statistically independent sources is solved by zeroing all cross-cumulants between independent sources: if properly set up, the issue then boils down to the diagonalization of a cumulant matrix (see Cardoso and Souloumiac [5], Kollo [9]).

Unlike moments – which are quite easily obtained by expectation of polynomial functions of the given random vector – there is no simple formula to compute the cumulants. Indeed, scalar cumulants are evaluated from lower-order moments by using a well-known polynomial formula (see Lukacs [3,2,10] or Mac Cullagh [11]). One calls it *Leonov and Shiryaev formula*, as is usually the case in ICA papers.

It is then easy to re-arrange the scalar (cross-)cumulants of a vector into a matrix. The main drawback of this usual approach is to involve lengthy formulae and, when calculation is of concern, a computational burden which becomes rapidly prohibitive as the dimension of the random vector increases. It is therefore advisable to search for direct and more concise expressions of cumulant matrices. One solution is actually returned by Neudecker matrix derivatives (Neudecker [12]) (see e.g. Kollo and von Rosen [13, pages 187–188]). Unfortunately, it faces two difficulties which limit its practical use:

(i) the resulting cumulant matrices are not square except on the second order;

(ii) matrix derivatives of the logarithm of the characteristic function are hardly tractable for orders higher than four and, according to the authors' knowledge, no formulae have ever been derived for those cases.

Limitation (i) can be overcome on the fourth order by using an algebraic technique such as proposed by Jinadasa and Tracy [14] and Tracy and Sultan [15] to obtain the matrix expression of the fourth-order moment of a Gaussian vector; the same result is found in Kollo [9], Loperfido [16]. However, point (ii) has not been solved till now. This paper gives the expressions of fifth and sixth-order cumulant matrices of a real-valued random vector. The expression of the sixth-order cumulant matrix is symmetric.

The paper is organized as follows. Section 2 resumes the basic definitions of moments and cumulants of a random vector and deduces Leonov and Shiryaev formula as a consequence of the Faà di Bruno formula (following the lines in Lukacs [3] and Hardy [17]). The main result of the paper – a theorem providing the formulae of cumulant matrices up to order six – is given in Section 3. At the same time a general strategy describes how to obtain similar results up to any order. The number of terms in each formula is also given. However, the cumbersome aspects of the proofs are postponed to the appendix. In Section 4, the formulae are used to get, without pain, i) upper bounds on the rank of cumulant matrices and ii) the sixth-order moment matrix of a Gaussian vector. Since the latter is a known formula (see for instance Magnus and Neudecker [18], Tracy and Sultan [15], and more recently Schott [19]), it somehow validates the expression of the cumulant matrix from which it is derived. Finally, Section 5 concludes on the advantages implied by the proposed matrix expressions as compared to the classical use of Leonov and Shiryaev formula.

2. Generalities on moment and cumulant matrices

This section first recalls the link between the scalar moments and cumulants of a random vector in order to pave the way for our main theorem given in Section 3.

2.1. Scalar moments and cumulants

Let $\mathbf{x} = (x_1, \ldots, x_p)'$ be a real-valued random vector of dimension $p \times 1$. Its characteristic function ϕ is defined on $\mathbb{R}^p$ as

$$\phi(\mathbf{u}) = \mathbb{E}\left[\exp(i(\mathbf{u}, \mathbf{x}))\right] = \mathbb{E}\left[\prod_{j=1}^{p} \exp(iu_j x_j)\right], \tag{1}$$

where $\mathbb{E}[\cdot]$ denotes the mathematical expectation, i is the imaginary unit, $\mathbf{u} = (u_1, \ldots, u_p)' \in \mathbb{R}^p$ and $(\cdot)'$ denotes the transposition operator.

Let $n \in \mathbb{N}$; vector $\mathbf{x}$ is said to admit moments of order n if

$$\mathbb{E}\left[|\mathbf{x}|^n\right] < \infty, \tag{2}$$

where $|\mathbf{x}| = \left(\sum_{j=1}^{p} x_j^2\right)^{1/2}$ stands for the Euclidean norm of $\mathbb{R}^p$. In that case, (i) ϕ is n times differentiable and (ii) all the scalar (cross-)moments of order $k \leq n$ of $\mathbf{x}$ are well defined as

$$\begin{aligned}&(\forall (j_1, \ldots, j_k) \in [\![1, p]\!]^k)\\ &\quad mom^{(k)}(x_{j_1}, \ldots, x_{j_k}) \stackrel{\text{def}}{=} \mathbb{E}\left[x_{j_1} \cdots x_{j_k}\right] = (-i)^k \frac{\partial^k \phi}{\partial u_{j_1} \cdots \partial u_{j_k}}(\mathbf{0}).\end{aligned} \tag{3}$$

Moreover, since ϕ is a continuous function on $\mathbb{R}^p$ and $\phi(\mathbf{0}) = 1$, it is necessarily non-zero in a neighborhood of $\mathbf{0} \in \mathbb{R}^p$. The second characteristic function (also referred to as the "cumulant function" in the literature),

$$\psi(\mathbf{u}) = \log\left(\phi(\mathbf{u})\right), \tag{4}$$

is thus well defined and, thanks to condition (2), it is n times differentiable in a neighborhood of $\mathbf{0}$. This makes possible the definition of scalar (cross-)cumulants of order $k \leq n$ of $\mathbf{x}$ as

$$(\forall (j_1, \ldots, j_k) \in [\![1, p]\!]^k) \qquad cum^{(k)}(x_{j_1}, \ldots, x_{j_k}) \stackrel{\text{def}}{=} (-i)^k \frac{\partial^k \psi}{\partial u_{j_1} \cdots \partial u_{j_k}}(\mathbf{0}). \tag{5}$$

Since Eq. (4) implicitly connects moments to cumulants, it suggests two methods to compute the latter in terms of the former:

1. Expansion up to order n of the following function:

$$\psi(\mathbf{u}) = \log\left(1 + \sum_{k=1}^{n} \frac{(i)^k}{k!} \sum_{(j_1,\ldots,j_k)\in[\![1,p]\!]^k} mom^{(k)}(x_{j_1}, \ldots, x_{j_k}) u_{j_1} \cdots u_{j_k} + o(|\mathbf{u}|^n)\right) \tag{6}$$

   where the sum in brackets is the Taylor series expansion up to order n of $\phi(\mathbf{u})$. This method was carried out for instance by Leonov and Shiryaev [2] and by Mac Cullagh [11]. It is well adapted to hand calculation, at least for low orders (say $n \leq 4$). Moreover this method leads also to general formulae which are nothing but special cases of the Faà di Bruno formula.
2. Direct use of the Faà di Bruno formula which returns the partial derivatives of a composition of applications. This second method was first used by Lukacs [3] in the case of random variables ($p = 1$).

Without surprise, these two methods give essentially the same results. However, this paper focuses on the second one because of its primary interest in higher orders (say $n > 4$). In fact, there are slightly different Faà di Bruno formulae which differ only by the way the terms are collected. One of the simplest formulae is provided by Hardy [17]; with the notations of the present paper, it reads

$$\frac{\partial^k}{\partial u_{j_1} \cdots \partial u_{j_k}} f \circ \phi(\mathbf{u}) = \sum_{\pi \in P([\![1,k]\!])} f^{(|\pi|)} \circ \phi(\mathbf{u}) \prod_{B \in \pi} \frac{\partial^{|B|}}{\prod_{l \in B} \partial u_{j_l}} \phi(\mathbf{u}) \tag{7}$$

where f is a function of one variable, $P([\![1,k]\!])$ denotes the set of partitions of the set $[\![1,k]\!] \stackrel{\text{def}}{=} \{1;2;\ldots;k\}$, and $|E|$ is the cardinality of set E. For example, with $k = 7$, $\pi = \{\{2;3;6\};\{1;4\};\{7\}\} \in P([\![1,7]\!])$ and $B = \{2;3;6\} \in \pi$, one gets

$$\frac{\partial^{|B|}}{\prod_{l \in B} \partial u_{j_l}} = \frac{\partial^3}{\partial u_{j_2} \partial u_{j_3} \partial u_{j_6}}. \tag{8}$$

The main advantage of Eq. (7) is to be fully developed, i.e. it no longer contains factorials assigned to the collection of equal partial derivatives when some indices j_l happen to coincide.

In order to compute cumulants in terms of moments, one sets $f = \log$, $\mathbf{u} = \mathbf{0}$ and $\phi(\mathbf{0}) = 1$ in Eq. (7). Since

$$f^{(n)}(1) = (-1)^{n-1}(n-1)! \tag{9}$$

and, in accordance with Eq. (3),

$$\frac{\partial^{|B|}}{\prod_{l \in B} \partial u_{j_l}} \phi(\mathbf{0}) = i^{|B|} mom^{(|B|)}(x_{j_l} \ : \ l \in B). \tag{10}$$

By noting that for any $\pi \in P([\![1,k]\!])$ it comes $\sum_{B \in \pi} |B| = k$, one eventually arrives at Leonov and Shiryaev formula:

$$cum^{(k)}(x_{j_1}, \ldots, x_{j_k}) = \sum_{\pi \in P([\![1,k]\!])} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{B \in \pi} mom^{(|B|)}(x_{j_l} \ : \ l \in B). \tag{11}$$

Alternatively, in order to compute moments in terms of cumulants, one sets $f = \exp$, ψ in place of Φ, $\mathbf{u} = \mathbf{0}$ and $\psi(\mathbf{0}) = 0$ in Eq. (7), so as to obtain

$$mom^{(k)}(x_{j_1}, \ldots, x_{j_k}) = \sum_{\pi \in P([\![1,k]\!])} \prod_{B \in \pi} cum^{(|B|)}(x_{j_l} \ : \ l \in B). \tag{12}$$

This formula will be used in Section 4.2 to find the expression of the sixth-order moment, $m_6(\mathbf{x})$, of a centered Gaussian random vector $\mathbf{x}$.

2.2. On the number of terms in Leonov and Shiryaev formula

To give an idea of the complexity of Leonov and Shiryaev formula, one counts the number of terms in the sum and product of Eq. (11). First of all, the number of partitions of the set $[\![1,k]\!]$ is known as the **Bell number** B_k,

$$B_k = \left|P([\![1,k]\!])\right|. \tag{13}$$

The Bell numbers can be computed by recurrence (see Cameron [20]),

$$(\forall k \geq 1) \qquad B_k = \sum_{l=1} \binom{k-1}{l-1} B_{k-l}, \tag{14}$$

even if an explicit formula is known (see later).

