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ON THE ESSENTIAL SPECTRUM OF N-BODY HAMILTONIANS WITH ASYMPTOTICALLY HOMOGENEOUS INTERACTIONS

June 11, 2015

VLADIMIR GEORGESCU AND VICTOR NISTOR

ABSTRACT. We determine the essential spectrum of N -body Hamiltonians with 2-body (or, more generally, k -body) potentials that have radial limits at infinity. The classical N -body Hamiltonians appearing in the well known HVZ-theorem are a particular case of this type of potentials corresponding to zero limits at infinity. Our result thus extends the HVZ-theorem that describes the essential spectrum of the usual N -body Hamiltonians. More precisely, if the configuration space of the system is a finite dimensional real vector space X , then let $\mathcal{E}(X)$ be the C^* -algebra of functions on X generated by the algebras $\mathcal{C}(\overline{X/Y})$, where Y runs over the set of all linear subspaces of X and $\mathcal{C}(\overline{X/Y})$ is the space of continuous functions on X/Y that have radial limits at infinity. This is the algebra used to define the potentials in our case, while in the classical case the $\mathcal{C}(\overline{X/Y})$ are replaced by $\mathcal{C}_0(X/Y)$. The proof of our main results is based on the study of the structure of the algebra $\mathcal{E}(X)$, in particular, we determine its character space and the structure of its cross-product $\mathcal{E}(X) := \mathcal{E}(X) \rtimes X$ by the natural action τ of X on $\mathcal{E}(X)$. Our techniques apply also to more general classes of Hamiltonians that have a many-body type structure. We allow, in particular, potentials with local singularities and more general behaviours at infinity. We also develop general techniques that may be useful for other operators and other types of questions, such as the approximation of eigenvalues.

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1. INTRODUCTION

Let X be a finite dimensional, real vector space. In this paper, we study Hamiltonians of the form

$$H := h(p) + \sum_Y v_Y \circ \pi_Y, \quad (1.1)$$

where $h : X^* \rightarrow \mathbb{R}$ is a continuous proper function, p is the momentum observable, π_Y is the canonical projection of X onto the quotient space X/Y , and v_Y is a suitable function on X/Y . The precise assumptions are given below. Our results lead, in particular, to a determination of the essential spectrum of H and of some more general operators. We also develop general techniques that may be useful for the study of other operators and of other types of questions, such as the approximation of eigenvalues.

To give a flavor of the nature of our results, let us state now one of our main results in its most elementary form. Let X^* denote the space dual to X , let \mathcal{F} denote the Fourier transform $L^2(X) \rightarrow L^2(X^*)$, and let M_h denote the operator of multiplication by h . Then $h(p)$ is the operator on $L^2(X)$ defined by

$$h(p) := \mathcal{F}^{-1} M_h \mathcal{F}. \quad (1.2)$$

Recall that a function $h : X^* \rightarrow \mathbb{R}$ is said to be *proper* if $h(k) \rightarrow \infty$ for $k \rightarrow \infty$. Throughout this paper, $h : X^* \rightarrow \mathbb{R}$ will be an arbitrary proper, continuous function.

Assume that, for each subspace $Y \subset X$, a bounded Borel function $v_Y : X/Y \rightarrow \mathbb{R}$ is given such that $v_Y = 0$ for all but a finite number of subspaces Y . We also assume that $\lim_{r \rightarrow +\infty} v_Y(ra)$ exists uniformly for a in a compact subset of $X/Y \setminus \{0\}$. Let us denote $\hat{a} := \{ra, r > 0\}$, with $a \in X$. Then \mathbb{S}_X is defined to be the set of half-lines in X , that is

$$\mathbb{S}_X := \{\hat{a}, a \in X, a \neq 0\}. \quad (1.3)$$

For $x \in X$, we denote by T_x the unitary translation operator on $L^2(X)$ defined by

$$(T_x f)(y) := f(y - x). \quad (1.4)$$

We also denote by $\overline{\cup S_\alpha}$ the *closure of the union* of a family of sets $\{S_\alpha\}$ and $V_Y := v_Y \circ \pi_Y$. In the following result, we assume that the potentials are bounded only for the sake of simplicity.

Theorem 1.1. *Let $H := h(p) + \sum_Y V_Y$, with h and V_Y as above. Then the strong limit $\hat{a}.H := \text{s-lim}_{r \rightarrow +\infty} T_{ra}^* H T_{ra}$ exists for any $a \in X$ and the essential spectrum of H is given by*

$$\sigma_{\text{ess}}(H) = \overline{\cup_{\alpha \in \mathbb{S}_X} \sigma(\alpha.H)}. \quad (1.5)$$

For each α , we have $\sigma(\alpha.H) = [E_\alpha, \infty)$, for some real E_α , and $\sigma_{\text{ess}}(H) = [\inf_\alpha E_\alpha, \infty)$.

The strong limit in the above theorem is defined in the usual sense of pointwise convergence on the domain of H (which is left invariant by the T_{ra} under the present assumptions). Operators the form (1.1) (and hence also Theorem 1.1 and its generalizations) cover many of the most interesting (from a physical point of view) Hamiltonians of N -body systems. Here are two typical examples. First, in the non-relativistic case, X is equipped with a Euclidean structure and a popular choice for h is $h(\xi) = |\xi|^2$, which gives $h(p) = -\Delta$, the positive Laplacian. Second, in the simplest relativistic case, we have again $X = (\mathbb{R}^3)^N$ and, writing the momentum p as $p = (p_1, \dots, p_N)$, a popular choice for h is $h(p) = \sum_{j=1}^N (p_j^2 + m_j^2)^{1/2}$ for some real m_j . We refer to [10] for a thorough study of the classical spectral and scattering theory of the non-relativistic N -body Hamiltonians with potentials v_Y that tend to zero at infinity.

In the main body of the paper, by using crossed-products of C^* -algebras, we shall obtain results more general than Theorem 1.1 in several ways. For example, we shall consider

also the case when an infinite number of non zero functions v_Y is allowed. Also, we shall allow for some unbounded functions v_Y .

Theorem 1.1 may be reformulated in a way that stresses the similarity with the HVZ theorem for usual N -body type Hamiltonians. To this end, we shall use the fact that, if $\alpha = \widehat{a} \not\subset Y$, then $\pi_Y(\alpha) \in \mathbb{S}_{X/Y}$ is well defined as the half-line determined by the non zero vector $\pi_Y(a)$ and we may naturally define

$$V_Y(\alpha) = \lim_{r \rightarrow +\infty} v_Y(\pi_Y(ra)).$$

Proposition 1.2. *For each $\alpha \in \mathbb{S}_X$, let*

$$H_\alpha := h(P) + \sum_{Y \supset \alpha} V_Y + \sum_{Y \not\supset \alpha} V_Y(\alpha). \quad (1.6)$$

Then $\alpha.H = H_\alpha$, and hence $\sigma_{\text{ess}}(H) = \overline{\cup_{\alpha \in \mathbb{S}_X} \sigma(H_\alpha)}$.

In our approach, the usual N -body type Hamiltonians are characterized by the condition that $\lim_{r \rightarrow +\infty} v_Y(ra) = 0$ for all $a \in X/Y, a \neq 0$. Assuming that this condition holds for our potentials v_Y , we obtain that $\alpha.H = h(P) + \sum_{Y \supset \alpha} V_Y$, and hence Proposition 1.2 becomes the usual version of the HVZ theorem.

Descriptions of the essential spectrum of various classes of self-adjoint operators in terms of limits at infinity of translates of the operators have already been obtained before, see for example [23, 33, 17, 25] (in historical order). Our approach is based on the ‘‘localization at infinity’’ technique developed in [17, 18] in the context of crossed-product C^* -algebras. See [7, 8, 34] for a general introduction to the basics of the problems studied here.

1.1. Statement of main results. We introduce now a framework which allows us to define and classify N -body Hamiltonians in terms of the complexity of the 2-body interactions.

If X is a finite dimensional vector space, we denote by $\mathcal{C}_b(X)$ the algebra of bounded continuous functions on X , by $\mathcal{C}_0(X)$ its ideal consisting of functions vanishing at infinity, and by $\mathcal{C}_b^u(X)$ the subalgebra of bounded *uniformly continuous* functions. Let $\mathcal{B}(X) := \mathcal{B}(L^2(X))$ be the algebra of bounded operators on $L^2(X)$ and $\mathcal{K}(X) := \mathcal{K}(L^2(X))$ the ideal of compact operators.

If Y is a subspace of X , we identify a function f on X/Y with the function $f \circ \pi_Y$ on X . In other terms, we can think of a function on X/Y as being a function on X that is invariant under translations by elements of Y . This clearly gives an embedding $\mathcal{C}_b^u(X/Y) \subset \mathcal{C}_b^u(X)$. The subalgebras of $\mathcal{C}_b^u(X/Y)$ can then be thought of as subalgebras of $\mathcal{C}_b^u(X)$. Thus $\mathcal{C}_0(X/Y)$ and the algebra $\mathcal{C}(\overline{X/Y})$ that we shall introduce below are both embedded in $\mathcal{C}_b^u(X)$.

Assume that, for each finite dimensional real vector space E , a norm closed, translation and conjugation invariant subalgebra $\mathcal{P}(E)$ of $\mathcal{C}_b^u(E)$ has been specified (the letter \mathcal{P} should suggest ‘‘potentials’’). Our assumptions on $\mathcal{P}(E)$ simply mean that it is a C^* -subalgebra of $\mathcal{C}_b^u(E)$. Then, for each subspace $Y \subset X$, we get a translation invariant subalgebra $\mathcal{P}(X/Y) \subset \mathcal{C}_b^u(X)$. Let us denote by $\langle A_\alpha, \alpha \in I \rangle$ the norm closed subalgebra generated by a family $\{A_\alpha\}_{\alpha \in I}$ of sets $A_\alpha \subset \mathcal{C}_b^u(X)$. Then we let

$$\mathcal{R}_{\mathcal{P}}(X) := \langle \mathcal{P}(X/Y), Y \subset X \rangle \quad \text{and} \quad \mathcal{B}_{\mathcal{P}}(X) := \mathcal{R}_{\mathcal{P}}(X) \rtimes X. \quad (1.7)$$

Thus $\mathcal{R}_{\mathcal{P}}(X)$ is the norm-closed subalgebra of $\mathcal{C}_b^u(X)$ generated by the $\mathcal{P}(X/Y)$, where Y runs over the set of *all linear subspaces* of X . It is also a translation invariant C^* -subalgebra of $\mathcal{C}_b^u(X)$. We shall regard the crossed product $\mathcal{B}_{\mathcal{P}}(X) := \mathcal{R}_{\mathcal{P}}(X) \rtimes X$, as a C^* -subalgebra of $\mathcal{B}(X)$. Its structure will play a crucial role in what follows. For instance, for our approach, it will be convenient to assume that $\mathcal{C}_0(E) \subset \mathcal{P}(E)$ and $\mathcal{P}(0) = \mathbb{C}$.

Then $\mathcal{R}_{\mathcal{P}}(X)$ contains $C^*(X)$ (group C^* -algebra) and $\mathcal{K}(X)$, since $\mathcal{C}_0(X) \subset \mathcal{P}(X)$ and $\mathcal{C}_0(X) \rtimes X = \mathcal{K}(X)$.

For reasons that will become apparent below, it will be natural to call $\mathcal{R}_{\mathcal{P}}(X)$ the *algebra of elementary interactions of type \mathcal{P}* and $\mathcal{R}_{\mathcal{P}}(X) \rtimes X$ the *algebra of N -body type Hamiltonians with interactions of type \mathcal{P}* . Indeed, $\mathcal{R}_{\mathcal{P}}(X)$ is the C^* -algebra of operators on $L^2(X)$ generated by the resolvents of the self-adjoint operators of the form $h(p) + V$, with $h : X^* \rightarrow \mathbb{R}_+$ continuous and proper, and $V \in \mathcal{R}_{\mathcal{P}}(X)$ [18, Proposition 3.3]. More generally, the *N -body Hamiltonians with interactions of type \mathcal{P}* that are of interest in this context are the self-adjoint operators affiliated to $\mathcal{R}_{\mathcal{P}}(X)$.

The ‘‘standard’’ N -body situation, as described for example in [9, Sec. 4] and [18, Sec. 6.5], corresponds to the choice $\mathcal{P}(E) = \mathcal{C}_0(E)$, the subalgebra of continuous functions on E that vanish at infinity. The algebra of elementary interactions $\mathcal{R}_{\mathcal{P}}(X) = \mathcal{R}_{\mathcal{C}_0}(X)$ in this case has a remarkable feature: it is graded by the ordered set of all linear subspaces of X , more precisely $\mathcal{R}_{\mathcal{C}_0}(X)$ is the norm closure of $\sum_{Y \subset X} \mathcal{C}_0(X/Y)$. This sum is direct and we have $\mathcal{C}_0(X/Y)\mathcal{C}_0(X/Z) \subset \mathcal{C}_0(X/(Y \cap Z))$. Let $\mathcal{R}_{\mathcal{C}_0}(X)$ the corresponding algebra of N -body Hamiltonians with interactions of type \mathcal{C}_0 , which inherits a graded C^* -algebra structure [27, 28]. The standard N -body Hamiltonians are self-adjoint operators affiliated to $\mathcal{R}_{\mathcal{C}_0}(X)$, and their analysis is greatly simplified by the existence of the grading.

For any real vector space E , let us denote by \overline{E} the spherical compactification of E and by $\overline{\mathcal{C}}(E) = \mathcal{C}(\overline{E})$. Our main goals in this paper is to consider the larger class of interactions $\mathcal{P}(E) = \overline{\mathcal{C}}(E)$ and to analyze the N -body Hamiltonians associated to them. We thus consider potentials that have radial limits at infinity. To make the notation clearer, when $\mathcal{P}(E) = \overline{\mathcal{C}}(E)$, we shall denote

$$\mathcal{E}(X) := \mathcal{R}_{\overline{\mathcal{C}}}(X) := \langle \overline{\mathcal{C}}(X/Y), Y \subset X \rangle \subset \mathcal{C}_b^u(X), \quad Y \subset X. \quad (1.8)$$

One of the main difficulties when $\mathcal{P}(E) = \overline{\mathcal{C}}(E) = \mathcal{C}(\overline{E})$ comes from the absence of a grading of the resulting algebra $\mathcal{E}(X)$ of elementary interactions, which requires more care in understanding its spectrum. We denote $\mathcal{E}(X) \rtimes X = \mathcal{R}_{\overline{\mathcal{C}}}(X)$ the associated C^* -algebra of N -body type Hamiltonians with interactions of type $\overline{\mathcal{C}}$. Another natural choice, which would give an even larger class of elementary interactions and of N -body type Hamiltonians, is to take as $\mathcal{P}(E)$ the algebra of slowly oscillating functions on E , a class of functions whose importance has been pointed out by H.O. Cordes [18, Sec. 6.2]. The general algebras $\mathcal{R}(X)$ and $\mathcal{R}(X) \rtimes X$ will not be used in this paper, but rather their particular versions $\mathcal{E}(X)$ and $\mathcal{E}(X) \rtimes X$ obtained by specializing $\mathcal{P}(X) = \overline{\mathcal{C}}(X) = \mathcal{C}(\overline{X})$.

A crucial observation is then that $\mathcal{E}(X)$ contains the ideal $\mathcal{C}_0(X) \rtimes X \simeq \mathcal{K}(X)$, where we recall that $\mathcal{K}(X)$ denotes the ideal of compact operators on $L^2(X)$. The algebra $\mathcal{E}(X)$ also contains the C^* -algebra $\mathcal{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$ consisting of two-body type operators; we call it the *spherical algebra*. For each $\alpha \in \mathbb{S}_X$ we shall denote by $[\alpha]$ the linear subspace (a line in this case) that it generates. An algebra that will play a role in the theory is the algebra $\mathcal{E}(X/[\alpha])$ defined for each $\alpha \in \mathbb{S}_X$ by

$$\mathcal{E}(X/[\alpha]) = C^*\text{-subalgebra of } \mathcal{C}_b^u(X) \text{ generated by } \mathcal{C}(\overline{X/Y}) \text{ with } \alpha \subset Y. \quad (1.9)$$

We recall that a self-adjoint operator H on a Hilbert space \mathcal{H} is *affiliated to a C^* -algebra* $\mathcal{C} \subset B(\mathcal{H})$ if $(H - z)^{-1} \in \mathcal{C}$ for some number z outside the spectrum of H . This notion and the meaning of the strong limit which defines $\tau_\alpha(H)$ below are further discussed at the end of Section 2, see Remark 2.9. The next theorem is the main result of this paper.