Second, it is seen that the number of terms in the product $\prod_{B\in\pi}$ of Eq. (11) is the number of parts of the partition π. Consequently it is useful to consider $P_r([\![1,k]\!])$, the set of partitions of the set $[\![1,k]\!]$ with r (non-empty) parts, so that Eq. (11) is rewritten into

$$cum^{(k)}(x_{j_1},\ldots,x_{j_k}) = \sum_{r=1}^{k} \sum_{\pi\in P_r([\![1,k]\!])} (-1)^{r-1}(r-1)! \prod_{B\in\pi} mom^{(|B|)}(x_{j_l} \ : \ l\in B). \tag{15}$$

The number of partitions of the set $[\![1,k]\!]$ with r (non-empty) parts is the (k,r)-th **Stirling number of the second kind** $S(k,r)$ (see Cameron [20]),

$$S(k,r) = \left|P_r([\![1,k]\!])\right|. \tag{16}$$

By noting that there are $r!S(k,r)$ surjective mappings from $[\![1,k]\!]$ to $[\![1,r]\!]$, the following formula is obtained:

$$S(k,r) = \frac{1}{r!}\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}j^k. \tag{17}$$

Since $B_k = \sum_{r=1}^{k} S(k,r)$, this gives an explicit formula for the Bell numbers

$$(\forall k \geq 1) \qquad B_k = \sum_{r=1}^{k}\frac{1}{r!}\sum_{j=1}^{r}(-1)^{r-j}\binom{r}{j}j^k. \tag{18}$$

Finally, the number of multiplications in Leonov and Shiryaev formula (11) is given by

$$N_{mult}(k) = \sum_{r=1}^{k} rS(k,r). \tag{19}$$

Values of $N_{mult}(k)$ from $k=1$ to 6 are given in Table 1.

Table 1
Number of additions and multiplications in Leonov and Shiryaev formula.

k	B_k	$N_{mult}(k)$
1	1	1
2	2	3
3	5	10
4	15	37
5	52	151
6	203	674

2.3. Moment and cumulant matrices

There are many different ways to arrange the p^k scalar moments of Eq. (3) in a vector (see Meijer [21]) or in a matrix. It is natural to select those arrangements which are maximally symmetric as is often practiced in the literature (see e.g. Schott [19], Jinadasa and Tracy [14], Tracy and Sultan [15]).

Definition 1 *(The moment matrix).* The moment matrix of a real-valued random vector $\mathbf{x} = (x_1, \ldots, x_p)'$, $\mathbb{E}\left[|\mathbf{x}|^n\right] < \infty$, $n \in \mathbb{N}$, of even order $2k \leq n$ is defined as

$$m_{2k}(\mathbf{x}) = \mathbb{E}\left[\otimes^k \mathbf{x}\mathbf{x}'\right] \qquad (\text{dimension } p^k \times p^k) \tag{20}$$

and that of odd order, $2k + 1 \leq n$, as

$$m_{2k+1}(\mathbf{x}) = \mathbb{E}\left[\otimes^k \mathbf{x}\mathbf{x}' \otimes \mathbf{x}\right] \qquad (\text{dimension } p^{k+1} \times p^k), \tag{21}$$

where $\otimes$ is the Kronecker product (see Eq. (26) below for a definition) and $\otimes^k \mathbf{x}\mathbf{x}' = \underbrace{\mathbf{x}\mathbf{x}' \otimes \cdots \otimes \mathbf{x}\mathbf{x}'}_{k \text{ matrices } p \times p}$.

It is easy to verify that all scalar (cross-)moments are included in Definition 1. To do so, let decompose $\mathbf{x} = \sum_{j \in [\![1,p]\!]} x_j \mathbf{e}_j$ onto the canonical basis of $\mathbb{R}^p$; thus, from the multilinearity property of the Kronecker product and the linearity of the expected value,

$$\begin{aligned} m_{2k}(\mathbf{x}) &= \mathbb{E}\left[\otimes^k \mathbf{x}\mathbf{x}'\right] = \mathbb{E}\left[\otimes^k \sum_{(j_1,j_2) \in [\![1,p]\!]^2} x_{j_1} x_{j_2} \mathbf{e}_{j_1} \mathbf{e}'_{j_2}\right] \\ &= \mathbb{E}\left[\sum_{(j_1,j_2) \in [\![1,p]\!]^2} x_{j_1} x_{j_2} \mathbf{e}_{j_1} \mathbf{e}'_{j_2} \otimes \cdots \otimes \sum_{(j_{2k-1},j_{2k}) \in [\![1,p]\!]^2} x_{j_{2k-1}} x_{j_{2k}} \mathbf{e}_{j_{2k-1}} \mathbf{e}'_{j_{2k}}\right] \\ &= \sum_{(j_1,j_2,\ldots,j_{2k-1},j_{2k}) \in [\![1,p]\!]^{2k}} \mathbb{E}\left[x_{j_1} x_{j_2} \cdots x_{j_{2k-1}} x_{j_{2k}}\right] \mathbf{e}_{j_1} \mathbf{e}'_{j_2} \otimes \cdots \otimes \mathbf{e}_{j_{2k-1}} \mathbf{e}'_{j_{2k}} \end{aligned}$$

$$= \sum_{(j_1,j_2,\ldots,j_{2k-1},j_{2k})\in[\![1,p]\!]^{2k}} mom^{(2k)}(x_{j_1},\ldots,x_{j_{2k}})\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_{2k-1}}\mathbf{e}'_{j_{2k}}. \tag{22}$$

Similarly, for odd orders,

$$m_{2k+1}(\mathbf{x}) = \sum_{(j_1,\ldots,j_{2k+1})\in[\![1,p]\!]^{2k+1}} mom^{(2k+1)}(x_{j_1},\ldots,x_{j_{2k+1}})\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\cdots \otimes\mathbf{e}_{j_{2k-1}}\mathbf{e}'_{j_{2k}}\otimes\mathbf{e}_{j_{2k+1}}. \tag{23}$$

It is then possible to define the cumulant matrix in a similar way as the moment matrix.

Definition 2 *(The cumulant matrix).* Let $\mathbf{x} = (x_1,\ldots,x_p)'$ be a real-valued random vector of dimension $p\times 1$, such that $\mathbb{E}\left[|\mathbf{x}|^n\right] < \infty$ for a given $n\in\mathbb{N}$. The cumulant matrix of even order, $2k\le n$, of $\mathbf{x}$ is a $p^k\times p^k$ matrix defined as

$$c_{2k}(\mathbf{x}) = \sum_{(j_1,\ldots,j_{2k})\in[\![1,p]\!]^{2k}} cum^{(2k)}(x_{j_1},\ldots,x_{j_{2k}})\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\cdots\otimes\mathbf{e}_{j_{2k-1}}\mathbf{e}'_{j_{2k}} \tag{24}$$

whereas the cumulant matrix of odd order, $2k+1\le n$, is a $p^{k+1}\times p^k$ matrix defined as

$$c_{2k+1}(\mathbf{x}) = \sum_{(j_1,\ldots,j_{2k+1})\in[\![1,p]\!]^{2k+1}} cum^{(2k+1)}(x_{j_1},\ldots,x_{j_{2k+1}})\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\cdots \otimes\mathbf{e}_{j_{2k-1}}\mathbf{e}'_{j_{2k}}\otimes\mathbf{e}_{j_{2k+1}}. \tag{25}$$

Remark 3. The matrix $c_n(\mathbf{x})$ is of course an unfolded version of the n-th cumulant tensor $\mathcal{T}_n = \left\{cum(x_{i_1},\ldots,x_{i_n})\right\}$ (see [22]).

Except for the special case of a Gaussian random vector, cumulants are much more difficult to compute than moments. Hitherto, the only way to compute the cumulant matrix $c_n(\mathbf{x})$ has consisted in

1. deriving the p^n scalar cumulants from the scalar moments of orders lower than n via Eq. (11),
2. arranging the scalar cumulants in a matrix according to Definition 2.

However, Eq. (11) tends to return lengthy expressions that are hardly tractable beyond order four or so (see Table 1). The goal of this paper is to compute the cumulant matrix $c_n(\mathbf{x})$ directly from the moment matrices of orders lower than n, thus providing formulae that are much more concise and convenient to handle.

3. Cumulant matrices from lower-order moment matrices

3.1. Kronecker product and commutation matrices

The main theorem of this paper makes a wide use of the **vec** operator and the Kronecker product. Furthermore, it requires some commutation matrices. The following definitions and properties are borrowed from Brewer [23], Magnus and Neudecker [18] or Wiens [24].

Let $\mathbf{A} = (a_{i,j})$ be a $p \times q$ matrix and $\mathbf{a}_j$ the j-th column of $\mathbf{A}$; then $\mathbf{vec}\,\mathbf{A}$ is the $pq \times 1$ vector

$$\mathbf{vec}\,\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_q \end{pmatrix}.$$

Let further $\mathbf{B}$ be an $r \times s$ matrix; then the Kronecker product $\mathbf{A} \otimes \mathbf{B}$ is defined as the $pr \times qs$ matrix

$$\mathbf{A} \otimes \mathbf{B} = \begin{pmatrix} a_{1,1}\mathbf{B} & \cdots & a_{1,q}\mathbf{B} \\ \vdots & \ddots & \vdots \\ a_{p,1}\mathbf{B} & \cdots & a_{p,q}\mathbf{B} \end{pmatrix}. \tag{26}$$

$\otimes$ is associative, but not commutative. Furthermore, it acts on the usual matrix product according to

$$(\mathbf{A} \otimes \mathbf{B})(\mathbf{C} \otimes \mathbf{D}) = (\mathbf{AC}) \otimes (\mathbf{BD}),$$

provided that $\mathbf{C}$ is $q \times t$ and $\mathbf{D}$ is $s \times u$. Use will also be made of the transposition property

$$(\mathbf{A} \otimes \mathbf{B})' = \mathbf{A}' \otimes \mathbf{B}'.$$

Definition 4 *(Three commutation matrices).* Let $p \geq 1$ be an integer. The commutation matrix $\mathbf{K}_{pp}$ of dimension $p^2 \times p^2$ is defined as

$$\mathbf{K}_{pp} = \sum_{(i,j) \in [\![1,p]\!]^2} \mathbf{e}_i \mathbf{e}_j' \otimes \mathbf{e}_j \mathbf{e}_i', \tag{27}$$

where $(\mathbf{e}_i)_{i \in [\![1,p]\!]}$ stands for the canonical basis of $\mathbb{R}^p$. Let us define two other related commutation matrices of dimension $p^3 \times p^3$:

$$\mathcal{P}_{3,2} = \mathbf{I}_p \otimes \mathbf{K}_{pp} \tag{28}$$

$$\mathcal{P}_{2,1} = \mathbf{K}_{pp} \otimes \mathbf{I}_p, \tag{29}$$

where $\mathbf{I}_p$ is the identity matrix.

Because of the lack of commutativity of the Kronecker product, these commutation matrices will be crucial in the proof of our theorem. In particular, if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are three vectors of dimension $p \times 1$, the following equalities hold (by straightforward calculations, see Brewer [23], Magnus and Neudecker [18] or Wiens [24]):

$$\mathbf{K}_{pp}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{b} \otimes \mathbf{a} \tag{30}$$

$$\mathcal{P}_{3,2}\,(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) = \mathbf{a} \otimes \mathbf{c} \otimes \mathbf{b} \tag{31}$$

$$\mathcal{P}_{2,1}\,(\mathbf{a} \otimes \mathbf{b} \otimes \mathbf{c}) = \mathbf{b} \otimes \mathbf{a} \otimes \mathbf{c}. \tag{32}$$

Since $\mathbf{K}'_{pp} = \mathbf{K}_{pp}$, it comes $\mathcal{P}'_{3,2} = \mathcal{P}_{3,2}$, $\mathcal{P}'_{2,1} = \mathcal{P}_{2,1}$, and

$$(\mathbf{a}' \otimes \mathbf{b}')\mathbf{K}_{pp} = \mathbf{b}' \otimes \mathbf{a}' \tag{33}$$

$$(\mathbf{a}' \otimes \mathbf{b}' \otimes \mathbf{c}')\,\mathcal{P}_{3,2} = \mathbf{a}' \otimes \mathbf{c}' \otimes \mathbf{b}' \tag{34}$$

$$(\mathbf{a}' \otimes \mathbf{b}' \otimes \mathbf{c}')\,\mathcal{P}_{2,1} = \mathbf{b}' \otimes \mathbf{a}' \otimes \mathbf{c}'. \tag{35}$$

Finally let us emphasize this very special case where no commutation matrix is needed,

$$\mathbf{a} \otimes \mathbf{b}' = \mathbf{b}' \otimes \mathbf{a} = \mathbf{a}\mathbf{b}'. \tag{36}$$

3.2. Statement of results for orders 2 *to* 6

One is now in a position to state the main result of the paper.

Theorem 1. *Let* $\mathbf{x} = (x_1, \ldots, x_p)'$ *be a real-valued random vector of dimension* $p \times 1$*, with assumed zero-mean for simplicity* ($\mathbb{E}[\mathbf{x}] = \mathbf{0}$)*. Let us further assume that* $\mathbb{E}[|\mathbf{x}|^n] < \infty$ *for a given integer* $n \geq 6$ *so as to guarantee the existence of moments and cumulants up to order six.*

Next let us define the four matrices:

$$\mathbf{K} = \mathbf{I}_{p^2} + \mathbf{K}_{pp} \tag{37}$$

$$\mathbf{P} = \mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2}\mathcal{P}_{2,1} \tag{38}$$

$$\mathbf{R} = \mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2} \tag{39}$$

$$\mathbf{Q} = \mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1} + \mathcal{P}_{2,1}\mathcal{P}_{3,2} + \mathcal{P}_{3,2}\mathcal{P}_{2,1} + \mathcal{P}_{2,1}\mathcal{P}_{3,2}\mathcal{P}_{2,1} \tag{40}$$

which depend only on dimension p*. Thus, the cumulant matrices of orders* 2 *to* 6 *read*

$$c_2(\mathbf{x}) = m_2(\mathbf{x}) = \mathbb{E}[\mathbf{x}\mathbf{x}'] \tag{41}$$

$$c_3(\mathbf{x}) = m_3(\mathbf{x}) = \mathbb{E}[\mathbf{x}\mathbf{x}' \otimes \mathbf{x}] \tag{42}$$

$$c_4(\mathbf{x}) = m_4(\mathbf{x}) - \mathbf{K}\Big(m_2(\mathbf{x}) \otimes m_2(\mathbf{x})\Big) - \mathbf{vec}\, m_2(\mathbf{x})\, \mathbf{vec}'\, m_2(\mathbf{x}) \tag{43}$$

$$c_5(\mathbf{x}) = m_5(\mathbf{x}) - \mathbf{R}\Big(m_3(\mathbf{x}) \otimes m_2(\mathbf{x})\Big)\mathbf{K} - \mathbf{P}\Big(m_3(\mathbf{x})' \otimes \mathbf{vec}\, m_2(\mathbf{x})\Big) - \mathbf{vec}\, m_3(\mathbf{x})\, \mathbf{vec}'\, m_2(\mathbf{x}) \tag{44}$$

$$c_6(\mathbf{x}) = m_6(\mathbf{x}) - \Gamma_{(4,2)}(\mathbf{x}) - \Gamma_{(3,3)}(\mathbf{x}) + 2\Gamma_{(2,2,2)}(\mathbf{x}), \tag{45}$$

respectively, where $\mathbf{vec}'\, m = (\mathbf{vec}\, m)'$ *and*

$$\Gamma_{(4,2)}(\mathbf{x}) = \mathbf{R}\Big(m_4(\mathbf{x}) \otimes m_2(\mathbf{x})\Big)\mathbf{R}' + \Big(\mathbf{M}_4(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x})\Big)\mathbf{P}' + \mathbf{P}\Big(\mathbf{M}_4(\mathbf{x})' \otimes \mathbf{vec}\, m_2(\mathbf{x})\Big)$$

$$\Gamma_{(3,3)}(\mathbf{x}) = \mathbf{R}\Big(m_3(\mathbf{x}) \otimes m_3(\mathbf{x})'\Big)\mathbf{P}' + \mathbf{vec}\, m_3(\mathbf{x})\, \mathbf{vec}'\, m_3(\mathbf{x})$$

$$\Gamma_{(2,2,2)}(\mathbf{x}) = \mathbf{Q}\Big(m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x})\Big) + \mathbf{P}\Big(m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x})\, \mathbf{vec}'\, m_2(\mathbf{x})\Big)\mathbf{P}'$$

$$\mathbf{M}_4(\mathbf{x}) = \mathbb{E}\left[\mathbf{x}\mathbf{x}' \otimes \mathbf{x} \otimes \mathbf{x}\right].$$

3.3. Outline of the proof and counting of the terms in the formulae

The sketch of the proof consists of

(a) injecting Eq. (11) that returns cumulants in terms of moments into Eqs. (24) and (25) which define cumulant matrices;
(b) expressing the terms of the resulting sum with moment matrices.

Let us detail the first step in the case of even orders $n = 2k$ (the principle is identical for odd orders):

$$\begin{aligned} c_n(\mathbf{x}) &= \sum_{(j_1,\ldots,j_n)\in[\![1,p]\!]^n} \left(\sum_{\pi\in P([\![1,n]\!])} (-1)^{|\pi|-1}(|\pi|-1)! \prod_{B\in\pi} mom^{(|B|)}(x_{j_l} : l \in B) \right) \\ &\quad \times \mathbf{e}_{j_1}\mathbf{e}'_{j_2} \otimes \cdots \otimes \mathbf{e}_{j_{n-1}}\mathbf{e}'_{j_n} \\ &= \sum_{s\in P(n)} (-1)^{|s|-1}(|s|-1)!\Gamma_s(\mathbf{x}) \end{aligned} \tag{46}$$

where $P(n)$ denotes the set of partitions of the integer n, that is the set of all $s = (s_1, \ldots, s_l)$, where $s_1 \geq \cdots \geq s_l \geq 1$ are integers such that $s_1 + \ldots + s_l = n$; $|s|$ denotes the length of s (that is the number l of integers in s) and

$$\Gamma_s(\mathbf{x}) = \sum_{\pi\in P_s([\![1,n]\!])} \underbrace{\left(\sum_{(j_1,\ldots,j_n)\in[\![1,p]\!]^n} \prod_{B\in\pi} mom^{(|B|)}(x_{j_l} : l \in B)\mathbf{e}_{j_1}\mathbf{e}'_{j_2} \otimes \cdots \otimes \mathbf{e}_{j_{n-1}}\mathbf{e}'_{j_n} \right)}_{T_\pi}, \tag{47}$$

Table 2
$p(n)$ (*resp.* $p(n) - p(n-1)$) number of terms (*resp.* non-zero terms) in formula (46).

n	$p(n,1)$	$p(n,2)$	$p(n,3)$	$p(n,4)$	$p(n,5)$	$p(n,6)$	$p(n)$	$p(n)-p(n-1)$
1	1						1	
2	1	1					2	1
3	1	1	1				3	1
4	1	2	1	1			5	2
5	1	2	2	1	1		7	2
6	1	3	3	2	1	1	11	4

with $P_s([\![1,n]\!])$ the set of partitions of the set $[\![1,n]\!]$ of type s. A partition of the set $[\![1,n]\!]$ of type s, where $s = (s_1, \ldots, s_l)$ is a partition of the integer n, is a partition of $[\![1,n]\!]$ with l parts $B_1, \ldots, B_l$ such that $|B_j| = s_j$ for all $j \in [\![1,l]\!]$. For instance, for $n = 4$,

$$P(4) = \{(4); (3,1); (2,2); (2,1,1); (1,1,1,1)\}, \tag{48}$$

and for $s = (2,2)$, $|s| = 2$ and

$$P_s([\![1,4]\!]) = \{\{\{1;2\};\{3;4\}\};\{\{1;3\};\{2;4\}\};\{\{1;4\};\{2;3\}\}\}. \tag{49}$$

Now let us compute the number of terms in the previous formulae.

Number of terms in formula (46). The number of partitions of n is usually named $p(n) = |P(n)|$. There is no simple formula for $p(n)$ and the easiest way to compute $p(n)$ is to consider $p(n,k)$, the number of partitions of n with k parts. Of course it comes

$$p(n) = \sum_{k=1}^{n} p(n,k), \tag{50}$$

$p(n,1) = 1$ and $p(n,n) = 1$. Moreover, this recurrence equation

$$p(n,k) = p(n-1,k-1) + p(n-k,k), \tag{51}$$

is obtained because $p(n-1,k-1)$ is also the number of partitions of n with k parts with smallest one equal to 1, whereas $p(n-k,k)$ is the number of partitions of n with k parts different from 1.