Theorem 1.3. *Let $A \in \mathcal{E}(X)$ see 1.8. Then, for each $\alpha \in \mathbb{S}_X$ and $a \in \alpha$, the limit $\alpha.A := s\text{-}\lim_{r \rightarrow +\infty} T_{ra}^* A T_{ra}$ exists and is independent of the choice of a . The resulting map $\tau_\alpha(A) := \alpha.A$ is a morphism of algebras and a linear projection of $\mathcal{E}(X)$ onto its*

subalgebra $\mathcal{E}(X/[\alpha])$. An operator $A \in \mathcal{E}(X)$ is compact if, and only if, $\tau_\alpha(A) = 0$ for all $\alpha \in \mathbb{S}_X$. Consequently, $\tau(A) := (\tau_\alpha(A))_{\alpha \in \mathbb{S}_X}$ induces an injective morphism

$$\mathcal{E}(X)/\mathcal{K}(X) \hookrightarrow \prod_{\alpha \in \mathbb{S}_X} \mathcal{E}(X/[\alpha]) \rtimes X. \quad (1.10)$$

If H is a self-adjoint operator affiliated to $\mathcal{E}(X)$ then for each $\alpha \in \mathbb{S}_X$ and $a \in \alpha$ the limit $\alpha.H = \text{s-lim}_{r \rightarrow +\infty} T_{ra}^* H T_{ra}$ exists and $\sigma_{\text{ess}}(H) = \overline{\bigcup_{\alpha \in \mathbb{S}_X} \sigma(\alpha.H)}$.

Remark 1.4. The union above may contain an infinite number of distinct terms $\sigma(\alpha.H)$ even in simple cases of N -body type. For example, this is the case if $H = \Delta + V$ with Δ the Laplacian associated to an Euclidean structure on X and $V \in \mathcal{E}(X)$ and also if V is a generic element of the closed subalgebra generated by the $\mathcal{C}_0(X/Y)$. We give a simple but nontrivial explicit example: let $X = \mathbb{R}^2$, \mathcal{Y} a countable set of lines whose union is dense in X , for each $Y \in \mathcal{Y}$ let $v_Y \in \mathcal{C}(\overline{X/Y})$ such that $\sum_Y \sup |v_Y| < \infty$, and $V = \sum_Y v_Y \circ \pi_Y$.

We give now examples of self-adjoint operators affiliated to $\mathcal{E}(X)$. If $k \in X^*$ we denote M_k the unitary operator on $L^2(X)$ given by $(M_k f)(x) = e^{i(x|k)} f(x)$. We denote $|\cdot|$ a quadratic norm on X^* . Theorem 1.1 is an immediate consequence of the following proposition and of Theorem 1.3.

Proposition 1.5. Let $H = h(p) + V$, where $h : X^* \rightarrow [0, \infty)$ is a continuous, proper function and $V = \sum_Y V_Y$ is a finite sum with V_Y bounded symmetric linear operators on $L^2(X)$ satisfying:

- (i) $\lim_{k \rightarrow 0} \|[M_k, V_Y]\| = 0$,
- (ii) $[T_y, V_Y] = 0$ for all $y \in Y$,
- (iii) $\text{s-lim}_{a \in X/Y, a \rightarrow \alpha} T_a^* V_Y T_a$ exists for each $\alpha \in \mathbb{S}_{X/Y}$.

Then H is affiliated to $\mathcal{E}(X)$.

We have to explain the meaning of the limit in (3) above. We have $\alpha \in \mathbb{S}_{X/Y}$, which is the boundary of X/Y in its compactification $\overline{X/Y}$. Note also that $T_x^* V_Y T_x$ depends a priori on the point x in X , and hence $T_a^* V_Y T_a$ makes no sense in general for $a \in X/Y$. However, if the condition (2) is satisfied, then $T_x^* V_Y T_x$ depends, in fact, only on the class $\pi_Y(x)$ of x in the quotient X/Y . Therefore we may set $T_{\pi_Y(x)}^* V_Y T_{\pi_Y(x)} = T_x^* V_Y T_x$ which gives a meaning to $T_a^* V_Y T_a$ for any $a \in X/Y$.

We now discuss a result that allows for certain unbounded interactions. We shall denote by $\mathcal{H}^s = \mathcal{H}^s(X)$ the usual Sobolev spaces on X defined for any real s .

Theorem 1.6. Let $h : X^* \rightarrow [0, \infty)$ be locally Lipschitz with derivative h' such that for some real numbers $c, s > 0$ and all $k \in X^*$ with $|k| > 1$ one has:

$$c^{-1}|k|^{2s} \leq h(k) \leq c|k|^{2s} \quad \text{and} \quad |h'(k)| \leq c(1 + |k|^{2s}). \quad (1.11)$$

Let $V = \sum V_Y$ be a finite sum with $V_Y : \mathcal{H}^s \rightarrow \mathcal{H}^{-s}$ symmetric operators satisfying:

- (i) for each $\mu > 0$ there is a real number ν that $V_Y \geq -\mu h(p) - \nu$,
- (ii) $\lim_{k \rightarrow 0} \|[M_k, V_Y]\|_{\mathcal{H}^s \rightarrow \mathcal{H}^{-s}} = 0$,
- (iii) $[T_y, V_Y] = 0$ for all $y \in Y$,
- (iv) $\text{s-lim}_{a \in \mathbb{S}_{X/Y}, a \rightarrow \alpha} T_a^* V_Y T_a$ exists in $B(\mathcal{H}^s, \mathcal{H}^{-s})$ for all $\alpha \in \mathbb{S}_{X/Y}$.

Then $h(p) + V$ is a symmetric operator $\mathcal{H}^s \rightarrow \mathcal{H}^{-s}$, which induces a self-adjoint operator H in $L^2(X)$ affiliated to $\mathcal{E}(X)$.

For the case when V_Y are multiplication operators, we obtain the following.

Remark 1.7. If V_Y is the operator of multiplication with a measurable function, then Condition (2) of Theorem 1.6 is automatically satisfied. On the other hand, Condition (3) gives that $V_Y(x + y) = V_Y(x)$ for all $x \in X$ and $y \in Y$. This means that $V_Y = v_Y \circ \pi_Y$ for a Borel measurable function $v_Y : X/Y \rightarrow \mathbb{R}$, which has to be such that the operator of multiplication by $v_Y \circ \pi_Y$ is a continuous map $\mathcal{H}^s(X) \rightarrow \mathcal{H}^{-s}(X)$. For this it suffices that the operator $v_Y(q_Y)$ of multiplication by v_Y be a continuous map $\mathcal{H}^s(X/Y) \rightarrow \mathcal{H}^{-s}(X/Y)$. The last condition then states that $\text{s-lim}_{a \rightarrow \alpha} v_Y(q_Y + a)$ exists strongly in $\mathcal{B}(\mathcal{H}^s(X/Y), \mathcal{H}^{-s}(X/Y))$.

Recall that a ‘‘two-body interaction’’ is a potential $V = V(q_i - q_j)$, that thus depends only on the relative positions q_i and q_j of the particles i and j . We thus see that our interactions are not necessarily two-body interactions but they are general k -body interactions.

Example 1.8. Let us assume that $X = \mathbb{R}^n$ and let $j = (j_1, \dots, j_n)$ be a multi-index. Then we set $|j| = j_1 + \dots + j_n$ and $p^j = p_1^{j_1} \dots p_n^{j_n}$ with $p_1 = -i\partial_1$, etc. Theorem 1.6 covers uniformly elliptic operators $H = \sum_{|j|, |k| \leq s} p^j a_{jk} p^k$ whose coefficients a_{jk} are finite sums of functions of the form $v_Y \circ \pi_Y$ with $v_Y : X/Y \rightarrow \mathbb{R}$ bounded measurable and such that $\lim_{z \rightarrow \alpha} v_Y(z)$ exists for each $\alpha \in \mathbb{S}_{X/Y}$. Note that we may allow the a_{jk} to be irregular (just bounded and measurable) in the principal part of the operator (terms with $|j| = |k| = s$). Clearly the coefficients of the lower order terms can be unbounded.

It is remarkable that the spherical algebra $\mathcal{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$ may be described in quite explicit terms. In the next theorem and in what follows, we adopt the following convention: if we write $S^{(*)}$ in a relation, then it means that that relation holds for $S^{(*)}$ replaced by either S or S^* .

Theorem 1.9. *The spherical algebra $\mathcal{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$ consists of the operators $S \in \mathcal{B}(X)$ that have the properties*

$$\lim_{x \rightarrow 0} \|(T_x - 1)S^{(*)}\| = 0, \quad \lim_{k \rightarrow 0} \|[M_k, S]\| = 0, \quad \text{and}$$

$$\text{s-lim}_{a \rightarrow \alpha} T_a^* S^{(*)} T_a \quad \text{exists for any } \alpha \in \mathbb{S}_X.$$

If $S \in \mathcal{S}(X)$ and $\alpha \in \mathbb{S}_X$ then $\text{s-lim}_{a \rightarrow \alpha} T_a^* S T_a = \tau_\alpha(S)$ and $\tau_\alpha(S) \in C^*(X)$. The map $\tau(S) : \alpha \mapsto \tau_\alpha(S)$ is norm continuous, so $\tau : \mathcal{S}(X) \rightarrow C(\mathbb{S}_X) \otimes C^*(X)$. This map τ is a surjective morphism and its kernel is the $\mathcal{K}(X)$. Hence we have a natural identification

$$\mathcal{S}(X)/\mathcal{K}(X) \cong C(\mathbb{S}_X) \otimes C^*(X) \cong \mathcal{C}_0(\mathbb{S}_X \times X^*). \quad (1.12)$$

Let H be a self-adjoint operator affiliated to $\mathcal{S}(X)$. Then for each $\alpha \in \mathbb{S}_X$ the limit $\alpha.H := \text{s-lim}_{a \rightarrow \alpha} T_a^* H T_a$ exists and $\sigma_{\text{ess}}(H) = \cup_\alpha \sigma(\alpha.H)$.

The next result is a general criterion of affiliation to $\mathcal{S}(X)$ for semi-bounded operators.

Theorem 1.10. *Let H be a bounded from below self-adjoint operator on $L^2(X)$ such that its form domain \mathcal{G} satisfies the following condition: the operators T_x and M_k leave \mathcal{G} invariant, the operators T_x are uniformly bounded in \mathcal{G} , and $\lim_{x \rightarrow 0} \|T_x - 1\|_{\mathcal{G} \rightarrow \mathcal{H}} = 0$. Assume that $\|[M_k, H]\|_{\mathcal{G} \rightarrow \mathcal{G}^*} \rightarrow 0$ as $k \rightarrow 0$ and that the limit $\alpha.H := \lim_{a \rightarrow \alpha} T_a^* H T_a$ exists strongly in $\mathcal{B}(\mathcal{G}, \mathcal{G}^*)$, for all $\alpha \in \mathbb{S}_X$. Then H is affiliated to $\mathcal{S}(X)$, for each $\alpha \in \mathbb{S}_X$ the operator in $L^2(X)$ associated to $\alpha.H$ is self-adjoint, and $\sigma_{\text{ess}}(H) = \cup_\alpha \sigma(\alpha.H)$.*

Observe that these two-body type results go beyond the usual Schrödinger operator framework hence could be useful in the study of dispersive partial differential equations, see for example the recent preprint [4] and references therein.

Let us notice an important difference between Theorems 1.3 and 1.10. In the first mentioned theorem, one considers limits over ra , with $r \rightarrow \infty$, that is limits along rays,

whereas in the second mentioned theorem, one considers general limits $a \rightarrow \alpha$ (not just along the ray corresponding to α). The stronger assumptions in Theorem 1.10 then lead to a stronger result.

1.2. Contents of the paper. Let us briefly describe the contents of the paper. In Section 2 we recall some facts concerning crossed products of translation invariant C^* -subalgebras of $C_b^u(X)$ by the action of X and the role of operators with the ‘‘position-momentum limit property’’ in this context. Then we discuss the question of the computation of the quotient with respect to the compacts of such crossed products. In Section 3 we briefly describe the topology and the continuous functions on the spherical compactification \overline{X} of a real vector space X . This allows us to introduce and study in Section 4 the spherical algebra $\mathcal{S}(X) := C(\overline{X}) \rtimes X$. We obtain an explicit description of the operators which belong to $\mathcal{S}(X)$ (Theorem 4.2) and we also give an explicit description of the quotient $\mathcal{S}(X)/\mathcal{K}(X)$ (Theorem 4.3). The canonical composition series of this algebra leads to Fredholm conditions and a determination of the essential spectrum of the operators affiliated to it. In Section 5 we give some general criteria for a self-adjoint operator to be affiliated to a general C^* -algebra and apply them to the case of $\mathcal{S}(X)$. The algebras $\mathcal{E}(X)$ and $\mathcal{E}^*(X)$ are studied in Section 6. At a technical level the main result here is the description of the spectrum of $\mathcal{E}(X)$ (Theorem 6.14). Subsection 6.3 is devoted to the study of the Hamiltonian algebra $\mathcal{E}^*(X)$: we prove there our main results, Theorems 6.18 and 6.21. In Subsection 6.4, we describe a general class of operators affiliated to $\mathcal{E}^*(X)$ (Theorem 6.28), for which we thus obtain an explicit description of the essential spectrum. Note that Theorem 6.24 gives a description of the algebras $C(\overline{X/Y}) \rtimes X$ that generate $\mathcal{E}^*(X)$ independent of their definition as crossed products.

This paper contains the full proofs of the results announced in [19], as well as some extensions of those results. Further results, especially related to the topology on the spectrum of the algebra $\mathcal{E}(X)$, will be included in [20].

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2. CROSSED PRODUCTS AND LOCALIZATIONS AT INFINITY

In this section, we review some needed results from [18] relating essential spectra of operators and the spectrum (or character space) of some algebras.

For any function u , we shall denote by M_u the operator of multiplication by u on suitable L^2 spaces. If $u : X \rightarrow \mathbb{C}$ and $v : X^* \rightarrow \mathbb{C}$ are measurable functions, then $u(q)$ and $v(p)$ are the operators on $L^2(X)$ defined as follows: $u(q) = M_u$, the multiplication operator by u , and $v(p) = \mathcal{F}^{-1}M_v\mathcal{F}$, where \mathcal{F} is the Fourier transform $L^2(X) \rightarrow L^2(X^*)$. If $x \in X$ and $k \in X^*$, then the unitary operators T_x and M_k are defined on $L^2(X)$ by

$$(T_x f)(y) := f(y - x) \quad \text{and} \quad (M_k f)(y) := e^{i\langle y, k \rangle} f(y), \quad (2.1)$$

and can be written in terms of p and q as $T_x = e^{-ixp}$ and $M_k = e^{ikq}$.

We shall denote by $C^*(X)$ the *group C^* -algebra* of X : this is the closed subspace of $\mathcal{B}(X)$ generated by the operators of convolution with continuous, compactly supported functions. The map $v \mapsto v(p)$ establishes an isomorphism between $C_0(X^*)$ and $C^*(X)$. We shall need the following general result about commutative C^* -algebras. Let A be a commutative C^* -algebra and \widehat{A} be its spectrum (or character space), consisting of non-zero algebra morphisms $\chi : A \rightarrow \mathbb{C}$. Then the Gelfand transform $\Gamma_A : A \rightarrow C_0(\widehat{A})$ is defined by $\Gamma_A(u)(\chi) := \chi(u)$ and is an isometric algebra isomorphism. In particular,

any commutative C^* -algebra is of the form $\mathcal{C}_0(\Omega)$ for some locally compact space (up to isomorphism). The characters of $\mathcal{C}_0(\Omega)$ are of the form χ_ω , $\omega \in \Omega$, where

$$\chi_\omega(u) := u(\omega) \quad u \in \mathcal{C}_0(\Omega). \quad (2.2)$$

If X acts on a C^* -algebra \mathcal{A} by automorphisms, we shall denote by $\mathcal{A} \rtimes X$ the cross-product algebra, see [32, 38]. Here the real vector space X is regarded as a locally compact, abelian group in the obvious way. Recall [17] that if \mathcal{A} is a translation invariant C^* -subalgebra of $\mathcal{C}_b^u(X)$, then an isomorphic realization of the cross-product algebra $\mathcal{A} \rtimes X$ is the norm closed linear subspace of $\mathcal{B}(X)$ generated by the operators of the form $u(q)v(p)$, where $u \in \mathcal{A}$ and $v \in \mathcal{C}_0(X^*)$. As a rule, we shall denote by τ_a the action of $a \in X$ on our algebras of functions.

Definition 2.1. *Let $A \in \mathcal{B}(X)$. We say that A has the position-momentum limit property if $\lim_{x \rightarrow 0} \|(T_x - 1)A^{(*)}\| = 0$ and $\lim_{k \rightarrow 0} \|[M_k, A]\| = 0$.*

A characterization of operators having the position-momentum limit property in terms of crossed products was given in [17]: it is shown that A has the position-momentum limit property if, and only if, $A \in \mathcal{C}_b^u(X) \rtimes X$.

If A is an operator on $L^2(X)$, then its translation by $x \in X$ is defined by the relation

$$\tau_x(A) := T_x^* A T_x \quad . \quad (2.3)$$

The notation $x.A := \tau_x(A)$ will often be more convenient. If u is a function on X we also denote $\tau_x(u) \equiv x.u$ its translation given by $(x.u)(y) = u(x+y)$. The notations are naturally related: $\tau_x(u(q)) = (x.u)(q) \equiv u(x+q)$. Note that $\tau_x(v(p)) = v(p)$.

By ‘‘point at infinity’’ of X , we shall mean a point in the boundary of X in a certain compactification of it. We shall next define *the translation by a point at infinity* χ for certain functions u and operators S defined on X . This construction will be needed for the description of the essential spectrum of operators of interest for us.

Let us fix a translation invariant C^* -algebra \mathcal{A} of bounded uniformly continuous functions on X containing the functions that have a limit at infinity: $\mathcal{C}_0(X) + \mathbb{C} \subset \mathcal{A} \subset \mathcal{C}_b^u(X)$. We denote $\widehat{\mathcal{A}}$ its character space (that is, the space of non-zero algebra morphisms $\mathcal{A} \rightarrow \mathbb{C}$). Then $\widehat{\mathcal{A}}$ is a compact topological space (for the weak topology). To every $x \in X$, there is associated the character χ_x , defined by $\chi_x(u) := u(x)$ for $u \in \mathcal{A}$ (recall Equation 2.2). Since $\mathcal{C}_0(X) \subset \mathcal{A} \subset \mathcal{C}_b^u(X)$, X is naturally embedded as an open dense subset in $\widehat{\mathcal{A}}$. Thus $\widehat{\mathcal{A}}$ is a compactification of X and

$$\delta(\mathcal{A}) := \widehat{\mathcal{A}} \setminus X, \quad (2.4)$$

the *boundary* of X in this compactification, is a compact set that can be characterized as the set of characters χ of \mathcal{A} whose restriction to $\mathcal{C}_0(X)$ is equal to zero.

Let us recall that if $x, y \in X$, then $(x.u)(y) = u(x+y) = \chi_x(y.u)$. If $u \in \mathcal{A}$, we extend the definition of $x.u$ by replacing in this relation χ_x with a character $\chi \in \widehat{\mathcal{A}}$.

Definition 2.2. *Let $u \in \mathcal{A}$ and $\chi \in \widehat{\mathcal{A}}$. Then we define*

$$(\chi.u)(y) := \chi(y.u), \quad \forall y \in X.$$

Since $u \in \mathcal{C}_b^u(X)$ (i.e. it is uniformly continuous), it is easy to check that $\tau_\chi(u) := \chi.u \in \mathcal{C}_b^u(X)$ and that $\tau_\chi : \mathcal{A} \rightarrow \mathcal{C}_b^u(X)$ is a unital morphism. We will say that τ_χ is the *morphism associated to the character* χ . We note that if the character χ corresponds to $x \in X$, then $\tau_\chi = \tau_x$, so our notation is consistent.

In particular, we get ‘‘translations at infinity’’ of $u \in \mathcal{A}$ by elements $\chi \in \delta(\mathcal{A})$. The function $\chi \mapsto \chi.u \in \mathcal{C}_b^u(X)$ defined on $\widehat{\mathcal{A}}$ is continuous if $\mathcal{C}_b^u(X)$ is equipped with the

topology of local uniform convergence, hence $\chi.u = \lim_{x \rightarrow \chi} x.u$ in this topology for any $\chi \in \delta(\mathcal{A})$. One has $u \in \mathcal{C}_0(X)$ if, and only if, $\chi.u = 0$ for all $\chi \in \delta(\mathcal{A})$. We mention that a translation $\chi.u$ by a point at infinity $\chi \in \delta(\mathcal{A})$ does not belong to \mathcal{A} in general. However, we shall see that this is true in the case $\mathcal{A} = \mathcal{E}(X)$ of interest for us, so in this case τ_χ is an endomorphism of \mathcal{A} .

If $A \in \mathcal{A} \rtimes X$, then we may also consider ‘‘translations at infinity’’ $\tau_\chi(A)$ by elements χ of the boundary $\delta(\mathcal{A})$ of X in $\widehat{\mathcal{A}}$ and we get a useful characterization of the compact operators. The following facts are proved in [18, Subsection 5.1].

Proposition 2.3. *For each $\chi \in \widehat{\mathcal{A}}$, there is a unique morphism $\tau_\chi : \mathcal{A} \rtimes X \rightarrow \mathcal{C}_b^u(X) \rtimes X$ such that*

$$\tau_\chi(u(q)v(p)) = (\chi.u)(q)v(p), \quad \text{for all } u \in \mathcal{A}, v \in \mathcal{C}_0(X).$$

If $A \in \mathcal{A} \rtimes X$, then $\chi \mapsto \tau_\chi(A)$ is a strongly continuous map $\widehat{\mathcal{A}} \rightarrow \mathcal{B}(X)$.

As before, we often abbreviate $\tau_\chi(A) = \chi.A$. This gives a meaning to the translation by χ of any operator $A \in \mathcal{A} \rtimes X$ and any character $\chi \in \widehat{\mathcal{A}}$. Observe that $\chi \mapsto \chi.A$ is just *the continuous extension to $\widehat{\mathcal{A}}$ of the strongly continuous map $X \ni x \mapsto x.A$* . In particular,

$$\tau_\chi(A) = \text{s-lim}_{x \rightarrow \chi} T_x^* A T_x \quad \text{for all } A \in \mathcal{A} \rtimes X \text{ and } \chi \in \delta(\mathcal{A}). \quad (2.5)$$

We have $\mathcal{K}(X) = \mathcal{C}_0(X) \rtimes X \subset \mathcal{A} \rtimes X$. Then [18, Theorem 1.15] gives:

Theorem 2.4. *An operator $A \in \mathcal{A} \rtimes X$ is compact if, and only if, $\tau_\chi(A) = 0 \forall \chi \in \delta(\mathcal{A})$. In other terms: $\bigcap_{\chi \in \delta(\mathcal{A})} \ker \tau_\chi = \mathcal{K}(X)$. The map $\tau(A) = (\tau_\chi(A))_{\chi \in \delta(\mathcal{A})}$ induces an injective morphism*

$$\mathcal{A} \rtimes X / \mathcal{K}(X) \hookrightarrow \prod_{\chi \in \delta(\mathcal{A})} \mathcal{C}_b^u(X) \rtimes X. \quad (2.6)$$

Remark 2.5. We emphasize the relation between this result and some facts from the theory of crossed products. The operation of taking the crossed product by the action of an amenable group transforms exact sequences in exact sequences [38, Proposition 3.19] hence we have an exact sequence

$$0 \rightarrow \mathcal{C}_0(X) \rtimes X \rightarrow \mathcal{A} \rtimes X \rightarrow (\mathcal{A}/\mathcal{C}_0(X)) \rtimes X \rightarrow 0. \quad (2.7)$$

Since $\mathcal{C}_0(X) \rtimes X \simeq \mathcal{K}(X)$ we get $\mathcal{A} \rtimes X / \mathcal{K}(X) \simeq (\mathcal{A}/\mathcal{C}_0(X)) \rtimes X$ which reduces the computation of the quotient $\mathcal{A} \rtimes X / \mathcal{K}(X)$ to the description of $\mathcal{A}/\mathcal{C}_0(X)$ which is isomorphic to $\mathcal{C}(\delta(\mathcal{A}))$. Moreover, we have $\tau_\chi = \tau_\chi \rtimes \text{id}_X$ where the morphisms τ_χ on the right hand side are those corresponding to \mathcal{A} . We complete this remark by noticing that if χ and χ_1 are obtained from each other by a translation by $x \in X$, then the corresponding morphisms τ_χ and τ_{χ_1} are unitarily equivalent by the unitary corresponding to x . In particular, in the above theorem and in the following corollary, it suffices to use one χ from each orbit of X acting on $\delta(X)$.

Remark 2.6. Let us notice that in view of the results in [11, 38], the above theorem provides nontrivial information on the cross-product algebra $\mathcal{C}(\delta(\mathcal{A})) \rtimes X$, and hence on the action of X on $\delta(\mathcal{A})$. It would be interesting to study the corresponding properties for a general Lie group G acting on $\mathcal{C}_b^u(G)$. Morphisms analogous to the τ_χ can be defined also in a groupoid framework [26, 30], but they do not have a similar, simple interpretation as strong limits. It would be interesting to understand the connections between the above theorem and the representation theory of groupoids [6, 7, 12, 24, 35]. Moreover, several important examples of non-compact manifolds that arise in other problems lead to groupoids that are *locally* of the form studied in this paper (but possibly replacing X by a general Lie group G , see [21, 22, 29] and many other papers).

Let A be a bounded operator A . We shall say, by definition, that $\lambda \notin \sigma_{\text{ess}}(A)$ if, and only if, $A - \lambda$ is Fredholm. For self-adjoint operators, this is equivalent to the usual definition. It means that the image $\widehat{A - \lambda}$ of $A - \lambda$ in the quotient $\mathcal{B}(X)/\mathcal{K}(X)$ is invertible. So $\sigma_{\text{ess}}(A) = \sigma(\widehat{A})$. On the other hand, the spectrum of a *normal* operator in a product of C^* -algebras is equal to the closure of the union of the spectra of its components. Thus the theorem above gives right away the following corollary.

Corollary 2.7. *If $A \in \mathcal{A} \rtimes X$ is a normal operator then $\sigma_{\text{ess}}(A) = \overline{\bigcup_{\chi \in \delta(\mathcal{A})} \sigma(\tau_{\chi}(A))}$.*

If $A \in \mathcal{B}(X)$, then the element $\widehat{A} \in \mathcal{B}(X)/\mathcal{K}(X)$ may be called *localization at infinity* of A . If $A \in \mathcal{A} \rtimes X$, then its localization at infinity can be identified with the element $\tau(A) = (\tau_{\chi}(A))_{\chi \in \delta(\mathcal{A})}$. Then the component $\tau_{\chi}(A) \in C_b^u(X) \rtimes X$ is called *localization of A at $\chi \in \delta(\mathcal{A})$* . Thus the essential spectrum of $A \in \mathcal{A} \rtimes X$ is the closure of the union of the spectra of all its localizations at infinity, where the “infinity” is determined by \mathcal{A} .

We extend now the notion of localization at infinity and the formula for the essential spectrum to certain self-adjoint operators related to $\mathcal{A} \rtimes X$. Recall that a self-adjoint operator H on a Hilbert space \mathcal{H} is *affiliated* to a C^* -algebra $\mathcal{C} \subset B(\mathcal{H})$ if $(H - z)^{-1} \in \mathcal{C}$ for some number z outside the spectrum of H . Clearly this implies $\varphi(H) \in \mathcal{C}$ for all $\varphi \in C_0(\mathbb{R})$. We shall make some more comments on this notion after the next corollary.

Corollary 2.8. *If H is a self-adjoint operator on $L^2(X)$ affiliated to $\mathcal{A} \rtimes X$ then for each $\chi \in \delta(\mathcal{A})$ the limit $\tau_{\chi}(H) := s\text{-}\lim_{x \rightarrow \chi} T_x^* H T_x$ exists and $\sigma_{\text{ess}}(H) = \overline{\bigcup_{\chi \in \delta(\mathcal{A})} \sigma(\tau_{\chi}(H))}$.*

The meaning of the limit above will be discussed below. Then the corollary is an immediate consequence of Theorem 2.4 if one thinks in terms of the functional calculus associated to H . Indeed, a real λ does not belong to $\sigma_{\text{ess}}(H)$ if and only if there is $\varphi \in C_0(\mathbb{R})$ with $\varphi(\lambda) \neq 0$ such that $\varphi(H)$ is compact.

For a detailed discussion of the notion of affiliation that we use in this paper we refer to [2, Sec. 8.1] (see also [9, Appendix A]). This notion is inspired by the quantum mechanical concept of observable as introduced by J. Von Neumann in the 1930s (see e.g. [36, Sec. 3.2] for a general and precise mathematical formulation) and later (1940s) developed in the Von Neumann algebra setting. A notion of affiliation in the C^* -algebra setting has also been introduced by S. Baaj and S.L. Woronowicz [3, 39] but it is different from that we use here: the contrary was erroneously stated in [17, p. 534], but has been corrected in [9, p. 278]. For example, any self-adjoint operator on a Hilbert space \mathcal{H} is affiliated to the algebra of compact operators $\mathcal{K}(\mathcal{H})$ in the sense of Baaj-Woronowicz, but a self-adjoint operator is affiliated to $\mathcal{K}(\mathcal{H})$ in our sense if and only if it has purely discrete spectrum.

According to our definition, a self-adjoint operator affiliated to an “abstract” C^* -algebra \mathcal{C} is the same thing as a real valued observable affiliated to \mathcal{C} , i.e. it is just a morphism $\Phi : C_0(\mathbb{R}) \rightarrow \mathcal{C}$. If $\mathcal{C} \subset B(\mathcal{H})$, then a densely defined self-adjoint operator H defines an observable by $\Phi(\varphi) = \varphi(H)$ for $\varphi \in C_0(\mathbb{R})$, and we say that H is affiliated to \mathcal{C} if this observable is affiliated to \mathcal{C} . But there are observables affiliated to \mathcal{C} that are not of this form: they are associated to self-adjoint operators K acting in closed subspaces $\mathcal{K} \subset \mathcal{H}$ as explained in the next remark. See [2, Sec. 8.1.2] for a more precise statement and proof.

We have to explain the meaning of the limit $s\text{-}\lim_{x \rightarrow \chi} T_x^* H T_x$ when H is an unbounded self-adjoint operator.