Number of non-zero terms in formula (46). As $\mathbb{E}[\mathbf{x}] = 0$ was assumed, one has $mom^{(1)}(x_j) = 0$ for all $j \in [\![1,p]\!]$. Thus $\Gamma_s(\mathbf{x}) = 0$ for all $s = (s_1, \ldots, s_l)$ with $s_l = 1$. Besides, $(s_1, \ldots, s_{l-1}, 1)$ is a partition of n if and only if $(s_1, \ldots, s_{l-1})$ is a partition of $n-1$. Finally, the number of non-zero terms in formula (46) is given by $p(n) - p(n-1)$.

Table 2 summarizes the previous results from $n = 1$ to 6.

Number of terms in the formula (47). Given $s = (s_1, \ldots, s_l)$, a partition of n, one has to compute $p_s(n) = |P_s([\![1,n]\!])|$, the number of partitions of the set $[\![1,n]\!]$ of type s. If

$s_i \neq s_{i'}$ for all $i \neq i'$, it is easy to see that $p_s(n) = \frac{n!}{s_1! \cdots s_l!}$. If not, one has to take in account permutations between the equal values of s_i, so that the general result is

$$p_s(n) = \frac{n!}{s_1! \cdots s_l! k_1! \cdots k_r!}, \tag{52}$$

where $k_1, \ldots, k_r$ are the respective numbers of the different values taken by the s_i. For example,

$$p_{(2,2)}(4) = |P_{(2,2)}([\![1,4]\!])| = \frac{4!}{2!2!2!} = 3, \tag{53}$$

as seen directly from Eq. (49).

For the second step, it is convenient to browse set $P(n)$ in the lexicographic order as suggested in Eq. (48) so that the first term in Eq. (46) will correspond to $s = (n)$, $|s| = 1$, $P_s([\![1,n]\!]) = \{\{\{1;\cdots;n\}\}\}$; that is the partition $\pi = \{\{1;\cdots;n\}\}$ with one block $B = \{1;\cdots;n\}$ such that

$$\begin{aligned}\Gamma_{(n)}(\mathbf{x}) = T_{\{\{1;\cdots;n\}\}} &= \sum_{(j_1,\ldots,j_n)\in[\![1,p]\!]^n} mom^{(n)}(x_{j_l} \ : \ l \in B)\mathbf{e}_{j_1}\mathbf{e}'_{j_2} \otimes \cdots \otimes \mathbf{e}_{j_{2k-1}}\mathbf{e}'_{j_n} \\ &= m_n(\mathbf{x}).\end{aligned} \tag{54}$$

In addition, in each product $\prod_{B\in\pi}$ of (47), blocks B will be browsed from the largest one to the smallest one, as suggested again by Eq. (48).

The calculation of $c_3(\mathbf{x})$ to $c_6(\mathbf{x})$ according to these principles is postponed to Appendix A.

4. Two immediate consequences

4.1. Upper bounds on the rank of cumulant matrices

Corollary 2. *Let* $\mathbf{x}$ *be a real-valued random vector of dimension* $p \times 1$ *such that* $\mathbb{E}\left[|\mathbf{x}|^n\right] < \infty$ *for a given integer* $n \geq 6$. *Then, its cumulant matrices of orders* 2 *to* 6 *have their rank upper-bounded as follows:*

$$\operatorname{rank}\Big(c_2(\mathbf{x})\Big) \leq p \tag{55}$$

$$\operatorname{rank}\Big(c_3(\mathbf{x})\Big) \leq p \tag{56}$$

$$\operatorname{rank}\Big(c_4(\mathbf{x})\Big) \leq \frac{p(p+1)}{2} \tag{57}$$

$$\operatorname{rank}\Big(c_5(\mathbf{x})\Big) \leq \frac{p(p+1)}{2} \tag{58}$$

$$\operatorname{rank}\Big(c_6(\mathbf{x})\Big) \leq \frac{p(p+1)(p+2)}{6}. \tag{59}$$

Proof. The first two inequalities are obvious. For the other ones, let us introduce

$$\widetilde{\mathbf{K}} = \frac{1}{2}\mathbf{K} \quad \text{and} \quad \widetilde{\mathbf{Q}} = \frac{1}{6}\mathbf{Q}, \tag{60}$$

where $\mathbf{K}$ and $\mathbf{Q}$ are matrices of dimension $p^2 \times p^2$ and $p^3 \times p^3$, as given in Eqs. (37) and (40), respectively. By using the symmetry properties of cumulants, it comes that

$$c_4(\mathbf{x}) = \widetilde{\mathbf{K}} c_4(\mathbf{x}), \quad c_5(\mathbf{x}) = c_5(\mathbf{x})\widetilde{\mathbf{K}}, \quad \text{and} \quad c_6(\mathbf{x}) = \widetilde{\mathbf{Q}} c_6(\mathbf{x}), \tag{61}$$

from which the following majorations immediately follows:

$$\begin{aligned} &\operatorname{rank}\left(c_4(\mathbf{x})\right) \leq \operatorname{rank}\left(\widetilde{\mathbf{K}}\right), \quad \operatorname{rank}\left(c_5(\mathbf{x})\right) \leq \operatorname{rank}\left(\widetilde{\mathbf{K}}\right), \quad \text{and} \\ &\operatorname{rank}\left(c_6(\mathbf{x})\right) \leq \operatorname{rank}\left(\widetilde{\mathbf{Q}}\right). \end{aligned} \tag{62}$$

Since matrices $\widetilde{\mathbf{K}}$ and $\widetilde{\mathbf{Q}}$ are idempotent (see Schott [19, Theorem 1]), their rank is returned by their trace. Thus,

$$\operatorname{rank}\left(\widetilde{\mathbf{K}}\right) = \operatorname{Tr}(\widetilde{\mathbf{K}}) = \frac{1}{2}\operatorname{Tr}(\mathbf{I}_{p^2} + \mathbf{K}_{pp}) = \frac{1}{2}\left(p^2 + \operatorname{Tr}(\mathbf{K}_{pp})\right). \tag{63}$$

The trace of the commutation matrices is given Magnus and Neudecker [18] by

$$\operatorname{Tr}(\mathbf{K}_{ab}) = 1 + \gcd(a-1, b-1), \tag{64}$$

where gcd stands for "greatest common divisor". This implies, in particular, $\operatorname{Tr}(\mathbf{K}_{pp}) = p$; therefore,

$$\operatorname{rank}\left(\widetilde{\mathbf{K}}\right) = \frac{1}{2}\left(p^2 + p\right) = \frac{p(p+1)}{2}. \tag{65}$$

Similarly,

$$\begin{aligned} \operatorname{rank}\left(\widetilde{\mathbf{Q}}\right) &= \operatorname{Tr}(\widetilde{\mathbf{Q}}) = \frac{1}{6}\operatorname{Tr}(\mathbf{Q}) \\ &= \frac{1}{6}\operatorname{Tr}\left(\mathbf{I}_{p^3} + \mathbf{I}_p \otimes \mathbf{K}_{pp} + \mathbf{K}_{pp} \otimes \mathbf{I}_p + \mathbf{K}_{pp^2} + \mathbf{K}_{p^2p} + (\mathbf{K}_{pp} \otimes \mathbf{I}_p)\mathbf{K}_{p^2p}\right) \\ &= \frac{1}{6}\left(p^3 + p^2 + p^2 + p + p + p^2\right) \\ &= \frac{p^3 + 3p^2 + 2p}{6} = \frac{p(p+1)(p+2)}{6}. \qquad \square \end{aligned} \tag{66}$$

4.2. The sixth-order moment matrix of a Gaussian vector

The matrix expression of the sixth-order moment of a Gaussian vector (so-called *Isserlis' theorem* or *Wick's theorem*) has been the aim of several research works. It has been provided by Neudecker [12], Magnus and Neudecker [18], Tracy and Sultan [15], and more recently by Schott [19]. It happens to be a direct corollary of Theorem 1.

Corollary 3. *Let $\mathbf{z} \sim \mathcal{N}(\mathbf{0}_p, \mathbf{\Sigma})$ be a zero-mean, real-valued, Gaussian vector. Its fourth and sixth-order moment matrix reads*

$$m_4(\mathbf{z}) = \mathbf{K}\Big(\mathbf{\Sigma} \otimes \mathbf{\Sigma}\Big) + \mathbf{vec}\,\mathbf{\Sigma}\mathbf{vec}'\,\mathbf{\Sigma} \tag{67}$$

$$m_6(\mathbf{z}) = \mathbb{E}(\mathbf{z}\mathbf{z}' \otimes \mathbf{z}\mathbf{z}' \otimes \mathbf{z}\mathbf{z}') = \mathbf{Q}\Big(\mathbf{\Sigma} \otimes \mathbf{\Sigma} \otimes \mathbf{\Sigma}\Big) + \mathbf{P}\Big(\mathbf{\Sigma} \otimes \mathbf{vec}\,\mathbf{\Sigma} \otimes \mathbf{vec}'\,\mathbf{\Sigma}\Big)\mathbf{P}' \tag{68}$$

where matrices $\mathbf{Q}$ and $\mathbf{P}$ are defined in Theorem 1.

Proof. Since $\mathbf{z}$ is Gaussian, all its cumulants of order greater than two are equal to zero, whence result (67). Therefore, the only non-zero term in Eq. (12), with $k = 6$, corresponds to the partition of type $(2, 2, 2)$. Hence,

$$m_6(\mathbf{z}) = \Gamma_{(2,2,2)}(\mathbf{z}), \tag{69}$$

where

$$\Gamma_{(2,2,2)}(\mathbf{z}) = \sum_{\pi \in P_{(2,2,2)}([1,6])} \left(\sum_{(j_1,\ldots,j_6)\in[1,p]^6} \prod_{B\in\pi} cum^{(2)}(z_{j_l} \ : \ l \in B)\mathbf{e}_{j_1}\mathbf{e}'_{j_2} \otimes \mathbf{e}_{j_3}\mathbf{e}'_{j_4} \otimes \mathbf{e}_{j_5}\mathbf{e}'_{j_6} \right). \tag{70}$$

This quantity is given by Eq. (A.70), wherein $m_2(\mathbf{z}) = c_2(\mathbf{z}) = \mathbf{\Sigma}$. □

Remark 5. In the scalar case ($p = 1$), Eq. (67) and Eq. (68) reduce to the well-known result (see for example Schott [19])

$$m_{2\alpha}(z) = \frac{(2\alpha)!}{2^\alpha \alpha!}\sigma^{2\alpha}. \tag{71}$$

Indeed, if $p = 1$, one obtains $\mathbf{\Sigma} = \mathbf{vec}\,\mathbf{\Sigma} = \mathbf{vec}'\,\mathbf{\Sigma} = \sigma^2$, $\mathbf{K} = 2$, $\mathbf{P} = 3$ and $\mathbf{Q} = 6$. Thus (67) and (68) become respectively $m_4(z) = 2\sigma^2\sigma^2 + \sigma^2\sigma^2 = 3\sigma^4$ and $m_6(z) = 6\sigma^2\sigma^2\sigma^2 + 9\sigma^2\sigma^2\sigma^2 = 15\sigma^6$ which are in accordance with the formula (71).