Remark 2.9. Let Y be a topological space, $z \in Y$ be a fixed point, and let H_y be a set self-adjoint operators (possibly unbounded), parametrized by $Y \setminus \{z\}$. The example that we have in mind is $H_x := T_x^* H T_x$, $x \in X$, and Y obtained from X by adding some point of a compactification. We say that $s\text{-}\lim_{y \rightarrow z} H_y$ exists if, by definition, the strong limit $\Phi(\varphi) := s\text{-}\lim_{y \rightarrow z} \varphi(H_y)$ exists for each function $\varphi \in C_0(\mathbb{R})$. It is easy to see that

this is equivalent to the existence of $s\text{-}\lim_{y \rightarrow z} (H_y - \lambda)^{-1}$ for some $\lambda \in \mathbb{C} \setminus \mathbb{R}$. But we emphasize that *this does not mean that there is a self-adjoint operator K on $L^2(X)$ such that $\Phi(\varphi) = \varphi(K)$ for all $\varphi \in \mathcal{C}_0(\mathbb{R})$* if the notion of self-adjointness is interpreted in the usual sense, which requires the domain to be dense in $L^2(X)$. However, the following is true: there is a closed subspace $\mathcal{K} \subset L^2(X)$ and a self-adjoint operator (in the usual sense) K in \mathcal{K} such that $\Phi(\varphi)\Pi_{\mathcal{K}} = \varphi(K)\Pi_{\mathcal{K}}$ and $\Phi(\varphi)\Pi_{\mathcal{K}}^\perp = 0$, where $\Pi_{\mathcal{K}}$ is the projection onto \mathcal{K} . The couple (\mathcal{K}, K) is uniquely defined and we write $s\text{-}\lim_x H_x = K$. One may have $\mathcal{K} = \{0\}$, in which case we write $s\text{-}\lim_x H_x = \infty$. See the Remark 5.5 for an example.

3. SPHERICAL COMPACTIFICATION

We now briefly discuss the spherical compactification of a vector spaces, its topology and space of continuous functions. As before, X is a finite dimensional real vector space. Recall that the sphere at infinity \mathbb{S}_X of X is the set of all half-lines $\alpha = \widehat{a} := \mathbb{R}_+ a$, with $\mathbb{R}_+ = (0, \infty)$ and $a \in X \setminus \{0\}$, equipped with the following topology: the open sets in \mathbb{S}_X are the sets of the form $\{\widehat{a}, a \in O\}$ with O open in $X \setminus \{0\}$. Let us denote by \overline{X} the disjoint union $X \cup \mathbb{S}_X$. If $|\cdot|$ is an arbitrary norm on X then \mathbb{S}_X is homeomorphic to the unit sphere $S_X := \{|\xi| = 1\}$ in X and $\overline{X} = X \cup \mathbb{S}_X$ can be endowed with a natural topology that makes it homeomorphic to the closed unit ball in X . The resulting topological space \overline{X} will be referred to as *the spherical compactification of X* and is discussed in detail in this subsection, since we need a good understanding of the continuous functions on \overline{X} .

It is convenient to have an explicit description of the topology of \overline{X} independent of the choice of a norm. A *cone* C (in X) is a subset of X stable under the action of \mathbb{R}_+ by multiplication. Put differently, C is a union of half-lines. A *truncated cone* (in X) is the intersection of a cone with the complement of a bounded set. A half-line α is *eventually in the truncated cone* C if there is $a \in \alpha$ such $\lambda a \in C$ if $\lambda \geq 1$. Let $C^\dagger \subset \mathbb{S}_X$ be the set of half-lines which are eventually in C . Then the sets of the form C^\dagger , with C an open truncated cone, form a base of the topology of \mathbb{S}_X . For any open truncated cone C in X , we denote $C^\ddagger := C \cup C^\dagger$. Then the open sets of X and the sets of the form C^\ddagger form a base of the topology of \overline{X} . It is easy to see that \overline{X} is a compact topological space in which X is densely and homeomorphically embedded. Moreover, \overline{X} induces on \mathbb{S}_X the (compact) topology we defined before.

By definition, a neighborhood of $\alpha \in \mathbb{S}_X$ in \overline{X} is a set that contains a subset of the form C^\ddagger . We denote $\tilde{\alpha}$ the set of traces on X of the neighborhoods of α in \overline{X} . Thus, a set belongs to $\tilde{\alpha}$ if and only if it contains an open truncated cone that eventually contains α . Let Y be a topological space and let $u : X \rightarrow Y$. If $\alpha \in \mathbb{S}_X$ and $y \in Y$, then the limit $\lim_{x \rightarrow \alpha} u(x)$ (or $\lim_\alpha u$) exists and is equal to y if, and only if, for each neighborhood V of y , there is a truncated cone C that eventually contains α such that $u(x) \in V$ if $x \in C$. We shall need the following simple lemma.

Lemma 3.1. *Let $u : X \rightarrow \mathbb{C}$ be such that the limit $U(\alpha) := \lim_{x \rightarrow \alpha} u(x)$ exists for each $\alpha \in \mathbb{S}_X$. Then U is a continuous function on \mathbb{S}_X . If u is continuous on X , then its extension by U on \mathbb{S}_X is continuous on \overline{X} .*

Proof. Let us notice first that $\lim_{\lambda \rightarrow \infty} u(\lambda a) = U(\alpha)$ for each $a \in \alpha \in \mathbb{S}_X$. Fix $\alpha \in \mathbb{S}_X$ and $\varepsilon > 0$. There is an open truncated cone C that eventually contains α (hence $\alpha \in C^\dagger$) such that $|u(x) - U(\alpha)| < \varepsilon$ for all $x \in C$. If β is another half-line eventually in C (that is, $\beta \in C^\dagger$), then $\lim_{x \rightarrow \beta} u = U(\beta)$. In particular, for each $b \in \beta$ we have $\lim_{\lambda \rightarrow \infty} u(\lambda b) = U(\beta)$. Since $\lambda b \in C$ for large λ , we get that $|U(\beta) - U(\alpha)| < \varepsilon$. Since $\beta \in C^\dagger$ is arbitrary and since the sets C^\dagger form a basis of the topology of \mathbb{S}_X , we see that U is continuous.

To prove the last statement and thus to complete the proof, let us extend u by U on \mathbb{S}_X . Then the reasoning used in the first half of the proof implies that $|u(x) - u(\alpha)| < \varepsilon$ for

all $x \in C^\ddagger$. Since the sets of the form C^\ddagger form a basis for the system of neighborhoods of $\alpha \in \mathbb{S}_X$ in \mathbb{S}_X and α is arbitrary, the extension of u to \overline{X} by U is continuous on \mathbb{S}_X . Hence if u is continuous on X , then its extension to \overline{X} is continuous everywhere. \square

Since X is a dense subset of \overline{X} , we may identify the algebra $\mathcal{C}(\overline{X})$ of continuous functions on \overline{X} with a subalgebra of $\mathcal{C}(X)$. We now give several descriptions of this subalgebra that are independent of the preceding construction of \overline{X} . Denote by $\mathcal{C}_h(X)$ the subalgebra of $\mathcal{C}_b^u(X)$ consisting of functions homogeneous of degree zero outside a compact set:

$$\mathcal{C}_h(X) := \{u \in \mathcal{C}(X), \exists K \subset X \text{ compact with } u(\lambda x) = u(x) \text{ if } x \notin K, \lambda \geq 1\}. \quad (3.1)$$

Lemma 3.2. *The algebra $\mathcal{C}(\overline{X})$ coincides with the closure of $\mathcal{C}_h(X)$ in $\mathcal{C}_b(X)$. Also,*

$$\mathcal{C}(\overline{X}) = \{u \in \mathcal{C}(X), \lim_{\lambda \rightarrow +\infty} u(\lambda a) \text{ exists uniformly in } \hat{a} \in \mathbb{S}_X\} \quad (3.2)$$

$$= \{u \in \mathcal{C}(X), \lim_{x \rightarrow \alpha} u(x) \text{ exists for each } \alpha \in \mathbb{S}_X\}. \quad (3.3)$$

Moreover, if $u \in \mathcal{C}(\overline{X})$ and if A, B are compact sets in X such that $0 \notin A$ then

$$\lim_{\lambda \rightarrow +\infty} u(\lambda a + b) = u(\hat{a}) \quad \text{uniformly in } a \in A \text{ and } b \in B. \quad (3.4)$$

Proof. The space \overline{X} is compact, and hence every continuous function on \overline{X} is uniformly continuous. This gives (3.2). Next, if $u \in \mathcal{C}(\overline{X})$, then it follows from the properties of continuous functions that the restriction of u to X satisfies the condition in (3.3). Conversely, if u is as in (3.3), then Lemma 3.1 implies $u \in \mathcal{C}(\overline{X})$. Thus we proved that $\mathcal{C}(\overline{X})$ is given by the relation (3.3).

We have $\mathcal{C}_h(X) \subset \mathcal{C}(\overline{X})$ by the definition of the topology on \overline{X} . Since $\mathcal{C}_h(X)$ separates the points of \overline{X} we see that $\mathcal{C}_h(X)$ is dense in $\mathcal{C}(\overline{X})$. Observe that the topology we introduced on \overline{X} could be introduced directly in terms of $\mathcal{C}_h(X)$: for example, $\tilde{\alpha}$ is the filter on X defined by the sets $\{x \in X, |u(x) - u(\alpha)| < 1\}$ when u runs over $\mathcal{C}_h(X)$.

Let us show that (3.4) holds for any $u \in \mathcal{C}(\overline{X})$. By the density property we have just proved, we may assume $u \in \mathcal{C}_h(X)$. Choose the norm $|\cdot|$ on X such that u is homogeneous of degree zero for $|x| \geq 1$ and identify \mathbb{S}_X with the corresponding unit sphere S_X . Let $\omega(\theta) := \sup |u(x) - u(y)|$ where $x, y \in S_X, |x - y| \leq \theta$. Then a simple geometric argument gives $|u(x) - u(y)| \leq \omega((2|y|/|x|)^{1/2})$ if $|x| \geq 1 + |y|$ which implies (3.4). \square

Note that $\mathcal{C}_h(X)$ is not stable under translations if the dimension of X is larger than one. But Equation (3.2), or a direct argument, immediately gives that $\mathcal{C}(\overline{X})$ is invariant under translations, and hence we may consider its crossed product $\mathcal{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$ by the action of X . This crossed product will be called the *spherical algebra* of X and we shall explicitly describe it in the next section.

4. THE SPHERICAL ALGEBRA

We study now the spherical algebra $\mathcal{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$ defined in the Introduction. We begin with a lemma which will be needed in the proof of Theorem 4.2. In order to clarify the statement of the following lemma and in order to prepare the ground for the use of filters in other proofs, we recall now some facts about filters [5].

A *filter* on X is a set ξ of subsets of X such that: (1) $X \in \xi$, (2) $\emptyset \notin \xi$, (3) if $\xi \ni F \subset G$, then $G \in \xi$, and (4) if $F, G \in \xi$ then $F \cap G \in \xi$. If Y is a topological space and $u : X \rightarrow Y$, then $\lim_{\xi} u = y$, or $\lim_{x \rightarrow \xi} u(x) = y$, means that $u^{-1}(V) \in \xi$ for any neighborhood V of y . The filter ξ on X is called *translation invariant* if, for each $F \in \xi$ and $x \in X$, we have $x + F \in \xi$. We say that ξ is *coarse* if, for each $F \in \xi$ and each compact K in X , there is

$G \in \xi$ such that $G + K \subset F$. Recall that we have denoted by $\tilde{\alpha}$ the set of traces on X of the neighborhoods of α in \overline{X} . Clearly $\tilde{\alpha}$ is a translation invariant and coarse filter on X for each $\alpha \in \mathbb{S}_X$.

Lemma 4.1. *Let ξ be a translation invariant filter in X , let K be a compact neighborhood of the origin, and $u \in C_b^u(X)$. Then*

$$\lim_{\xi} u = 0 \Leftrightarrow \lim_{a \rightarrow \xi} \int_{a+K} |u(x)| dx = 0 \Leftrightarrow \text{s-lim}_{a \rightarrow \xi} u(q+a) = 0. \quad (4.1)$$

Proof. Recall that $u(q)$ denotes the operator of multiplication by u and $u(q+a)$ is its translation by a . We have $\text{s-lim}_{a \rightarrow \xi} u(q+a) = 0$ if, and only if, $\int |u(x+a)f(x)|^2 dx \rightarrow 0$ as $a \rightarrow \xi$ for all $f \in L^2(X)$, by the definition of the strong limit. By taking f to be the characteristic function of the compact set K and by using the Cauchy-Schwartz inequality, we obtain $\lim_{a \rightarrow \xi} \int_{a+K} |u(x)| dx = 0$. Reciprocally, if this relation is satisfied then it is also satisfied with K replaced by any of its translates because ξ is translation invariant. By summing a finite number of such relations, we get $\lim_{a \rightarrow \xi} \int_{a+M} |u(x)| dx = 0$ for any compact M . Since u is bounded, we also obtain $\lim_{a \rightarrow \xi} \int_{a+M} |u(x)| dx = 0$ for any compact M , and hence $\lim_{a \rightarrow \xi} \int |u(x)f(x)|^2 dx = 0$, for any simple function f . Using again the boundedness of u , we then obtain $\lim_{a \rightarrow \xi} \int |u(x)f(x)|^2 dx = 0$ for $f \in L^2(X)$.

We now show that $\lim_{\xi} u = 0$ is equivalent to $\lim_{a \rightarrow \xi} \int_{a+K} |u(x)| dx = 0$. We may assume $u \geq 0$ and since u and $a \mapsto \int_{a+K} u(x) dx$ are bounded uniformly continuous functions, we may also assume that ξ is coarse[†]. If $\lim_{\xi} u = 0$, then $\{u < \varepsilon\} \in \xi$, for any $\varepsilon > 0$. Since ξ is coarse, there is $F \in \xi$ such that $F + K \subset \{u < \varepsilon\}$, hence, if $a \in F$, then $\int_{a+K} u(x) dx \leq \varepsilon|a+K| = \varepsilon|K|$. Thus we have $\lim_{a \rightarrow \xi} \int_{a+K} u(x) dx = 0$. Conversely, assume that this last condition is satisfied and let $\varepsilon > 0$. Since u is uniformly continuous, there is a compact symmetric neighborhood $L \subset K$ of zero such that $|u(x) - u(y)| < \varepsilon$ if $x, y \in L$. Then

$$u(a)|L| = \int_{a+L} (u(a) - u(x)) dx + \int_{a+L} u(x) dx \leq \varepsilon|L| + \int_{a+L} u(x) dx$$

hence $\limsup_{a \rightarrow \xi} u(a) \leq \varepsilon|L|$. \square

Recall that $\tau_a(S) = T_a^* S T_a$, where the unitary translation operators T_a are defined in (2.1).

Theorem 4.2. *The algebra $\mathcal{S}(X) := C(\overline{X}) \rtimes X$ consists of the $S \in \mathcal{B}(X)$ that have the position-momentum limit property and are such that $\text{s-lim}_{a \rightarrow \alpha} \tau_a(S)^{(*)}$ exists $\forall \alpha \in \mathbb{S}_X$.*

Proof. Let \mathcal{A} be the set of bounded operators that have the properties in the statement of the theorem. We first show that $\mathcal{S}(X) \subset \mathcal{A}$. Recall that in the concrete realization we mentioned above, $C_b^u(X) \rtimes X$ is identified with the norm closed linear space generated by the operators $S = u(q)v(p)$ with $u \in C_b^u(X)$ and $v \in C_0(X^*)$, while $C(\overline{X}) \rtimes X$ is the norm closed subspace generated by the same type of operators, but with $u \in C(\overline{X})$. It follows that an operator $S = u(q)v(p)$, with $u \in C(\overline{X})$, has the position-momentum limit property and

$$\text{s-lim}_{a \rightarrow \alpha} T_a^* S T_a = \text{s-lim}_{a \rightarrow \alpha} u(q+a)v(p) = u(\alpha)v(p), \quad (4.2)$$

because of relation (3.4). Thus $\mathcal{S}(X) \subset \mathcal{A}$ and it remains to prove the opposite inclusion.

It is clear that \mathcal{A} is a C^* -algebra. From [18, Theorem 3.7] it follows that \mathcal{A} is a crossed product $\mathcal{A} = \mathcal{A} \rtimes X$ with $\mathcal{A} \subset C_b^u(X)$ if, and only if, $\mathcal{A} \subset C_b^u(X) \rtimes X$ and

$$x \in X, k \in X^*, S \in \mathcal{A} \Rightarrow T_x S \in \mathcal{A} \text{ and } M_k S M_k^* \in \mathcal{A}. \quad (4.3)$$

[†]This follows from [18, Lemma 2.2] and a simple argument, which shows that the round envelope of a translation invariant filter is coarse. We do not include the details since in our applications $\xi = \tilde{\alpha}$ which is coarse.