Table 3
Number of additions and multiplications in Leonov and Shiryaev formula (LS) *versus* the concise formulae (CF).

	Additions		Multiplications	
	(LS)	(CF)	(LS)	(CF)
$c_4(\mathbf{x})$	$15p^4$	$(p^2-1)p^4$	$37p^4$	$(p^2+1)p^4$
$c_5(\mathbf{x})$	$52p^5$	$(2p^3+p^2)p^5$	$151p^5$	$(2p^3+p^2+3)p^5$
$c_6(\mathbf{x})$	$203p^6$	$(9p^3-2)p^6$	$674p^6$	$(9p^3+7)p^6$

5. Computational cost of the concise formulae

Let us compare the computational cost of the proposed concise formulae (CF) of Theorem 1 with the classical repeated used of Leonov and Shiryaev formula (LS). Suppose that all moments and moment matrices of the zero-mean random vector $\mathbf{x}$ of dimension $p \times 1$ have been already computed.

With LS, any of the p^n terms of $c_n(\mathbf{x})$ requires the same number of additions and multiplications as given in Table 1.

With CF, one has to take into account the matrix products, the Kronecker products and the sums of matrices. The product of a $p \times q$ matrix with a $q \times r$ matrix requires $pr(q-1)$ additions and prq multiplications, whereas the Kronecker product of a $p \times q$ matrix with an $r \times s$ matrix requires no additions but $pqrs$ multiplications (see Eq. (26)).

The results for $c_4(\mathbf{x})$, $c_5(\mathbf{x})$ and $c_6(\mathbf{x})$ are gathered in Table 3.

Without surprise the number of operations required for $c_n(\mathbf{x})$ are $O(p^n)$ with LS and $O(p^{n+2})$ or $O(p^{n+3})$ with CF because of the products of "big matrices" in the concise formulae. This shows that the advantages of the concise formulae do not reside in computational complexity. However the construction of the matrix $c_n(\mathbf{x})$ in (LS) cases requires n imbricated loops whereas it is realized by usual matrix operations with CF. Finally, the proposed formulae have a definite advantage when they come to be coded with high-level matrix/array programming language, such as Matlab®. By way of an example (Ould-Baba [25]), it has been verified by the authors that the numerical computation of the empirical cumulant matrices of orders five and six with dimension $p = 6$ are about 150 times faster than with coded with LS formula (version 7 of Matlab used on a laptop computer with 2.10GHz clock and 4Go RAM).

6. Conclusion

This paper has introduced a methodology for deriving explicit formulae for the cumulant matrices of a random vector in terms of moment matrices of lower orders. The methodology reproduces known formulae up to order four and returns new ones at orders five and six, which are all embodied in Theorem 1. In principle, it can also be applied to address higher orders. Such formulae of the cumulant matrices have several advantages. First, from the notational point of view, they are quite concise. Second, from the theoretical point of view, they evidence a direct link with moment matrices. As a consequence,

the moment matrices of a random Gaussian vector – which have been of some interest in the literature – are easily found by setting the cumulant matrix to zero (e.g. see Corollary 3). In addition, they make possible to upper-bound the rank of the cumulant matrices as described in Corollary 2. These properties have considerable importance in applications such as independent component analysis (ICA) Comon [4], which strongly rely on the use of the cumulant matrices of order four – the Quadricovariance Albera et al. [6] – or six – the Hexacovariance Albera et al. [8]. In this case, the upper-bound on the rank of the cumulant matrices indicates the maximum number of independent sources that can be separated. In addition, the majority of results introduced in reference Loperfido [26], Loperfido [27] for the third and fourth moment and cumulant matrices can be easily generalized to the fifth and sixth moment and cumulant matrices.

Appendix A. Proof of Theorem 1 (page 401)

The formulae are obtained under the assumption of a zero-mean process but are generally true after centering a non-zero-mean process.

A.1. Proof for order 3

The only partition of 3 which does not involve 1 is (3): $P_{(3)}([\![1,3]\!]) = \{\{\{1;2;3\}\}\}$. Thus

$$c_3(\mathbf{x}) = \Gamma_{(3)}(\mathbf{x}) = \sum_{(j_1,j_2,j_3)\in[\![1,p]\!]^3} mom^{(3)}(x_{j_1}, x_{j_2}, x_{j_3})\mathbf{e}_{j_1}\mathbf{e}'_{j_2} \otimes \mathbf{e}_{j_3} = m_3(\mathbf{x}). \quad \text{(A.1)}$$

A.2. Proof for order 4

The only partitions of 4 (see Eq. (48)) which do not involve 1 are (4) and (2, 2):

$$P_{(4)}([\![1,4]\!]) = \{\{\{1;2;3;4\}\}\}$$
$$P_{(2,2)}([\![1,4]\!]) = \{\{\{1;2\};\{3;4\}\};\{\{1;3\};\{2;4\}\};\{\{1;4\};\{2;3\}\}\}.$$

In order to avoid any ambiguity, each term of $\Gamma_{(2,2)}(\mathbf{x})$ will be denoted by T_π where π is the corresponding partition in (47). Therefore,

$$\begin{aligned} c_4(\mathbf{x}) &= m_4(\mathbf{x}) - \Gamma_{(2,2)}(\mathbf{x}) \qquad \text{(A.2)} \\ &= m_4(\mathbf{x}) - \underbrace{m_2(\mathbf{x}) \otimes m_2(\mathbf{x})}_{T_{\{\{1;2\};\{3;4\}\}}} \\ &\quad - \underbrace{\sum_{(j_1,\ldots,j_4)\in[\![1,p]\!]^4} mom^{(2)}(x_{j_1}, x_{j_3}) mom^{(2)}(x_{j_2}, x_{j_4})\mathbf{e}_{j_1}\mathbf{e}'_{j_2} \otimes \mathbf{e}_{j_3}\mathbf{e}'_{j_4}}_{T_{\{\{1;3\};\{2;4\}\}}} \end{aligned}$$

$$\underbrace{-\sum_{(j_1,\ldots,j_4)\in[\![1,p]\!]^4} mom^{(2)}(x_{j_1},x_{j_4})mom^{(2)}(x_{j_2},x_{j_3})\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_3}\mathbf{e}'_{j_4}}_{T_{\{\{1;4\};\{2;3\}\}}}. \quad \text{(A.3)}$$

For

$$\begin{aligned}\mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_3}\mathbf{e}'_{j_4} &= \mathbf{e}_{j_1}\otimes\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_3}\otimes\mathbf{e}'_{j_4} = \mathbf{e}_{j_1}\otimes\mathbf{e}_{j_3}\otimes\mathbf{e}'_{j_2}\otimes\mathbf{e}'_{j_4}\\ &= (\mathbf{e}_{j_1}\otimes\mathbf{e}_{j_3})\otimes(\mathbf{e}_{j_2}\otimes\mathbf{e}_{j_4})' \qquad \text{(A.4)}\\ &= K_{pp}(\mathbf{e}_{j_3}\otimes\mathbf{e}_{j_1})(\mathbf{e}_{j_2}\otimes\mathbf{e}_{j_4})' = K_{pp}(\mathbf{e}_{j_3}\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_1}\mathbf{e}'_{j_4}), \qquad \text{(A.5)}\end{aligned}$$

it comes

$$\begin{aligned}T_{\{\{1;3\};\{2;4\}\}} &= \left(\sum_{(j_1,j_3)\in[\![1,p]\!]^2} mom^{(2)}(x_{j_1},x_{j_3})\mathbf{e}_{j_1}\otimes\mathbf{e}_{j_3}\right)\\ &\quad\times\left(\sum_{(j_2,j_4)\in[\![1,p]\!]^2} mom^{(2)}(x_{j_2},x_{j_4})\mathbf{e}_{j_2}\otimes\mathbf{e}_{j_4}\right)'\\ &= \mathbf{vec}\, m_2(\mathbf{x})\,\mathbf{vec}'\, m_2(\mathbf{x}), \qquad \text{(A.6)}\end{aligned}$$

$$\begin{aligned}T_{\{\{1;4\};\{2;3\}\}} &= K_{pp}\left(\sum_{(j_2,j_3)\in[\![1,p]\!]^2} mom^{(2)}(x_{j_2},x_{j_3})\mathbf{e}_{j_3}\mathbf{e}'_{j_2}\right)\\ &\quad\otimes\left(\sum_{(j_1,j_4)\in[\![1,p]\!]^2} mom^{(2)}(x_{j_1},x_{j_4})\mathbf{e}_{j_1}\mathbf{e}'_{j_4}\right)\\ &= K_{pp}m_2(\mathbf{x})\otimes m_2(\mathbf{x}). \qquad \text{(A.7)}\end{aligned}$$

Upon collecting $m_2(\mathbf{x})\otimes m_2(\mathbf{x})$, one finally arrives at

$$c_4(\mathbf{x}) = m_4(\mathbf{x}) - \mathbf{K}m_2(\mathbf{x})\otimes m_2(\mathbf{x}) - \mathbf{vec}\, m_2(\mathbf{x})\,\mathbf{vec}'\, m_2(\mathbf{x}). \qquad \text{(A.8)}$$

A.3. Proof for order 5

The only partitions of 5 which do not involve 1 are (5) and (3, 2):

$$\begin{aligned}P_{(5)}([\![1,5]\!]) &= \{\{\{1;2;3;4;5\}\}\}\\ P_{(3,2)}([\![1,5]\!]) &= \{\{\{1;2;3\};\{4;5\}\};\{\{1;2;4\};\{3;5\}\};\{\{1;3;4\};\{2;5\}\};\{\{2;3;4\};\{1;5\}\};\\ &\quad\{\{1;2;5\};\{3;4\}\};\{\{1;3;5\};\{2;4\}\};\{\{2;3;5\};\{1;4\}\};\{\{1;4;5\};\{2;3\}\};\\ &\quad\{\{2;4;5\};\{1;3\}\};\{\{3;4;5\};\{1;2\}\}\}.\end{aligned}$$