By the definition of \mathcal{A} , the condition $\mathcal{A} \subset \mathcal{C}_b^u(X) \rtimes X$ is obviously satisfied. Moreover, we have $T_a^* T_x S T_a = T_x T_a^* S T_a$ and $T_a^* M_k S M_k^* T_a = M_k T_a^* S T_a M_k^*$, and hence the last two conditions in (4.3) are also satisfied. Therefore \mathcal{A} is a crossed product. Theorem 3.7 from [18] gives more: the unique translation invariant C^* -subalgebra $\mathcal{A} \subset \mathcal{C}_b^u(X)$ such that $\mathcal{A} = \mathcal{A} \rtimes X$ is the set of $u \in \mathcal{C}_b^u(X)$ such that $u(q)v(p)$ and $\overline{u}(q)v(p)$ belong to \mathcal{A} if $v \in \mathcal{C}_0(X^*)$. In our case, we see that \mathcal{A} is the set of all $u \in \mathcal{C}_b^u(X)$ such that $s\text{-}\lim_{a \rightarrow \alpha} T_a^* u(q)^{(*)} T_a v(p)$ exists for all $\alpha \in \mathbb{S}_X$ and $v \in \mathcal{C}_0(X^*)$. But the operators $T_a^* u(q)^{(*)} T_a = u^{(*)}(q+a)$ are normal and uniformly bounded and the union of the ranges of the operators $v(p)$ is dense in $L^2(X)$, hence

$$\mathcal{A} = \{u \in \mathcal{C}_b^u(X), \exists s\text{-}\lim_{a \rightarrow \alpha} u(q+a) \forall \alpha \in \mathbb{S}_X\}.$$

Let us fix α and let $u \in \mathcal{C}_b^u(X)$ be such that the limit $s\text{-}\lim_{a \rightarrow \alpha} u(q+a)$ exists. This limit is a function, but since the filter $\tilde{\alpha}$ is translation invariant, this function must be in fact a constant c . Applying Lemma 4.1 to $u - c$ we get $\lim_{\alpha} u = c$. Lemma 3.2 then gives

$$\mathcal{A} = \{u \in \mathcal{C}_b^u(X), \exists \lim_{x \rightarrow \alpha} u(x) \forall \alpha \in \mathbb{S}_X\} = \mathcal{C}(\overline{X}).$$

This proves the theorem. \square

For each $\alpha \in \mathbb{S}_X$ and $S \in \mathcal{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$, we then define

$$\tau_\alpha(S) := s\text{-}\lim_{a \rightarrow \alpha} T_a^* S T_a. \quad (4.4)$$

Theorem 4.3. *If $S \in \mathcal{S}(X)$ and $\alpha \in \mathbb{S}_X$, then $\tau_\alpha(S) \in C^*(X)$ and the map $\tau(S) : \alpha \mapsto \tau_\alpha(S)$ is norm continuous, hence $\tau : \mathcal{S}(X) \rightarrow C(\mathbb{S}_X) \otimes C^*(X)$. The resulting morphism τ is a surjective morphism and its kernel is the set $\mathcal{K}(X) = \mathcal{C}_0(X) \rtimes X$ of compact operators on $L^2(X)$. Hence we have a natural identification*

$$\mathcal{S}(X)/\mathcal{K}(X) \cong C(\mathbb{S}_X) \otimes C^*(X) \cong \mathcal{C}_0(\mathbb{S}_X \times X^*). \quad (4.5)$$

Proof. If $S = u(q)v(p)$, then, from (4.2), we get $\tau_\alpha(u(q)v(p)) = u(\alpha)v(p)$, and thus $\tau(S) = \tilde{u} \otimes v(p)$, where \tilde{u} is the restriction of $u : \overline{X} \rightarrow \mathbb{C}$ to \mathbb{S}_X . The first assertion of the theorem then follows from the density in $\mathcal{S}(X)$ of the linear space generated by the operators of the form $u(q)v(p)$. The fact that τ_α are morphisms follows from their definition as strong limits, and it implies the fact that τ is a morphism. Since the range of a morphism is closed and $u \mapsto \tilde{u}$ is a surjective map $\mathcal{C}(\overline{X}) \rightarrow C(\mathbb{S}_X)$, we get the surjectivity of τ . It remains to show that $\ker \tau = \mathcal{C}_0(X) \rtimes X$. By what we have proved, $\mathcal{A}_0 = \ker \tau$ is the set of operators S that have the position-momentum property and are such that $s\text{-}\lim_{a \rightarrow \alpha} T_a S T_a = 0$ for all $\alpha \in \mathbb{S}_X$. The argument of the proof of Theorem 4.2 with \mathcal{A} replaced by \mathcal{A}_0 shows that $\mathcal{A}_0 = \mathcal{A}_0 \rtimes X$, with \mathcal{A}_0 equal to the set of all $u \in \mathcal{C}_b^u(X)$ such that $\lim_{x \rightarrow \alpha} u(x) = 0$ for all $\alpha \in \mathbb{S}_X$, and hence $\mathcal{A}_0 = \mathcal{C}_0(X)$. \square

Remark 4.4. The fact that $\tau_\alpha(S)$ belongs to $C^*(X)$ can be understood more generally as follows. Since the filter α is translation invariant, if S is an arbitrary bounded operator such that the limit $S_\alpha := s\text{-}\lim_{a \rightarrow \alpha} T_a^* S T_a$ exists, then S_α commutes with all the U_x , and hence S is of the form $v(p)$, for some $v \in L^\infty(X^*)$. If S has the position-momentum limit property, then it is clear that S_α also has the position-momentum limit property, which forces $v \in \mathcal{C}_0(X^*)$.

We have the following consequences of the above theorem. We do not need closure in the union since $\tau_\alpha(S)$ depends *norm* continuously on α . See [31] for a general discussion of the need of closures of the unions in results of this type.

Corollary 4.5. *Let $S \in \mathcal{S}(X)$ be a normal element. Then $\sigma_{\text{ess}}(S) = \cup_\alpha \sigma(\tau_\alpha(S))$.*

Corollary 4.6. *Let H be a self-adjoint operator affiliated to $\mathcal{S}(X)$. Then for each $\alpha \in \mathbb{S}_X$ the limit $\alpha.H := s\text{-}\lim_{a \rightarrow \alpha} T_a^* H T_a$ exists and $\sigma_{\text{ess}}(H) = \cup_\alpha \sigma(\alpha.H)$.*

For the proof and for the meaning of the limit above, see Remark 2.9.

We give now the simplest concrete application of Corollary 4.6. The boundedness condition on V can be eliminated, but this requires some technicalities, which will be discussed in the next section.

Proposition 4.7. *Let $H = h(p) + V$, where $h : X^* \rightarrow \mathbb{R}$ is a continuous proper function and V is a bounded symmetric linear operator on $L^2(X)$ satisfying*

- (i) $\lim_{k \rightarrow 0} \|[M_k, V]\| = 0$,
- (ii) $\alpha.V := \text{s-lim}_{a \rightarrow \alpha} T_a^* V T_a$ exists for each $\alpha \in \mathbb{S}_X$.

Then H is affiliated to $\mathcal{S}(X)$, we have $\alpha.H = h(p) + \alpha.V$, and $\sigma_{\text{ess}}(H) = \cup_{\alpha} \sigma(\alpha.H)$. Moreover, for each $\alpha \in \mathbb{S}_X$, there is a function $v_{\alpha} \in C_b^u(X^)$ such that $\alpha.V = v_{\alpha}(p)$.*

Proof. First we have to check that the self-adjoint operator H is affiliated to $\mathcal{S}(X)$. For this, it suffices to prove that there is a number z such that the operator $S = (H - z)^{-1}$ satisfies the conditions of Theorem 4.2. To check the position-momentum limit property we have to prove that $(T_x - 1)S$ and $[M_k, S]$ tend to zero in norm when $x \rightarrow 0$ and $k \rightarrow 0$ (the condition involving S^* will then also be satisfied since S^* is of the same form as S). Since the range of S is the domain of $h(p)$, the first condition is clearly satisfied. If we denote $S_0 = (h(p) - z)^{-1}$ and choose z such that $\|V S_0\| < 1$, then we have $S = S_0(1 + V S_0)^{-1}$ and $S_0 \in C^*(X)$ hence $[M_k, S_0]$ tends to zero in norm as $k \rightarrow 0$. It remains to be shown that $(1 + V S_0)^{-1}$ also satisfies this condition: but this is clear because the set of bounded operators A such that $\|[M_k, A]\| \rightarrow 0$ is a C^* -algebra, hence a full subalgebra of $\mathcal{B}(X)$.

The fact that $\text{s-lim}_{a \rightarrow \alpha} T_a^* S T_a$ exists and is equal to $\alpha.S = (\alpha.H - z)^{-1}$ for each $\alpha \in \mathbb{S}_X$ is an easy consequence of the relation $T_a^* S T_a = S_0(1 + T_a^* V T_a S_0)^{-1}$.

Finally, to show that $\alpha.V = v_{\alpha}(p)$, for some $v_{\alpha} \in C_b^u(X^*)$, we use the argument of Remark 4.4. Indeed, we shall have this representation for some bounded Borel function v_{α} which must be uniformly continuous because $\lim_{k \rightarrow 0} \|[M_k, \alpha.V]\| = 0$. \square

Example 4.8. A typical example is when V is the operator of multiplication by a bounded Borel function $V : X \rightarrow \mathbb{R}$ such that $V(\alpha) := \lim_{x \rightarrow \alpha} V(x)$ exists for each $\alpha \in \mathbb{S}_X$. Then $\alpha.V$ is the operator of multiplication by the number $V(\alpha)$. Note that by Lemma 3.1 the limit function $\alpha \mapsto V(\alpha)$ is continuous on \mathbb{S}_X , even if V is not continuous on X .

5. AFFILIATION CRITERIA

We now recall, for the benefit of the reader, a little bit of the formalism that we shall use below. If H is a self-adjoint operator on a Hilbert space \mathcal{H} , then the domain of $|H|^{1/2}$ equipped with the graph topology is called the *form domain* of H . If we denote it \mathcal{G} , then we have natural continuous embedding $\mathcal{G} \subset \mathcal{H} \subset \mathcal{G}^*$, where \mathcal{G}^* is the space adjoint of \mathcal{G} (the space of conjugate linear continuous forms on \mathcal{G}). The operator $H : D(H) \rightarrow \mathcal{H}$ extends to a continuous symmetric operator $\widehat{H} \in \mathcal{B}(\mathcal{G}, \mathcal{G}^*)$, which has the following property: a complex number z belongs to the resolvent set of H if, and only if, $\widehat{H} - z$ is a bijective map $\mathcal{G} \rightarrow \mathcal{G}^*$. In this case, $(H - z)^{-1}$ coincides with the restriction of $(\widehat{H} - z)^{-1}$ to \mathcal{H} . Conversely, let \mathcal{G} be a Hilbert space densely and continuously embedded in \mathcal{H} . If $S : \mathcal{G} \rightarrow \mathcal{G}^*$ is a symmetric operator, then the *operator induced by S in \mathcal{H}* is the operator H in \mathcal{H} whose domain is the set of $u \in \mathcal{G}$ such that $Su \in \mathcal{H}$ given by $H = S|_D(H)$. If $S - z : \mathcal{G} \rightarrow \mathcal{G}^*$ is a bijective map for some complex z , then $D(H)$ is a dense subspace of \mathcal{H} , the operator H is self-adjoint, and $\widehat{H} = S$. If S is bounded from below, then \mathcal{G} coincides with the form domain of H . From now on, we shall drop the ‘‘hat’’ from the notation \widehat{H} and write simply H for the extended operator when there is no danger of confusion.

Lemma 5.1. *Let \mathcal{G} be a Hilbert space densely and continuously embedded in $L^2(X)$. Then the following conditions are equivalent:*

- *The operators T_x and M_k leave invariant \mathcal{G} , we have $\|T_x\|_{\mathcal{B}(\mathcal{G})} \leq C$ for a number C independent of x , and $\lim_{x \rightarrow 0} \|T_x - 1\|_{\mathcal{G} \rightarrow \mathcal{H}} = 0$.*
- *$\mathcal{G} = D(w(p))$ for some proper Borel function $w : X^* \rightarrow [1, \infty)$ such that there exists a compact neighborhood L of zero in X^* and $c > 0$ such that $\sup_{\ell \in L} w(k + \ell) \leq cw(k)$ for all $k \in X^*$.*

Proof. We discuss only the nontrivial implication. Denote $\mathcal{H} = L^2(X)$. Since $\{T_x\}_{x \in X}$ is a strongly continuous unitary group in \mathcal{H} that leaves \mathcal{G} invariant, the restrictions $T_x|_{\mathcal{G}}$ form a C_0 -group in \mathcal{G} , which by assumption is (uniformly) bounded. It is well known that this implies that there is a Hilbert structure on \mathcal{G} , equivalent to the initial one, for which the operators $T_x|_{\mathcal{G}}$ are unitary (indeed, \mathbb{R} is amenable). Thus, from now on, we may assume that the operators T_x are unitary in \mathcal{G} . Then, by the Friedrichs theorem, there exists a unique self-adjoint operator G on \mathcal{H} with the following properties:

- (i) $G \geq c > 0$ for some number c ;
- (ii) $\mathcal{G} = D(G)$;
- (iii) for all $g \in \mathcal{G}$, we have $\|g\|_{\mathcal{G}} = \|Gg\|$.

By hypothesis, the unitary operator T_x leaves invariant the domain of G and $\|g\|_{\mathcal{G}} = \|GT_x g\| = \|T_x^* G T_x g\|$ for all $g \in D(G)$ and $x \in X$. By the uniqueness of G , we have $T_x^* G T_x = G$, and hence G commutes with all translations. It follows that there is a Borel function $w : X^* \rightarrow [c, \infty)$ such that $G = w(p)$. We have

$$\begin{aligned} \|(T_x - 1)\|_{\mathcal{G} \rightarrow \mathcal{H}} &= \|(T_x - 1)G^{-1}\|_{\mathcal{H} \rightarrow \mathcal{H}} = \|(e^{ixP} - 1)w^{-1}(P)\|_{\mathcal{H} \rightarrow \mathcal{H}} \\ &= \text{ess sup}_{p \in X^*} |(e^{ixp} - 1)w^{-1}(p)| \end{aligned}$$

and w^{-1} is a bounded Borel function. It follows that w^{-1} tends to zero at infinity.

Now we shall use the fact that the M_k also leave invariant \mathcal{G} . Then the group induced by $\{M_k\}$ in \mathcal{G} is of class C_0 . In particular, $\|w(p)M_\ell g\| \leq C\|w(p)g\|$ if $\ell \in L$ and $g \in \mathcal{G}$. Since $M_\ell^* w(p)M_\ell = w(p + \ell)$, we get $\|w(p + \ell)w(p)^{-1}f\| \leq C\|f\|$ for $\ell \in L$ and $f \in \mathcal{H}$, which means that $w(k + \ell)w(k)^{-1} \leq C$ for all $k \in X^*$ and $\ell \in L$. Thus for each fixed k , w is bounded on $k + L$, hence w is bounded on any compact. \square

The next result is a general criterion of affiliation to $\mathcal{S}(X)$ for semi-bounded operators.