One may check that $P_{(3,2)}([\![1,5]\!])$ contains $\frac{5!}{3!2!} = 10$ elements. As before, each term of $\Gamma_{(3,2)}(\mathbf{x})$ will be denoted as T_π where π is the corresponding partition in Eq. (47). Thus

$$c_5(\mathbf{x}) = m_5(\mathbf{x}) - \Gamma_{(3,2)}(\mathbf{x}), \tag{A.9}$$

where $\Gamma_{(3,2)}(\mathbf{x}) = \sum_{\pi \in P_{(3,2)}([\![1,5]\!])} T_\pi$. Next, the T_π's are computed by factoring terms of order 3 in first positions. The principle is illustrated for instance on $T_{\{\{2;3;5\};\{1;4\}\}}$:

$$\begin{aligned}
&T_{\{\{2;3;5\};\{1;4\}\}} \\
&\quad = \sum_{(j_1,\dots,j_5)\in[\![1,p]\!]^5} mom^{(3)}(x_{j_2},x_{j_3},x_{j_5})mom^{(2)}(x_{j_1},x_{j_4})\mathbf{e}_{j_1}\otimes\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_3}\otimes\mathbf{e}'_{j_4}\otimes\mathbf{e}_{j_5} \\
&\quad = \sum_{(j_1,\dots,j_5)\in[\![1,p]\!]^5} mom^{(3)}(x_{j_2},x_{j_3},x_{j_5})mom^{(2)}(x_{j_1},x_{j_4})(\mathbf{e}_{j_1}\otimes\mathbf{e}_{j_3}\otimes\mathbf{e}_{j_5})\otimes(\mathbf{e}'_{j_2}\otimes\mathbf{e}'_{j_4}) \\
&\quad = \sum_{(j_1,\dots,j_5)\in[\![1,p]\!]^5} mom^{(3)}(x_{j_2},x_{j_3},x_{j_5})mom^{(2)}(x_{j_1},x_{j_4})\mathcal{P}_{2,1}\mathcal{P}_{3,2}(\mathbf{e}_{j_3}\otimes\mathbf{e}_{j_5}\otimes\mathbf{e}_{j_1}) \\
&\qquad \otimes(\mathbf{e}'_{j_2}\otimes\mathbf{e}'_{j_4}) \\
&\quad = \mathcal{P}_{2,1}\mathcal{P}_{3,2}\sum_{(j_1,\dots,j_5)\in[\![1,p]\!]^5}\underbrace{mom^{(3)}(x_{j_2},x_{j_3},x_{j_5})}_{=mom^{(3)}(x_{j_3},x_{j_2},x_{j_5})}mom^{(2)}(x_{j_1},x_{j_4})(\mathbf{e}_{j_3}\otimes\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_5}) \\
&\qquad \otimes(\mathbf{e}_{j_1}\otimes\mathbf{e}'_{j_4}) \\
&\quad = \mathcal{P}_{2,1}\mathcal{P}_{3,2}m_3(\mathbf{x})\otimes m_2(\mathbf{x}).
\end{aligned} \tag{A.10}$$

Following similar lines,

$$T_{\{\{1;2;3\};\{4;5\}\}} = m_3(\mathbf{x})\otimes m_2(\mathbf{x}) \tag{A.11}$$

$$T_{\{\{1;2;4\};\{3;5\}\}} = m_3(\mathbf{x})'\otimes \mathbf{vec}\, m_2(\mathbf{x}) \tag{A.12}$$

$$T_{\{\{1;3;4\};\{2;5\}\}} = m_3(\mathbf{x})\otimes m_2(\mathbf{x})\mathbf{K}_{pp} \tag{A.13}$$

$$T_{\{\{2;3;4\};\{1;5\}\}} = \mathcal{P}_{2,1}m_3(\mathbf{x})'\otimes \mathbf{vec}\, m_2(\mathbf{x}) \tag{A.14}$$

$$T_{\{\{1;2;5\};\{3;4\}\}} = \mathcal{P}_{3,2}m_3(\mathbf{x})\otimes m_2(\mathbf{x}) \tag{A.15}$$

$$T_{\{\{1;3;5\};\{2;4\}\}} = \mathbf{vec}\, m_3(\mathbf{x})\, \mathbf{vec}'\, m_2(\mathbf{x}) \tag{A.16}$$

$$T_{\{\{1;4;5\};\{2;3\}\}} = \mathcal{P}_{3,2}m_3(\mathbf{x})\otimes m_2(\mathbf{x})\mathbf{K}_{pp} \tag{A.17}$$

$$T_{\{\{2;4;5\};\{1;3\}\}} = \mathcal{P}_{3,2}\mathcal{P}_{2,1}m_3(\mathbf{x})'\otimes \mathbf{vec}\, m_2(\mathbf{x}) \tag{A.18}$$

$$T_{\{\{3;4;5\};\{1;2\}\}} = \mathcal{P}_{2,1}\mathcal{P}_{3,2}m_3(\mathbf{x})\otimes m_2(\mathbf{x}).\mathbf{K}_{pp}. \tag{A.19}$$

The last step is to collect terms in $m_3(\mathbf{x})\otimes m_2(\mathbf{x})$, $m_3(\mathbf{x})'\otimes \mathbf{vec}\, m_2(\mathbf{x})$, and $\mathbf{vec}\, m_3(\mathbf{x})\, \mathbf{vec}'\, m_2(\mathbf{x})$, respectively:

$$\begin{aligned}
\Gamma_{(3,2)}(\mathbf{x}) &= \underbrace{\left(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2}\right)}_{\mathbf{R}} m_3(\mathbf{x}) \otimes m_2(\mathbf{x}) \\
&\quad + \underbrace{\left(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2}\right)}_{\mathbf{R}} m_3(\mathbf{x}) \otimes m_2(\mathbf{x})\mathbf{K}_{pp} \\
&\quad + \underbrace{\left(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2}\mathcal{P}_{2,1}\right)}_{\mathbf{P}} m_3(\mathbf{x})' \otimes \mathbf{vec}\, m_2(\mathbf{x}) + \mathbf{vec}\, m_3(\mathbf{x})\, \mathbf{vec}'\, m_2(\mathbf{x}) \\
&= \mathbf{R} m_3(\mathbf{x}) \otimes m_2(\mathbf{x}) \underbrace{\left(\mathbf{I}_{p^2} + \mathbf{K}_{pp}\right)}_{\mathbf{K}} + \mathbf{P} m_3(\mathbf{x})' \otimes \mathbf{vec}\, m_2(\mathbf{x}) \\
&\quad + \mathbf{vec}\, m_3(\mathbf{x})\, \mathbf{vec}'\, m_2(\mathbf{x}) \\
&= \mathbf{R} m_3(\mathbf{x}) \otimes m_2(\mathbf{x})\mathbf{K} + \mathbf{P} m_3(\mathbf{x})' \otimes \mathbf{vec}\, m_2(\mathbf{x}) \\
&\quad + \mathbf{vec}\, m_3(\mathbf{x})\, \mathbf{vec}'\, m_2(\mathbf{x}).
\end{aligned} \tag{A.20}$$

A.4. Proof for order 6

The 4 partitions of 6 which do not involve 1 are: (6), $(4,2)$, $(3,3)$ and $(2,2,2)$. Therefore $c_6(\mathbf{x})$ is a sum of four terms,

$$c_6(\mathbf{x}) = m_6(\mathbf{x}) - \Gamma_{(4,2)}(\mathbf{x}) - \Gamma_{(3,3)}(\mathbf{x}) + 2\Gamma_{(2,2,2)}(\mathbf{x}), \tag{A.21}$$

the last three of which are detailed hereafter.

<u>Expression of $\Gamma_{(4,2)}(\mathbf{x})$</u>:

It is seen that $P_{(4,2)}([\![1,6]\!])$ contains $\frac{6!}{4!2!} = 15$ elements:

$$\begin{aligned}
P_{(4,2)}([\![1,6]\!]) = \{&\{\{1;2;3;4\};\{5;6\}\};\{\{1;2;3;5\};\{4;6\}\};\{\{1;2;4;5\};\{3;6\}\}; \\
&\{\{1;3;4;5\};\{2;6\}\};\{\{2;3;4;5\};\{1;6\}\};\{\{1;2;3;6\};\{4;5\}\}; \\
&\{\{1;2;4;6\};\{3;5\}\};\{\{1;3;4;6\};\{2;5\}\};\{\{2;3;4;6\};\{1;5\}\}; \\
&\{\{1;2;5;6\};\{3;4\}\};\{\{1;3;5;6\};\{2;4\}\};\{\{2;3;5;6\};\{1;4\}\}; \\
&\{\{1;4;5;6\};\{2;3\}\};\{\{2;4;5;6\};\{1;3\}\};\{\{3;4;5;6\};\{1;2\}\}\}
\end{aligned} \tag{A.22}$$

$$\Gamma_{(4,2)}(\mathbf{x}) = \sum_{\pi \in P_{(4,2)}([\![1,6]\!])} T_\pi. \tag{A.23}$$