Theorem 5.2. *Let H be a self-adjoint operator on $L^2(X)$ that is bounded from below and its form domain \mathcal{G} satisfies the conditions of Lemma 5.1. Assume that $\|[M_k, H]\|_{\mathcal{G} \rightarrow \mathcal{G}^*} \rightarrow 0$ as $k \rightarrow 0$ and that the limit $\alpha.H := \lim_{a \rightarrow \alpha} T_a^* H T_a$ exists strongly in $\mathcal{B}(\mathcal{G}, \mathcal{G}^*)$, for all $\alpha \in \mathbb{S}_X$. Then H is affiliated to $\mathcal{S}(X)$, for each $\alpha \in \mathbb{S}_X$ the operator in $L^2(X)$ associated to $\alpha.H$ is self-adjoint, and $\sigma_{\text{ess}}(H) = \cup_{\alpha} \sigma(\alpha.H)$.*

Proof. We shall use Theorem 4.2 and then Corollary 4.6. We first check that the first condition of Theorem 4.2 is satisfied. Let us denote $R = (H + i)^{-1}$. We have $\|(T_x - 1)\|_{\mathcal{G} \rightarrow \mathcal{H}} = \|(T_x - 1)|R|^{1/2}\|$, and hence $\lim_{x \rightarrow 0} \|(T_x - 1)R\| = 0$. As explained above, R extends uniquely to an operator $\widehat{R} \in \mathcal{B}(\mathcal{G}^*, \mathcal{G})$. The operators M_k leave \mathcal{G} invariant and thus extend continuously to \mathcal{G}^* . Consequently, we have $[M_k, \widehat{R}] = \widehat{R}[H, M_k]\widehat{R}$. Hence we get $\lim_{k \rightarrow 0} \|[M_k, \widehat{R}]\|_{\mathcal{G}^* \rightarrow \mathcal{G}} = 0$, which is more than enough to show that H has the position-momentum limit property.

To finish the proof of the proposition, it is enough to check the last condition of Theorem 4.2 and then use Corollary 4.6. Clearly $\alpha.H : \mathcal{G} \rightarrow \mathcal{G}^*$ satisfies $\langle g | \alpha.H g \rangle =$

$\lim_{a \rightarrow \alpha} \langle T_a g | H T_a g \rangle$ for each $g \in \mathcal{G}$. Note that since we assumed H bounded from below, we may assume that $H \geq 1$ (otherwise we add to it a sufficiently large number). Then, if w is as in Lemma 5.1, the norm $\|w(p)g\|$ defines the topology of \mathcal{G} , and hence $\langle u | H u \rangle \geq c \|w(p)u\|^2$ for some number c and all $u \in \mathcal{G}$. This implies $\langle T_a g | H T_a g \rangle \geq c \|w(p)T_a g\|^2 = c \|w(p)g\|^2$. Thus we get $\langle g | \alpha.H g \rangle \geq c \|w(p)g\|^2$, and hence $\alpha.H$ is a bijective map $\mathcal{G} \rightarrow \mathcal{G}^*$. Next, to simplify the notation, we set $H_a = T_a^* H T_a$, $H_\alpha = \alpha.H$, and note that since these operators are isomorphisms $\mathcal{G} \rightarrow \mathcal{G}^*$, we have $H_a^{-1} - H_\alpha^{-1} = H_a^{-1}(H_\alpha - H_a)H_a^{-1}$ as operators $\mathcal{G}^* \rightarrow \mathcal{G}$, which clearly implies $s\text{-}\lim_{a \rightarrow \alpha} T_a^* H^{-1} T_a = H_\alpha^{-1}$ in $\mathcal{B}(\mathcal{G}^*, \mathcal{G})$, which is more than enough to prove the convergence of the self-adjoint operators $T_a^* H T_a$ to the self-adjoint operator $\alpha.H$ in $L^2(X)$ in the sense required in Corollary 4.6. \square

In the next theorem, we consider operators of the form $h(p) + V$, with V unbounded, and impose on h the simplest conditions that ensure that the form domain of $h(p)$ is stable under the operators M_k . Obviously, much more general conditions could have been used to obtain the same result, however, these conditions are well adapted to elliptic operators with non-smooth coefficients. For any real number s , let $\mathcal{H}^s \equiv \mathcal{H}^s(X)$ be the Sobolev space of order s on X . Also, let $|\cdot|$ be any norm on X^* .

Theorem 5.3. *Let $h : X^* \rightarrow [0, \infty)$ be a locally Lipschitz function with derivative h' such that, for some real numbers $c, s > 0$ and all $k \in X^*$ with $|k| > 1$, we have:*

$$c^{-1}|k|^{2s} \leq h(k) \leq c|k|^{2s} \quad \text{and} \quad |h'(k)| \leq c(1 + |k|^{2s}). \quad (5.6)$$

Let $V : \mathcal{H}^s \rightarrow \mathcal{H}^{-s}$ symmetric such that $V \geq -\mu h(p) - \nu$, for some numbers μ, ν , with $\mu < 1$. We assume that V satisfies the following two conditions:

- (i) $\lim_{k \rightarrow 0} \|[M_k, V]\|_{\mathcal{H}^s \rightarrow \mathcal{H}^{-s}} = 0$,
- (ii) $\forall \alpha \in \mathbb{S}_X$ the limit $\alpha.V := s\text{-}\lim_{a \rightarrow \alpha} T_a^* V T_a$ exists strongly in $B(\mathcal{H}^s, \mathcal{H}^{-s})$.

Then $h(p) + V$ and $h(p) + \alpha.V$ are symmetric operators $\mathcal{H}^s \rightarrow \mathcal{H}^{-s}$ and the operators H and $\alpha.H$ associated to them in $L^2(X)$ are self-adjoint and affiliated to $\mathcal{S}(X) := \mathcal{C}(\overline{X}) \rtimes X$. Moreover, the essential spectrum of H is given by the relation $\sigma_{\text{ess}}(H) = \cup_\alpha \sigma(\alpha.H)$.

Proof. If we denote $w = \sqrt{1 + h}$, then the form domain \mathcal{G} of $h(p)$ is $\mathcal{G} = D(w(p)) = \mathcal{H}^s$. The second condition of Lemma 5.1 will be satisfied if $\sup_{|\ell| < 1} h(k + \ell) \leq c(1 + h(k))$ for some number $c > 0$, which is clearly true under our assumptions on h . Then note that we have $h(p) + V + \nu + 1 \geq (1 - \mu)h(p) + 1$ as operators $\mathcal{G} \rightarrow \mathcal{G}^*$ and this estimate remains true if V is replaced by $\alpha.V$. It follows that $h(p) + V + \nu + 1 : \mathcal{G} \rightarrow \mathcal{G}^*$ is bijective hence the operator H induced by $h(p) + V$ in $L^2(X)$ is self-adjoint. The same method applies to $\alpha.H$. Thus the conditions of Theorem 5.2 are satisfied and we may use it to get the results of the present theorem. \square

Example 5.4. The simplest examples which are covered by the preceding result are the usual elliptic symmetric operators $\sum_{|s|, |t| \leq m} p^s g_{st} p^t$ with bounded measurable coefficients g_{st} such that $\lim_{a \rightarrow \alpha} g_{st}(x + a) = g_{st}^\alpha$ exists for each $x \in X$ and $\alpha \in \mathbb{S}_X$. Here $X = \mathbb{R}^n$, the indices s, t belong to \mathbb{N}^n with for example $|s| = s_1 + \dots + s_n$, and $p^s = p_1^{s_1} \dots p_n^{s_n}$ with $p_j = -i\partial_{x_j}$. The operator in $L^2(X)$ associated with the preceding differential expression will be self-adjoint and bounded from below with the usual Sobolev space \mathcal{H}^m as form domain if $\sum_{|s|=|t|=m} \langle p^s u | g_{st} p^t u \rangle \geq \mu \|u\|_{\mathcal{H}^m}^2 - \nu \|u\|^2$ for some numbers $\mu, \nu > 0$. Then the localizations at infinity will be the operators $\alpha.H$ of the same form, but with g_{st} replaced by g_{st}^α . Note that we could allow the lower order coefficients g_{st} to be more singular and all the g_{st} could be non-local operators.

Remark 5.5. The situations considered in Example 5.4 could give the wrong impression that the localizations at infinity $\alpha.H$ are self-adjoint operators in the usual sense on $L^2(X)$. The following example shows that this is not true even in simple situations. Let $H = p^2 + v(q)$ in $L^2(\mathbb{R})$ with $v(x) = 0$ if $x < 0$ and $v(x) = x$ if $x \geq 0$. It is clear that H has the position-momentum limit property and if $R = (H + 1)^{-1}$ it is not difficult to check that $\text{s-lim}_{a \rightarrow +\infty} T_a^* R T_a = 0$ and $\text{s-lim}_{a \rightarrow -\infty} T_a^* R T_a = (p^2 + 1)^{-1}$. Indeed, the translated potentials $v_a(x) = (T_a^* v(q) T_a)(x) = v(x + a)$ form an increasing family, i.e. $v_a \leq v_b$ if $a \leq b$, such that $v_a(x) \rightarrow +\infty$ if $a \rightarrow +\infty$ and $v_a(x) \rightarrow 0$ if $a \rightarrow -\infty$. Thus $H_{+\infty} = \infty$, in the sense that its domain is equal to $\{0\}$, and $H_{-\infty} = p^2$.

Remark 5.6. In view of the Remark 5.5, it is tempting to see what happens in the case of the Stark Hamiltonian $H = p^2 + q$. In fact the situation is much worse: H has not the position-momentum property (both conditions of Definition 2.1 are violated by the resolvent of H) and we have $\text{s-lim}_{|a| \rightarrow \infty} T_a^* H T_a = \infty$ and $\text{s-lim}_{|k| \rightarrow \infty} M_k^* H M_k = \infty$, while the essential spectrum of H is \mathbb{R} . So the localizations of H in the regions $|p| \sim \infty$ and $|q| \sim \infty$ say nothing about the essential spectrum of H .

We now recall some definitions and a result from [9] that can be also be used for operators that are not semi-bounded and that will be especially useful in the more general context of N -body Hamiltonians.

Let H_0 be a self-adjoint operator on a Hilbert space \mathcal{H} with form domain \mathcal{G} . We say that a continuous sesquilinear form V on \mathcal{G} (i.e. a symmetric linear map $V : \mathcal{G} \rightarrow \mathcal{G}^*$) is a *standard form perturbation* of H_0 if there are positive numbers μ, ν with $\mu < 1$ such that either $\pm V \leq \mu |H_0| + \nu$ or H_0 is bounded from below and $V \geq -\mu H_0 - \nu$. In this case the operator in H in \mathcal{H} associated to $H_0 + V : \mathcal{G} \rightarrow \mathcal{G}^*$ is self-adjoint (see the comments at the beginning of this section).

We do not need to recall the definition of *strict affiliation*, but we use the following alternative definition: a self-adjoint operator H is strictly affiliated to a C^* -algebra \mathcal{C} of operators on \mathcal{H} if, and only if, there is $\theta \in C_0(\mathbb{R})$ with $\theta(0) = 1$ such that $\lim_{\varepsilon \rightarrow 0} \|\theta(\varepsilon H)C - C\| = 0$, for all $C \in \mathcal{C}$. The following is a consequence of Theorem 2.8 and Lemma 2.9 in [9].

Theorem 5.7. *Let H_0 be a self-adjoint operator, V a standard form perturbation of H_0 , and $H = H_0 + V$ the self-adjoint operator defined above. Assume that H_0 is strictly affiliated to a C^* -algebra \mathcal{C} of operators on \mathcal{H} . If there is $\phi \in C_0(\mathbb{R})$ with $\phi(x) \sim |x|^{-1/2}$ for large x such that $\phi(H_0)^2 V \phi(H_0) \in \mathcal{C}$, then H is also strictly affiliated to \mathcal{C} .*

We may of course replace $\phi(H_0)^2 V \phi(H_0) \in \mathcal{C}$ by the more symmetric and simpler looking condition $\phi(H_0) V \phi(H_0) \in \mathcal{C}$, but this will not cover in the applications the case when the operator V is of the same order as H_0 .

The next proposition is an immediate consequence of Theorem 5.7. Note that below the form domain of $h(p)$ is the domain of $k(p)$, where k is the function $|h|^{1/2}$. It is clear that if h is a proper continuous function, then $h(p)$ is strictly affiliated to $\mathcal{S}(X)$.

Proposition 5.8. *Let $H = h(p) + V$, where $h : X^* \rightarrow \mathbb{R}$ is a continuous proper function and V is a standard form perturbation of $h(p)$. If $(1 + |h(p)|)^{-1} V (1 + |h(p)|)^{-1/2}$ belongs to $\mathcal{S}(X)$ then H is strictly affiliated to $\mathcal{S}(X)$.*

We may replace above $(1 + |h|)^{-1/2}$ by any function of the form $\theta \circ h$ with θ as in Theorem 5.7. Indeed, $\mathcal{C}_b(X^*)$ is obviously included in the multiplier algebra of $\mathcal{S}(X)$.

For $0 \leq s \leq 1$ let $\mathcal{G}^s = D(|h(p)|^s)$ equipped with the graph topology and let \mathcal{G}^{-s} be its adjoint space. So $\mathcal{G}^1 = \mathcal{G}$, $\mathcal{G}^0 = \mathcal{H}$ and $\mathcal{G}^{-1} = \mathcal{G}^*$. If V is a continuous symmetric form on \mathcal{G} such that $V \mathcal{G}^1 \subset \mathcal{G}^{-s}$ for some $s < 1$ then for each $\mu > 0$ there is a real ν such that $\pm V \leq \mu |h(p)| + \nu$, hence V is a standard form perturbation of $h(p)$ and H is well defined.

Corollary 5.9. *Let $H = h(p) + V$, where $h : X^* \rightarrow \mathbb{R}$ is a continuous proper function, and let V be a continuous symmetric form on \mathcal{G} such that $V\mathcal{G}^1 \subset \mathcal{G}^{-s}$ with $s < 1$. Let ϕ be a smooth function such that $\phi(x) \sim |x|^{-1/2}$ for large x and denote $L = \phi(H_0)V\phi(H_0)$. If $\lim_{k \rightarrow 0} \|[M_k, L]\| = 0$ and $\alpha.V = s\text{-}\lim_{a \rightarrow \alpha} T_a^*VT_a$ exists in $\mathcal{B}(\mathcal{G}, \mathcal{G}^*)$ for each $\alpha \in \mathbb{S}_X$, then H is affiliated to $\mathcal{S}(X)$, we have $\alpha.H = h(p) + \alpha.V$, and $\sigma_{\text{ess}}(H) = \cup_{\alpha} \sigma(\alpha.H)$.*

Two more comments in connection with the results of this section (and of the next one). We stated the applications of the abstract theorems in a way adapted to elliptic operators, but the extension to hypoelliptic operators is easy: it suffices to consider functions $h \in C^m$ with derivatives of order m bounded and to replace the Sobolev spaces by spaces associated to weights of the form $\sum_{|\gamma| \leq m} |h^{(\gamma)}(k)|$. On the other hand, it is obvious that Theorem 5.7 covers matrix differential operators, e.g. the Dirac operator (which are not semi-bounded). Indeed, it suffices to replace $L^2(X)$ by $L^2(X) \otimes E$ with E a finite dimensional Hilbert space and to consider the algebra $\mathcal{S}(X) \times \mathcal{B}(E)$. A more general and natural object is $\mathcal{S}(X) \otimes \mathcal{K}(E)$ where E can be infinite dimensional [18, Section 4].

6. N-BODY TYPE INTERACTIONS

In this section we introduce and study the algebra of potentials (or elementary interactions) in the N -body case.

6.1. The algebra of elementary interactions. We recall that, for each linear subspace $Y \subset X$, we have denoted by $\pi_Y : X \rightarrow X/Y$ the canonical surjection. The map $u \mapsto u \circ \pi_Y$ then gives an embedding $C_b^u(X/Y) \subset C_b^u(X)$, which will be systematically used below. Consequently, the subalgebras of $C_b^u(X/Y)$ will be thought of as subalgebras of $C_b^u(X)$, for example $C_0(X/Y)$ and $\mathcal{C}(\overline{X/Y})$ are regarded as embedded in $C_b^u(X)$.