In the above equation, each T_π is computed so that factors of order 4 come first (e.g. there will be no such term as $m_2(\mathbf{x}) \otimes m_4(\mathbf{x})$). For instance,

$$
\begin{aligned}
&T_{\{\{1;3;5;6\};\{2;4\}\}} \\
&= \sum_{(j_1,\ldots,j_6)\in[\![1,p]\!]^6} mom^{(4)}(x_{j_1},x_{j_3},x_{j_5},x_{j_6}) mom^{(2)}(x_{j_2},x_{j_4}) \mathbf{e}_{j_1}\mathbf{e}'_{j_2}\otimes\mathbf{e}_{j_3}\mathbf{e}'_{j_4}\otimes\mathbf{e}_{j_5}\mathbf{e}'_{j_6} \\
&= \sum_{(j_1,\ldots,j_6)\in[\![1,p]\!]^6} mom^{(4)}(x_{j_1},x_{j_3},x_{j_5},x_{j_6}) \\
&\quad \times mom^{(2)}(x_{j_2},x_{j_4})(\mathbf{e}_{j_1}\otimes\mathbf{e}_{j_3}\otimes\mathbf{e}_{j_5})(\mathbf{e}'_{j_2}\otimes\mathbf{e}'_{j_4}\otimes\mathbf{e}'_{j_6}) \\
&= \sum_{(j_1,\ldots,j_6)\in[\![1,p]\!]^6} mom^{(4)}(x_{j_1},x_{j_3},x_{j_5},x_{j_6}) \\
&\quad \times mom^{(2)}(x_{j_2},x_{j_4})(\mathbf{e}_{j_1}\otimes\mathbf{e}_{j_3}\otimes\mathbf{e}_{j_5})(\mathbf{e}'_{j_6}\otimes\mathbf{e}'_{j_2}\otimes\mathbf{e}'_{j_4})\mathcal{P}_{2,1}\mathcal{P}_{3,2} \\
&= \sum_{(j_1,\ldots,j_6)\in[\![1,p]\!]^6} mom^{(4)}(x_{j_1},x_{j_3},x_{j_5},x_{j_6}) \\
&\quad \times mom^{(2)}(x_{j_2},x_{j_4})(\mathbf{e}_{j_1}\mathbf{e}'_{j_6}\otimes\mathbf{e}_{j_3}\otimes\mathbf{e}_{j_5})\otimes(\mathbf{e}'_{j_2}\otimes\mathbf{e}'_{j_4})\mathcal{P}_{2,1}\mathcal{P}_{3,2} \\
&= \mathbf{M}_4(\mathbf{x})\otimes\mathbf{vec}'\, m_2(\mathbf{x})\mathcal{P}_{2,1}\mathcal{P}_{3,2}. \qquad \text{(A.24)}
\end{aligned}
$$

Following similar lines,

$$T_{\{\{1;2;3;4\};\{5;6\}\}} = m_4(\mathbf{x})\otimes m_2(\mathbf{x}) \qquad \text{(A.25)}$$

$$T_{\{\{1;2;3;5\};\{4;6\}\}} = \mathbf{M}_4(\mathbf{x})\otimes\mathbf{vec}'\, m_2(\mathbf{x}) \qquad \text{(A.26)}$$

$$T_{\{\{1;2;4;5\};\{3;6\}\}} = \mathcal{P}_{3,2}m_4(\mathbf{x})\otimes m_2(\mathbf{x}) \qquad \text{(A.27)}$$

$$T_{\{\{1;3;4;5\};\{2;6\}\}} = \mathbf{M}_4(\mathbf{x})\otimes\mathbf{vec}'\, m_2(\mathbf{x})\mathcal{P}_{2,1} \qquad \text{(A.28)}$$

$$T_{\{\{2;3;4;5\};\{1;6\}\}} = \mathcal{P}_{2,1}\mathcal{P}_{3,2}m_4(\mathbf{x})\otimes m_2(\mathbf{x}) \qquad \text{(A.29)}$$

$$T_{\{\{1;2;3;6\};\{4;5\}\}} = m_4(\mathbf{x})\otimes m_2(\mathbf{x})\mathcal{P}_{3,2} \qquad \text{(A.30)}$$

$$T_{\{\{1;2;4;6\};\{3;5\}\}} = \mathbf{M}_4(\mathbf{x})'\otimes\mathbf{vec}\, m_2(\mathbf{x}) \qquad \text{(A.31)}$$

$$T_{\{\{1;3;4;6\};\{2;5\}\}} = m_4(\mathbf{x})\otimes m_2(\mathbf{x})\mathcal{P}_{3,2}\mathcal{P}_{2,1} \qquad \text{(A.32)}$$

$$T_{\{\{2;3;4;6\};\{1;5\}\}} = \mathcal{P}_{2,1}\mathbf{M}_4(\mathbf{x})'\otimes\mathbf{vec}\, m_2(\mathbf{x}) \qquad \text{(A.33)}$$

$$T_{\{\{1;2;5;6\};\{3;4\}\}} = \mathcal{P}_{3,2}m_4(\mathbf{x})\otimes m_2(\mathbf{x})\mathcal{P}_{3,2} \qquad \text{(A.34)}$$

$$T_{\{\{2;3;5;6\};\{1;4\}\}} = \mathcal{P}_{2,1}\mathcal{P}_{3,2}m_4(\mathbf{x})\otimes m_2(\mathbf{x})\mathcal{P}_{3,2} \qquad \text{(A.35)}$$

$$T_{\{\{1;4;5;6\};\{2;3\}\}} = \mathcal{P}_{3,2}m_4(\mathbf{x})\otimes m_2(\mathbf{x})\mathcal{P}_{3,2}\mathcal{P}_{2,1} \qquad \text{(A.36)}$$

$$T_{\{\{2;4;5;6\};\{1;3\}\}} = \mathcal{P}_{3,2}\mathcal{P}_{2,1}\mathbf{M}_4(\mathbf{x})'\otimes\mathbf{vec}\, m_2(\mathbf{x}) \qquad \text{(A.37)}$$

$$T_{\{\{3;4;5;6\};\{1;2\}\}} = \mathcal{P}_{2,1}\mathcal{P}_{3,2}m_4(\mathbf{x})\otimes m_2(\mathbf{x})\mathcal{P}_{3,2}.\mathcal{P}_{2,1}. \qquad \text{(A.38)}$$

The last step is to collect terms in $m_4(\mathbf{x})\otimes m_2(\mathbf{x})$, $\mathbf{M}_4(\mathbf{x})\otimes\mathbf{vec}'\, m_2(\mathbf{x})$, and $\mathbf{M}_4(\mathbf{x})'\otimes\mathbf{vec}\, m_2(\mathbf{x})$ respectively:

$$\begin{aligned}
\Gamma_{(4,2)}(\mathbf{x}) = {} & \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{R}}\, m_4(\mathbf{x}) \otimes m_2(\mathbf{x}) \\
& + \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{R}}\, m_4(\mathbf{x}) \otimes m_2(\mathbf{x})\mathcal{P}_{3,2}\mathcal{P}_{2,1} \\
& + \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{R}}\, m_4(\mathbf{x}) \otimes m_2(\mathbf{x})\mathcal{P}_{3,2} \\
& + \mathbf{M}_4(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x})\, \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{P}'} \\
& + \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2}\mathcal{P}_{2,1})}_{\mathbf{P}}\, \mathbf{M}_4(\mathbf{x})' \otimes \mathbf{vec}\, m_2(\mathbf{x}) \\
= {} & \mathbf{R}\Big(m_4(\mathbf{x}) \otimes m_2(\mathbf{x})\Big)\mathbf{R}' + \Big(\mathbf{M}_4(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x})\Big)\mathbf{P}' \\
& + \mathbf{P}\Big(\mathbf{M}_4(\mathbf{x})' \otimes \mathbf{vec}\, m_2(\mathbf{x})\Big). \qquad \text{(A.39)}
\end{aligned}$$

<u>Expression of $\Gamma_{(3,3)}(\mathbf{x})$</u>:

It is seen that $P_{(3,3)}([\![1,6]\!])$ contains $\frac{6!}{3!3!2!} = 10$ elements:

$$\begin{aligned}
P_{(3,3)}([\![1,6]\!]) = {} & \{\{\{1;2;3\};\{4;5;6\}\};\{\{1;2;4\};\{3;5;6\}\};\{\{1;3;4\};\{2;5;6\}\}; \\
& \{\{2;3;4\};\{1;5;6\}\};\{\{1;2;5\};\{3;4;6\}\};\{\{1;3;5\};\{2;4;6\}\}; \\
& \{\{2;3;5\};\{1;4;6\}\};\{\{1;4;5\};\{2;3;6\}\};\{\{2;4;5\};\{1;3;6\}\}; \\
& \{\{3;4;5\};\{1;2;6\}\}\} \qquad \text{(A.40)}
\end{aligned}$$

$$\Gamma_{(3,3)}(\mathbf{x}) = \sum_{\pi \in P_{(3,3)}([1,6]]}} T_\pi. \qquad \text{(A.41)}$$

In the above equation, each T_π is computed so that factor $m_3(\mathbf{x})$ comes first (e.g. there will be no such term as $m_3(\mathbf{x})' \otimes m_3(\mathbf{x})$):

$$T_{\{\{1;2;3\};\{4;5;6\}\}} = m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \qquad \text{(A.42)}$$

$$T_{\{\{1;2;4\};\{3;5;6\}\}} = \mathcal{P}_{2,1}\mathcal{P}_{3,2} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})'\mathcal{P}_{2,1}\mathcal{P}_{3,2} \qquad \text{(A.43)}$$

$$T_{\{\{1;3;4\};\{2;5;6\}\}} = m_3(\mathbf{x}) \otimes m_3(\mathbf{x})'\mathcal{P}_{2,1} \qquad \text{(A.44)}$$

$$T_{\{\{2;3;4\};\{1;5;6\}\}} = \mathcal{P}_{3,2} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})'\mathcal{P}_{2,1}\mathcal{P}_{3,2} \qquad \text{(A.45)}$$

$$T_{\{\{1;2;5\};\{3;4;6\}\}} = \mathcal{P}_{3,2} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \qquad \text{(A.46)}$$

$$T_{\{\{1;3;5\};\{2;4;6\}\}} = \mathbf{vec}\, m_3(\mathbf{x}) \otimes \mathbf{vec}'\, m_3(\mathbf{x}) \qquad \text{(A.47)}$$

$$T_{\{\{2;3;5\};\{1;4;6\}\}} = \mathcal{P}_{2,1}\mathcal{P}_{3,2} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \qquad \text{(A.48)}$$

$$T_{\{\{1;4;5\};\{2;3;6\}\}} = \mathcal{P}_{3,2} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})'\mathcal{P}_{2,1} \qquad \text{(A.49)}$$

$$T_{\{\{2;4;5\};\{1;3;6\}\}} = m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \mathcal{P}_{2,1} \mathcal{P}_{3,2} \tag{A.50}$$

$$T_{\{\{3;4;5\};\{1;2;6\}\}} = \mathcal{P}_{2,1} \mathcal{P}_{3,2} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \mathcal{P}_{2,1}. \tag{A.51}$$