As explained in Subsection 1.1 in the Introduction, the (abelian) algebra of elementary interactions of type $\overline{\mathcal{C}}$ is defined by:

$$\mathcal{E}(X) := \langle \mathcal{C}(\overline{X/Y}), Y \subset X \rangle := C^*\text{-subalgebra of } C_b^u(X) \text{ generated by } \mathcal{C}(\overline{X/Y}), \quad (6.1)$$

where Y ranges through *all* subspaces of X . The algebra $\mathcal{E}(X)$ will play a leading role in our approach. From the definition, it follows that $\mathcal{E}(X)$ is a translation invariant subalgebra since the generating subspaces $\mathcal{C}(\overline{X/Y})$ are already translation invariant. The “ N -body type Hamiltonians” we are interested in turn out to be self-adjoint operators affiliated to the crossed product $\mathcal{E}(X) \rtimes X$. We shall thus study general self-adjoint operators affiliated to the crossed product $\mathcal{E}(X) \rtimes X$. The algebra $\mathcal{E}(X)$ is not graded, as in the standard N -body framework of the algebra $\mathcal{R}_{C_0}(X)$ defined in the Introduction, but has a natural filtration that plays an important role in our analysis.

Let us fix a linear subspace $Z \subset X$. Then X/Z is a finite dimensional real vector space, and hence the C^* -algebra $\mathcal{E}(X/Z) \subset C_b^u(X/Z)$ is well defined and the embedding $C_b^u(X/Z) \subset C_b^u(X)$ allows us to identify $\mathcal{E}(X/Z)$ with a C^* -subalgebra of $\mathcal{E}(X)$. If $Y \supset Z$ is another linear subspace then $Y/Z \subset X/Z$ and we may identify $X/Y = (X/Z)/(Y/Z)$. Therefore we can identify

$$\mathcal{E}(X/Z) = C^*\text{-subalgebra of } \mathcal{E}(X) \text{ generated by } \cup_{Z \subset Y} \mathcal{C}(\overline{X/Y}). \quad (6.2)$$

Thus, the C^* -algebra $\mathcal{E}(X)$ is equipped with a family of C^* -subalgebras $\mathcal{E}(X/Y)$, where Y runs over the set of linear subspaces of X , such that, for $0 \subset Z \subset Y \subset X$, we have

$$\mathbb{C} = \mathcal{E}(0) = \mathcal{E}(X/X) \subset \mathcal{E}(X/Y) \subset \mathcal{E}(X/Z) \subset \mathcal{E}(X). \quad (6.3)$$

Recall now that \mathbb{S}_X consists of the half-lines of X . We shall denote by $[\alpha]$ the one dimensional subspace generated by a half-line $\alpha \in \mathbb{S}_X$. Observe that the algebras $\mathcal{E}(X/[\alpha])$ are maximal among the non-trivial subalgebras of $\mathcal{E}(X)$ of the form $\mathcal{E}(X/Y)$.

Translation at infinity along a direction $\alpha = \mathbb{R}_+ a \in \mathbb{S}_X$ gives us a linear projection τ_α of $\mathcal{E}(X)$ onto the subalgebra $\mathcal{E}(X/[\alpha])$ as follows. Let us define for $u \in \mathcal{E}(X)$ and $a \in X \setminus \{0\}$

$$\tau_\alpha(u)(x) := \lim_{r \rightarrow +\infty} u(ra + x). \quad (6.4)$$

Lemma 6.1. *Let $Y \subset X$ be a real, linear subspace and $u \in \mathcal{C}(\overline{X/Y})$. Then*

$$\tau_\alpha(u) = \begin{cases} u(\pi_Y(\alpha)) \in \mathbb{C} & \text{if } \alpha \notin Y \\ u & \text{if } \alpha \subset Y. \end{cases}$$

Proof. If $\alpha \notin Y$, $\pi_Y(\alpha)$ is a half line in X/Y , and hence $u(\pi_Y(\alpha))$ is defined. The fact that the limit is as stated follows from the definition. \square

Note that in the above lemma $\tau_\alpha(u)$ is a constant if $\alpha \notin Y$. The lemma gives right away the following.

Proposition 6.2. *If $\alpha \in \mathbb{S}_X$ and $u \in \mathcal{E}(X)$, then, for any $a \in \alpha$, the limit $\tau_\alpha(u)(x)$ exists for all $x \in X$, is independent of the choice of $a \in \alpha$, and $\tau_\alpha(u) \in \mathcal{E}(X)$. The map $\tau_\alpha : \mathcal{E}(X) \rightarrow \mathcal{E}(X)$ is an algebra morphism with range $\mathcal{E}(X/[\alpha])$ and $\tau_\alpha(u) = u$ for all $u \in \mathcal{E}(X/[\alpha])$.*

Proof. Lemma 6.1 shows that the map τ_α maps $\mathcal{C}(\overline{X/Y})$ to itself, if $\alpha \subset Y$, and maps $\mathcal{C}(\overline{X/Y})$ to \mathbb{C} otherwise. The subspace of $B \subset \mathcal{E}(X)$ for which the limit $\tau_\alpha(u)(x)$ exists for any x is a norm closed, conjugation invariant subalgebra of $\mathcal{E}(X)$. Since B contains the generators of $\mathcal{E}(X)$, we obtain that $B = \mathcal{E}(X)$. Consequently, the limit $\tau_\alpha(u)(x)$ exists for all $u \in \mathcal{E}(X)$ and all $x \in X$. Also, we obtain that τ_α maps the generators of $\mathcal{E}(X)$ to a system of generators of $\mathcal{E}(X/[\alpha]) \subset \mathcal{E}(X)$, and hence τ_α maps $\mathcal{E}(X)$ onto $\mathcal{E}(X/[\alpha])$ surjectively.

To complete the proof, we notice that $\tau_\alpha \circ \tau_\alpha = \tau_\alpha$ on the standard system of generators of $\mathcal{E}(X)$, and hence $\tau_\alpha = id$ on the range of τ_α , that is, on $\mathcal{E}(X/[\alpha])$. \square

Remark 6.3. Thus for each $\alpha \in \mathbb{S}_X$ the relation (6.4) defines a unital endomorphism τ_α of $\mathcal{E}(X)$ which is also a linear projection of $\mathcal{E}(X)$ onto the subalgebra $\mathcal{E}(X/[\alpha])$. We note that τ_α does not commute with τ_β in general: if a subspace Z does not contain α and β and $u \in \mathcal{C}(\overline{X/Z})$ then $\tau_\alpha \tau_\beta(u) = u(\pi_Z(\beta))$ and $\tau_\beta \tau_\alpha(u) = u(\pi_Z(\alpha))$.

Remark 6.4. For the purpose of this paper, the elements of $\mathcal{E}(X)$ should be thought as multiplication operators on the space $L^2(X)$. If, according to the notational conventions from the beginning of Section 2, we denote by $u(q)$ the operator of multiplication by $u \in \mathcal{E}(X)$ and, if we set $\tau_\alpha(u(q)) = \tau_\alpha(u)(q)$, then we get an expression similar to (4.4):

$$\tau_\alpha(u(q)) = \text{s-lim}_{r \rightarrow +\infty} T_{ra}^* u(q) T_{ra} = \text{s-lim}_{r \rightarrow +\infty} u(ra + q). \quad (6.5)$$

We emphasize however that $\text{s-lim}_{a \rightarrow \alpha} T_a^* u(q) T_a$ does not exist for general $u \in \mathcal{E}(X)$.

The next few results concern the subalgebras $\mathcal{E}(X/Y)$.

Proposition 6.5. *Let $\bar{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ be a system of half-lines, which generate a subspace Y of X . Then*

$$\mathcal{E}(X/Y) = \mathcal{E}(X/[\alpha_1]) \cap \dots \cap \mathcal{E}(X/[\alpha_n]). \quad (6.6)$$

The morphism $\tau_{\bar{\alpha}} := \tau_{\alpha_1} \tau_{\alpha_2} \dots \tau_{\alpha_n}$ is a linear projection of $\mathcal{E}(X)$ onto $\mathcal{E}(X/Y)$.

Proof. If $u \in \mathcal{C}(\overline{X/Z})$, for some Z , then Lemma 6.1 gives $\tau_{\overline{\alpha}}(u) = u$ if $Y \subset Z$ and $\tau_{\overline{\alpha}}(u) \in \mathbb{C}$ otherwise. In any case, $\tau_{\overline{\alpha}}(u) \in \mathcal{E}(X/Y)$. Since $\tau_{\overline{\alpha}}$ is a morphism, we see that the range of $\tau_{\overline{\alpha}}$ is included in $\mathcal{E}(X/Y)$ and $\tau_{\overline{\alpha}}(u) = u$ if $u \in \mathcal{E}(X/Y)$. Thus $\tau_{\overline{\alpha}}$ is a linear projection of $\mathcal{E}(X)$ onto $\mathcal{E}(X/Y)$. Hence if $u \in \mathcal{E}(X)$ we have $u \in \mathcal{E}(X/Y)$ if and only if $\tau_{\overline{\alpha}}(u) = u$. If u belongs to the right hand side of (6.6) then $\tau_{\overline{\alpha}}(u) = u$, so $u \in \mathcal{E}(X/Y)$. \square

Note that a permutation of the $\alpha_1, \alpha_2, \dots, \alpha_n$ will give a different projection onto $\mathcal{E}(X/Y)$ (see Remark 6.3). More generally, if $\overline{\beta} = (\beta_1, \dots, \beta_m)$ is a second system of half-lines which generates Y , then $\tau_{\overline{\beta}}$ is a projection $\mathcal{E}(X) \rightarrow \mathcal{E}(X/Y)$ distinct from $\tau_{\overline{\alpha}}$ in general.

By using (6.3) and (6.6) we get

$$\mathcal{E}(X/Y) = \bigcap_{\alpha \subset Y} \mathcal{E}(X/[\alpha]) = \{u \in \mathcal{E}(X), \tau_{\alpha}(u) = u \forall \alpha \subset Y\} \quad (6.7)$$

from which we get

$$\mathcal{E}(X/Y) = \{u \in \mathcal{E}(X), u(x+y) = u(x) \forall y \in Y\} = \mathcal{E}(X) \cap \mathcal{C}_b^u(X/Y). \quad (6.8)$$

Indeed, if C is the middle term in (6.8), then $\mathcal{E}(X/Y) \subset C$, by the definition of $\mathcal{E}(X/Y)$ and the definition of τ_{α} shows that C is included in the right hand side of (6.7).

Proposition 6.6. *If Y, Z are subspaces of X then $\mathcal{E}(X/(Y+Z)) = \mathcal{E}(X/Y) \cap \mathcal{E}(X/Z)$.*

Proof. Let Y', Z' be supplements of $Y \cap Z$ in Y and Z respectively. Choose a basis a_1, \dots, a_n of $Y+Z$ such that a_1, \dots, a_i is a basis of Y' , then a_{i+1}, \dots, a_j is a basis of $Y \cap Z$, and a_{j+1}, \dots, a_n is a basis of Z' . Denote α_k the half-line determined by a_k . From (6.6) we get

$$\mathcal{E}(X/Y) = \bigcap_{k < j} \mathcal{E}(X/[\alpha_k]) \quad \text{and} \quad \mathcal{E}(X/Z) = \bigcap_{k > i} \mathcal{E}(X/[\alpha_k])$$

hence $\mathcal{E}(X/Y) \cap \mathcal{E}(X/Z) = \bigcap_{k=1}^n \mathcal{E}(X/[\alpha_k])$ which is $\mathcal{E}(X/(Y+Z))$ by (6.6). \square

6.2. The character space. We now turn to the study of the spectrum (or character space) of the algebra $\mathcal{E}(X)$ of elementary interactions. We begin with an elementary remark.

Let $x \in X$. Then to x there corresponds the character $\chi_x(u) = u(x)$ on $\mathcal{C}_b^u(X)$. The character χ_x is completely determined by its restriction to the ideal $\mathcal{C}_0(X)$ of $\mathcal{C}_b^u(X)$. Similarly, if $\alpha \in \overline{X}$, then α defines a character $\chi_{\alpha} : \mathcal{C}(\overline{X}) \rightarrow \mathbb{C}$ by $\chi_{\alpha}(u) = u(\alpha)$.

The following lemma and its corollary will provide a crucial ingredient in the proof of Theorem 6.14 identifying the spectrum of $\mathcal{E}(X)$, which is one of our main results.

Lemma 6.7. *Let $Y \subset X$ be a subspace, let B be the C^* -algebra generated by $\mathcal{C}(\overline{X})$ and $\mathcal{C}(\overline{X/Y})$ in $\mathcal{C}_b^u(X)$, and let $\alpha \in \mathbb{S}_X \setminus \mathbb{S}_Y$. Then the character χ_{α} of $\mathcal{C}(\overline{X})$ extends to a unique character of B . This extension is the restriction of τ_{α} to B .*

Proof. Recall that the canonical projection $\pi_Y : X \rightarrow X/Y$ extends to a continuous map $\pi_Y : \overline{X} \setminus \mathbb{S}_Y \rightarrow \overline{X/Y}$, which sends $\mathbb{S}_X \setminus \mathbb{S}_Y$ onto \mathbb{S}_Y . Thus $\beta := \pi_Y(\alpha) \in \mathbb{S}_Y$ and χ_{β} is a character of $\mathcal{C}(\overline{X/Y})$. Let χ be a character of B such that $\chi|_{\mathcal{C}(\overline{X})} = \chi_{\alpha}$. We shall verify now that $\chi|_{\mathcal{C}(\overline{X/Y})} = \chi_{\beta}$.

To prove that $\chi|_{\mathcal{C}(\overline{X/Y})} = \chi_{\beta}$, it suffices to show that the kernel of χ_{β} is included in that of χ , which means that for $u \in \mathcal{C}(\overline{X/Y})$ with $u(\beta) = 0$, we should have $\chi(u) = 0$. By a density argument, it suffices to assume that $u = 0$ on a neighborhood V of β in $\overline{X/Y}$. It is clear that we can find $v \in \mathcal{C}(\overline{X})$ with $v(\alpha) = 1$ with support in the $\pi_Y^{-1}(V)$, hence $uv = 0$. Since $u, v \in B$, we have

$$0 = \chi(uv) = \chi(u)\chi(v) = \chi(u)\chi_{\alpha}(v) = \chi(u)v(\alpha) = \chi(u).$$

This proves that $\chi|_{\mathcal{C}(\overline{X/Y})} = \chi_\beta$, as claimed.

From the relation $\chi|_{\mathcal{C}(\overline{X/Y})} = \chi_\beta$ just proved, we obtain the uniqueness of χ , since $\mathcal{C}(\overline{X})$ and $\mathcal{C}(\overline{X/Y})$ generate B . To complete the proof, let us notice that the restriction of τ_α to $\mathcal{C}(\overline{X})$ is χ_α and its restriction to $\mathcal{C}(\overline{X/Y})$ is also character, because $\alpha \not\subset Y$. Thus τ_α is a character on B and we get $\chi = \tau_\alpha|_B$ by uniqueness. This completes the proof. \square

Corollary 6.8. *Let χ_1 and χ_2 be characters of $\mathcal{E}(X)$. Let us assume that $\chi_1 = \chi_2$ on $\mathcal{E}(X/[\alpha])$ and that there exists $\alpha \in \mathbb{S}_X$ such that $\chi_1(u) = \chi_2(u) = u(\alpha)$ for all $u \in \mathcal{C}(\overline{X})$. Then $\chi_1 = \chi_2$.*

Proof. It is enough to show that $\chi_1 = \chi_2$ on each of the algebras $\mathcal{C}(\overline{X/Y})$, since the later generate $\mathcal{E}(X)$, by definition. Since $\chi_1 = \chi_2 = \chi_\alpha$ on $\mathcal{C}(\overline{X})$, we obtain $\chi_1 = \chi_2$ on all $\mathcal{C}(\overline{X/Y})$ with $\alpha \not\subset Y$, by Lemma 6.7. Since $\mathcal{C}(X/[\alpha])$ contains (indeed, it is generated by) all $\mathcal{C}(\overline{X/Y})$ with $\alpha \subset Y$, the result follows. \square

We now proceed to the construction of the characters of $\mathcal{E}(X)$. We begin with a remark concerning the simplest nontrivial case that helps to understand the general case.