Therefore,

$$\begin{aligned}
\Gamma_{(3,3)}(\mathbf{x}) = {} & \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{R}} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \\
& + \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{R}} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \mathcal{P}_{2,1} \\
& + \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2})}_{\mathbf{R}} m_3(\mathbf{x}) \otimes m_3(\mathbf{x})' \mathcal{P}_{2,1}\mathcal{P}_{3,2} + \mathbf{vec}\, m_3(\mathbf{x}) \otimes \mathbf{vec}'\, m_3 \\
= {} & \mathbf{R}\Big(m_3(\mathbf{x}) \otimes m_3(\mathbf{x})'\Big)\mathbf{P}' + \mathbf{vec}\, m_3(\mathbf{x}) \otimes \mathbf{vec}'\, m_3.
\end{aligned} \tag{A.52}$$

<u>Expression of $\Gamma_{(2,2,2)}(\mathbf{x})$:</u>

It is seen that $P_{(2,2,2)}([\![1,6]\!])$ contains $\frac{6!}{2!2!2!3!} = 15$ elements:

$$\begin{aligned}
P_{(2,2,2)}([\![1,6]\!]) = {} & \{\{\{1;2\};\{3;4\};\{5;6\}\};\{\{1;3\};\{2;4\};\{5;6\}\};\{\{2;3\};\{1;4\};\{5;6\}\}; \\
& \{\{1;2\};\{3;5\};\{4;6\}\};\{\{1;3\};\{2;5\};\{4;6\}\};\{\{2;3\};\{1;5\};\{4;6\}\}; \\
& \{\{1;2\};\{4;5\};\{3;6\}\};\{\{1;4\};\{2;5\};\{3;6\}\};\{\{2;4\};\{1;5\};\{3;6\}\}; \\
& \{\{1;3\};\{4;5\};\{2;6\}\};\{\{1;4\};\{3;5\};\{2;6\}\};\{\{3;4\};\{1;5\};\{2;6\}\}; \\
& \{\{2;3\};\{4;5\};\{1;6\}\};\{\{2;4\};\{3;5\};\{1;6\}\};\{\{3;4\};\{2;5\};\{1;6\}\}\}
\end{aligned} \tag{A.53}$$

$$\Gamma_{(2,2,2)}(\mathbf{x}) = \sum_{\pi \in P_{(2,2,2)}([\![1,6]\!])} T_\pi. \tag{A.54}$$

In the above equation, each T_π is computed by imposing the ordering $m_2(\mathbf{x}) < \mathbf{vec}\, m_2(\mathbf{x}) < \mathbf{vec}'\, m_2(\mathbf{x})$, which amounts to placing first blocks with polarities {even}{odd}, then {odd}{odd}, and finally {even}{even}:

$$T_{\{\{1;2\};\{3;4\};\{5;6\}\}} = m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \tag{A.55}$$

$$T_{\{\{1;3\};\{2;4\};\{5;6\}\}} = \mathcal{P}_{3,2}\mathcal{P}_{2,1} m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x}) \mathcal{P}_{2,1}\mathcal{P}_{3,2} \tag{A.56}$$

$$T_{\{\{2;3\};\{1;4\};\{5;6\}\}} = \mathcal{P}_{2,1} m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \tag{A.57}$$

$$T_{\{\{1;2\};\{3;5\};\{4;6\}\}} = m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x}) \tag{A.58}$$

$$T_{\{\{1;3\};\{2;5\};\{4;6\}\}} = \mathcal{P}_{3,2}\mathcal{P}_{2,1} m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x}) \tag{A.59}$$

$$T_{\{\{2;3\};\{1;5\};\{4;6\}\}} = \mathcal{P}_{2,1} m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x}) \tag{A.60}$$

$$T_{\{\{1;2\};\{4;5\};\{3;6\}\}} = \mathcal{P}_{3,2} m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \tag{A.61}$$

$$T_{\{\{1;4\};\{2;5\};\{3;6\}\}} = \mathcal{P}_{3,2}\mathcal{P}_{2,1} m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \tag{A.62}$$

$$T_{\{\{2;4\};\{1;5\};\{3;6\}\}} = \mathcal{P}_{2,1} m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x}) \mathcal{P}_{2,1}\mathcal{P}_{3,2} \tag{A.63}$$

$$T_{\{\{1;3\};\{4;5\};\{2;6\}\}} = \mathcal{P}_{3,2}\mathcal{P}_{2,1} m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x}) \mathcal{P}_{2,1} \tag{A.64}$$

$$T_{\{\{1;4\};\{3;5\};\{2;6\}\}} = m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x}) \mathcal{P}_{2,1} \tag{A.65}$$

$$T_{\{\{3;4\};\{1;5\};\{2;6\}\}} = \mathcal{P}_{2,1} m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x}) \mathcal{P}_{2,1} \tag{A.66}$$

$$T_{\{\{2;3\};\{4;5\};\{1;6\}\}} = \mathcal{P}_{2,1}\mathcal{P}_{3,2} m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \tag{A.67}$$

$$T_{\{\{2;4\};\{3;5\};\{1;6\}\}} = m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x}) \mathcal{P}_{2,1}\mathcal{P}_{3,2} \tag{A.68}$$

$$T_{\{\{3;4\};\{2;5\};\{1;6\}\}} = \mathcal{P}_{2,1}\mathcal{P}_{3,2}\mathcal{P}_{2,1} m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}). \tag{A.69}$$

Therefore,

$$\begin{aligned}
\Gamma_{(2,2,2)}(\mathbf{x}) = {} & \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2} + \mathcal{P}_{3,2}\mathcal{P}_{2,1} + \mathcal{P}_{2,1}\mathcal{P}_{3,2} + \mathcal{P}_{2,1}\mathcal{P}_{3,2}\mathcal{P}_{2,1})}_{\mathbf{Q}} \\
& \times m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \\
& + \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2}\mathcal{P}_{2,1})}_{\mathbf{P}} m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x}) \\
& + \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2}\mathcal{P}_{2,1})}_{\mathbf{P}} m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x}) \mathcal{P}_{2,1} \\
& + \underbrace{(\mathbf{I}_{p^3} + \mathcal{P}_{2,1} + \mathcal{P}_{3,2}\mathcal{P}_{2,1})}_{\mathbf{P}} m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x}) \mathcal{P}_{2,1}\mathcal{P}_{3,2} \\
= {} & \mathbf{Q}\Big(m_2(\mathbf{x}) \otimes m_2(\mathbf{x}) \otimes m_2(\mathbf{x})\Big) + \mathbf{P}\Big(m_2(\mathbf{x}) \otimes \mathbf{vec}\, m_2(\mathbf{x}) \otimes \mathbf{vec}'\, m_2(\mathbf{x})\Big)\mathbf{P}'.
\end{aligned} \tag{A.70}$$

This completes the proof of Theorem 1.

References

[1] A. Hald, The early history of the cumulants and the Gram–Charlier series, Int. Stat. Rev. 68 (2000) 137–153.

[2] V. Leonov, A. Shiryaev, On a method of calculation of semi-invariants, Theory Probab. Appl. 4 (1959) 319–329.

[3] E. Lukacs, Applications of Faà di Bruno's formula in mathematical statistics, Amer. Math. Monthly 62 (1955) 340–348.

[4] P. Comon, Independent component analysis, a new concept?, Signal Process. 36 (1994) 287–314.

[5] J.-F. Cardoso, A. Souloumiac, Blind beamforming for non gaussian signals, IEE Proc. F 140 (1993) 362–370.

[6] L. Albera, A. Ferreol, P. Chevalier, P. Comon, ICAR, a tool for blind source separation using fourth-order statistics only, IEEE Trans. Signal Process. 53 (2005) 3633–3643.

[7] L. Albera, A. Ferreol, P. Comon, P. Chevalier, Sixth order blind identification of underdetermined mixtures (birth) of sources, in: 4th International Symposium on Independent Component Analysis and Blind Signal Separation (ICA2003), Nara, Japan, pp. 909–914.
[8] L. Albera, A. Ferreol, P. Comon, P. Chevalier, Blind identification of over-complete mixtures of sources (BIOME), Linear Algebra Appl. 391 (2004) 1–30.
[9] T. Kollo, Multivariate skewness and kurtosis measures with an application in ICA, J. Multivariate Anal. 99 (2008) 2328–2338.
[10] T.P. Speed, Cumulants and partition lattices, Aust. J. Stat. 25 (1983) 378–388.
[11] P. Mac Cullagh, Tensor Methods in Statistics, Monogr. Statist. Appl. Probab., Chapman and Hall, 1987.
[12] H. Neudecker, Some theorems on matrix differentiation with special reference to Kronecker matrix products, J. Amer. Statist. Assoc. 64 (1969) 953–963.
[13] T. Kollo, D. von Rosen, Advanced Multivariate Statistics with Matrices, Springer, 2005.
[14] K.G. Jinadasa, D.S. Tracy, Higher order moments of random vectors using matrix derivatives, Stoch. Anal. Appl. 4 (1986) 399–407.
[15] D.S. Tracy, S.A. Sultan, Higher order moments of multivariate normal distribution using matrix derivatives, Stoch. Anal. Appl. 11 (1993) 337–348.
[16] N. Loperfido, A note on the fourth cumulant of a finite mixture distribution, J. Multivariate Anal. 123 (2014) 386–394.
[17] M. Hardy, Combinatorics of partial derivatives, Electron. J. Combin. 13 (2006) 1–13.
[18] J.R. Magnus, H. Neudecker, The commutation matrix: some properties and applications, Ann. Statist. 7 (1979) 381–394.
[19] J.R. Schott, Kronecker product permutation matrices and their application to moment matrices of the normal distribution, J. Multivariate Anal. 87 (2003) 177–190.
[20] P.J. Cameron, Combinatorics: Topics, Techniques, Algorithms, Cambridge University Press, 1994.
[21] E. Meijer, Matrix algebra for higher order moments, Linear Algebra Appl. 410 (2005) 112–134.
[22] L. Qi, W. Sun, Y. Wang, Numerical multilinear algebra and its applications, Front. Math. China 2 (2007) 501–526.
[23] J.W. Brewer, Kronecker products and matrix calculus in system theory, IEEE Trans. Circuits Syst. CAS-25 (1978) 772–781.
[24] D.P. Wiens, On moments of quadratic forms in non-spherically distributed variables, Statistics 23 (1992) 265–270.
[25] H. Ould-Baba, Analyse en composantes indépendantes multidimensionnelles via des cumulants d'ordres variés, PhD thesis, Université de Technologie de Compiègne, 2014.
[26] N. Loperfido, Spectral analysis of the fourth moment matrix, Linear Algebra Appl. 435 (2011) 1837–1844.
[27] N. Loperfido, Singular value decomposition of the third multivariate moment, Linear Algebra Appl. 473 (2014) 202–216.