Remark 6.9. If $\alpha \in \mathbb{S}_X$ and $\beta \in \mathbb{S}_{X/[\alpha]}$, then $[\beta]$ is the one dimensional subspace generated by β in $X/[\alpha]$, and hence $\pi_{[\alpha]}^{-1}([\beta])$ is a two dimensional subspace of X that we shall denote by $[\alpha, \beta]$. Note that we may and shall identify $(X/[\alpha])/[\beta]$ with $X/[\alpha, \beta]$. Then Proposition 6.2 gives us two morphisms $\tau_\alpha : \mathcal{E}(X) \rightarrow \mathcal{E}(X/[\alpha])$ and $\tau_\beta : \mathcal{E}(X/[\alpha]) \rightarrow \mathcal{E}(X/[\alpha, \beta])$ that are linear projections. Thus $\tau_\beta \tau_\alpha : \mathcal{E}(X) \rightarrow \mathcal{E}(X/[\alpha, \beta])$ is a morphism and a projection and if $a \in X/[\alpha, \beta]$ then $u \mapsto (\tau_\beta \tau_\alpha u)(a)$ is a character of $\mathcal{E}(X)$.

We now extend the construction of the above remark to an arbitrary number of half-lines. However, it will be convenient first to introduce the following notations.

Notations 6.10. Our construction involves finite sequences $\vec{\alpha} := (\alpha_1, \alpha_2, \dots, \alpha_n)$ with $0 \leq n \leq \dim(X)$ and linear subspaces $[\vec{\alpha}] := [\alpha_1, \alpha_2, \dots, \alpha_n]$ of X associated to them. If $n = 0$, then we define $\vec{\alpha}$ as the empty set and we associate to it the subspace of X reduced to zero: $[\emptyset] = \{0\}$. If $n = 1$ then $\vec{\alpha} = (\alpha_1)$ with $\alpha_1 \in \mathbb{S}_X$ and, as before, $[\alpha_1]$ is the one dimensional subspace of X generated by α_1 . The case $n = 2$ is treated in the Remark 6.9 and we extend the notation to $n \geq 3$ by induction: $\alpha_n \in \mathbb{S}_{X/[\alpha_1, \dots, \alpha_{n-1}]}$ and $[\alpha_1, \dots, \alpha_n] = \pi_Y^{-1}([\alpha_n])$ is an n -dimensional subspace of X (here $Y = [\alpha_1, \dots, \alpha_{n-1}]$). Note that we may identify $X/[\alpha_1, \dots, \alpha_n] = (X/[\alpha_1, \dots, \alpha_{n-1}])/[\alpha_n]$. We denote $\tilde{\Omega}_X^{(n)}$ the set of the just defined $\vec{\alpha}$ of length n and

$$\Omega_X^{(n)} := \{(a, \vec{\alpha}), \vec{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \tilde{\Omega}_X^{(n)}, a \in X/[\alpha_1, \dots, \alpha_n]\}.$$

In particular, $\Omega_X^{(0)} \equiv X$ and $\Omega_X^{(N)} \equiv \tilde{\Omega}_X^{(N)}$ if $N = \dim(X)$, since $[\alpha_1, \dots, \alpha_N] = X$. Let

$$\Omega_X = \bigcup_{n=0}^{\dim(X)} \Omega_X^{(n)}. \quad (6.9)$$

Definition 6.11. If $(a, \vec{\alpha}) \in \Omega_X^{(n)}$, then we define

$$\tau_{\vec{\alpha}} = \tau_{\alpha_n} \tau_{\alpha_{n-1}} \dots \tau_{\alpha_1} \quad \text{and} \quad \tau_{a, \vec{\alpha}} = \tau_a \tau_{\vec{\alpha}}, \quad (6.10)$$

which are endomorphisms of $\mathcal{E}(X)$. We agree that τ_\emptyset is the identity of $\mathcal{E}(X)$.

In particular, the range of $\tau_{\vec{\alpha}}$ is $\mathcal{E}(X/[\vec{\alpha}])$ and $\tau_{\vec{\alpha}}$ is an endomorphism of $\mathcal{E}(X)$ and a linear projection of $\mathcal{E}(X)$ onto the subalgebra $\mathcal{E}(X/[\vec{\alpha}])$. The morphisms of the form $\tau_{\vec{\alpha}}$ considered in Proposition 6.5 also have these properties, but they may be distinct from the $\tau_{\vec{\alpha}}$, the objects $\vec{\alpha}$ and $\vec{\alpha}$ being different in nature. Note also that, since $a \in X/[\vec{\alpha}]$, translation by a is a morphism τ_a of $\mathcal{E}(X/[\vec{\alpha}])$, hence $\tau_{a, \vec{\alpha}}$ is well defined.

We now introduce what will turn out to be a parametrization of the characters of $\mathcal{E}(X)$.

Definition 6.12. If $(a, \vec{\alpha}) \in \Omega_X$, we define the character $\chi_{a, \vec{\alpha}}$ of $\mathcal{E}(X)$ by the formula

$$\chi_{a, \vec{\alpha}}(u) := \chi_a(\tau_{\vec{\alpha}}(u)) = \tau_{\vec{\alpha}}(u)(a). \quad (6.11)$$

We need to explain what happens in the limit case $n = \dim(X)$.

Remark 6.13. Let $n = \dim(X)$ and $(a, \vec{\alpha}) \in \Omega_X^{(n)}$. Then $[\vec{\alpha}] = X$, and hence $X/[\vec{\alpha}] = 0$, so the only possible choice for a is $a = 0$. Moreover, $\tau_{\vec{\alpha}} : \mathcal{E}(X) \rightarrow \mathbb{C}$ is already a character. Since $\tau_0 = \text{id}$, we get $\chi_{0, \vec{\alpha}} = \tau_{\vec{\alpha}}$.

We are ready now to prove one of our main results, which is a description of all the characters of the algebra $\mathcal{E}(X)$. Recall that we denote by $\widehat{\mathcal{E}(X)}$ the character space of $\mathcal{E}(X)$.

Theorem 6.14. The map $\Omega_X \rightarrow \widehat{\mathcal{E}(X)}$ defined by $(a, \vec{\alpha}) \mapsto \chi_{a, \vec{\alpha}}$ is bijective.

Proof. The preceding construction shows that $\chi_{a, \vec{\alpha}}$ is a character, therefore we only need to show that every character χ of $\mathcal{E}(X)$ is of this form and that the pair $(a, \vec{\alpha})$ is uniquely determined. To this end, we look at the restriction of χ to the subalgebra $\mathcal{C}(\overline{X})$ and proceed by induction on the dimension of X .

Every character of $\mathcal{C}(\overline{X})$ is of the form $u \mapsto u(x) = \chi_x$ for some $x \in \overline{X}$. Hence there is a unique $x \in \overline{X}$ such that $\chi|_{\mathcal{C}(\overline{X})} = \chi_x$. If $x = a \in X$, then $\chi(u) = u(a)$ for all $u \in \mathcal{E}(X)$, and thus $\chi = \chi_{a, \emptyset}$. The characters χ of this form are characterized by the fact that the restriction of χ to $\mathcal{C}_0(X)$ is non-zero. The value of a is then determined by restriction to $\mathcal{C}_0(X)$, since there is a one-to-one correspondence between the characters of $\mathcal{C}_0(X)$ and the points of X . Thus all the characters $\chi_{a, \emptyset}$, $a \in X$, are distinct.

Now assume that $x = \alpha \in \mathbb{S}_X$ and that the assertion of the theorem is true for all vector spaces of dimension strictly less than that of X (induction hypothesis). Then the theorem holds for the space $X/[\alpha]$, so there is $\vec{\beta} = (\beta_1, \dots, \beta_k)$ with

$$\beta_1 \in X/[\alpha], \beta_2 \in X/[\alpha, \beta_1], \dots, \beta_k \in X/[\alpha, \beta_1, \dots, \beta_{k-1}],$$

such that the restriction of χ to $\mathcal{E}(X/[\alpha])$ is given by $\chi(u) = (\tau_{\vec{\beta}}u)(b)$ for some $b \in (X/[\alpha])/\vec{\beta}$. That is, $\chi = \chi_{b, \vec{\beta}}$ on $\mathcal{E}(X/[\alpha])$. Let $a = b$ and let $\vec{\alpha}$ be obtained by including α in front of the sequence $\vec{\beta}$, more precisely $\vec{\alpha} = (\alpha, \beta_1, \dots, \beta_k)$. Then $\chi_{a, \vec{\alpha}}(u) = (\tau_{\vec{\beta}} \circ \tau_{\alpha}u)(b)$ and the characters χ and $\chi_{a, \vec{\alpha}}$ coincide on $\mathcal{E}(X/[\alpha])$. On the other hand, on $\mathcal{C}(\overline{X})$, the characters χ and $\chi_{a, \vec{\alpha}}$ coincide with the character $\chi_{\alpha} : \mathcal{E}(X/[\alpha]) \rightarrow \mathbb{C}$. Therefore $\chi = \chi_{a, \vec{\alpha}}$ by Corollary 6.8.

The same argument can be used to show that we obtain a one-to-one parametrization of all these characters. We shall proceed once more by induction on the length of $\vec{\alpha}$. If $\chi_{a, \vec{\alpha}} = \chi_{b, \vec{\beta}}$, we have two possibilities: first that their restrictions to $\mathcal{C}_0(X)$ is non-zero and, second, that their restrictions to $\mathcal{C}_0(X)$ is zero. In the first case, we must have $\vec{\alpha} = \emptyset$ and $\vec{\beta} = \emptyset$, by the discussion earlier in the proof. By restricting to $\mathcal{C}_0(X)$, we also obtain $a = b \in X$. Let us assume that $\vec{\alpha} \neq \emptyset$, then $\chi_{a, \vec{\alpha}}$ restricts to zero on $\mathcal{C}_0(X)$ and hence $\vec{\beta} \neq \emptyset$ as well. Since the restrictions of $\chi_{a, \vec{\alpha}}$ and $\chi_{b, \vec{\beta}}$ to $\mathcal{C}(\overline{X}/\overline{Y})$ are χ_{α_1} and χ_{β_1} respectively, we obtain $\alpha_1 = \beta_1$. The proof is completed by induction using the restrictions of these characters to $\mathcal{E}(X/[\alpha_1])$, as in the first part of the proof. \square

We shall describe now the morphism τ_{χ} on $\mathcal{E}(X)$ defined as the translation by a character $\chi = \chi_{a, \vec{\alpha}} \in \widehat{\mathcal{E}(X)}$, see Section 2, Definition 2.2.

Theorem 6.15. The translation morphism associated to the character $\chi_{a, \vec{\alpha}}$ by Definition 2.2 is the unital endomorphism $\tau_{a, \vec{\alpha}}$ of $\mathcal{E}(X)$ introduced in Definition 6.11.

Proof. If $\chi = \chi_a \equiv \tau_{a, \emptyset}$ for some $a \in X$, then this is just the usual translation by a , i.e. $\tau_{\chi_a}(u) = \tau_a(u) = a.u$ is the function $x \mapsto u(a+x)$. In general, we have to use the definition in Definition 2.2, that is, $(\tau_\chi(u))(y) = \chi(y.u)$ for all $y \in X$. Thus, if $\chi = \chi_{a, \vec{\alpha}}$ as above, then from Definition 6.12 we get

$$(\tau_\chi(u))(x) = \chi_{a, \vec{\alpha}}(x.u) = \chi_a(\tau_{\vec{\alpha}}(x.u)).$$

It is clear that X acts by translation on each of the algebras $\mathcal{E}(X/Y)$ and that the morphism $\tau_{\vec{\alpha}} : \mathcal{E}(X) \rightarrow \mathcal{E}(X/\vec{\alpha})$ is covariant for this action, that is, $\tau_{\vec{\alpha}}(x.u) = x.(\tau_{\vec{\alpha}}(u))$. Thus

$$(\tau_\chi(u))(x) = \chi_a(x.(\tau_{\vec{\alpha}}(u))) = (x.(\tau_{\vec{\alpha}}(u)))(a) = (\tau_{\vec{\alpha}}(u))(x+a),$$

and hence we get $\tau_\chi(u) = \tau_a \tau_{\vec{\alpha}}(u)$, which is (6.10). \square

Remark 6.16. Although we shall not use this here, let us mention that in view of Remark 2.5 and of Theorem 6.15, it is interesting to notice that the action of X on the space of characters of $\mathcal{E}(X)$ is given by $\tau_x(\chi_{a, \vec{\alpha}}) = \chi_{a-\pi_{\vec{\alpha}}(x), \vec{\alpha}}$, where $\pi_{\vec{\alpha}}$ is the canonical map $X \rightarrow X/[\vec{\alpha}]$. Hence, for the determination of the essential spectrum, it is enough to consider the characters $\chi_{0, \vec{\alpha}}$ and their associated translations $\tau_{\chi_{0, \vec{\alpha}}} = \tau_{0, \vec{\alpha}} = \tau_{\vec{\alpha}}$. This issue, as well as further results, especially related to the topology on the spectrum of the algebra $\mathcal{E}(X)$, will be discussed in [20].

6.3. The Hamiltonian algebra. We now apply the results we have proved to the study of essential spectra. Since $\mathcal{E}(X)$ is a translation invariant C^* -subalgebra of $\mathcal{C}_b^u(X)$ such that $\mathcal{C}_0(X) + \mathbb{C} \subset \mathcal{E}(X)$, we may take $\mathcal{A} = \mathcal{E}(X)$ in Section 2. The algebra generated by the Hamiltonians that are of interest for us is the crossed product

$$\mathcal{E}(X) := \mathcal{E}(X) \rtimes X. \quad (6.12)$$

As explained in Section 2, $\mathcal{E}(X)$ can be thought as the closed linear subspace of $\mathcal{B}(X)$ generated by the operators of the form $u(q)v(p)$ with $u \in \mathcal{E}(X)$ and $v \in \mathcal{C}_0(X^*)$. On the other hand, since $\mathcal{C}(\overline{X/Y})$ is a translation invariant C^* -subalgebra of $\mathcal{C}_b^u(X)$ we may also consider the crossed product $\mathcal{C}(\overline{X/Y}) \rtimes X$ and we clearly have

$$\mathcal{E}(X) = C^*\text{-subalgebra of } \mathcal{B}(X) \text{ generated by } \bigcup_{Y \subset X} \mathcal{C}(\overline{X/Y}) \rtimes X. \quad (6.13)$$

Similarly, for any subspace $Y \subset X$ we may consider the crossed product $\mathcal{E}(X/Y) = \mathcal{E}(X/Y) \rtimes X$. We thus obtain a family of C^* -subalgebras of $\mathcal{E}(X)$ which, as a consequence of (6.3), has the following property: if $Z \subset Y$ then

$$C^*(X) = \mathcal{E}(0) = \mathcal{E}(X/X) \subset \mathcal{E}(X/Y) \subset \mathcal{E}(X/Z) \subset \mathcal{E}(X). \quad (6.14)$$

From the general facts described in Section 2, and by taking into account the properties of $\mathcal{E}(X)$ established in the preceding subsection, we see that for any $A \in \mathcal{E}(X)$ the map $x \mapsto \tau_x(A) = T_x^* A T_x$ extends to a strongly continuous map $\chi \mapsto \tau_\chi(A) \in \mathcal{E}(X)$ on the spectrum of $\mathcal{E}(X)$ such that

$$\tau_\chi(u(q)v(p)) = \tau_\chi(u(q))v(p) \quad \text{for all } u \in \mathcal{E}(X) \text{ and } v \in \mathcal{C}_0(X^*).$$

Here $\chi \in \widehat{\mathcal{E}(X)}$ hence it is of the form described in Theorem 6.14 and the associated endomorphism τ_χ of $\mathcal{E}(X)$ is described in (6.10). Note that, in virtue of Theorem 2.4, we are only interested in the characters that belong to the boundary $\delta(\mathcal{E}(X))$ of X in $\widehat{\mathcal{E}(X)}$, which are those with $\vec{\alpha} \neq \emptyset$. Then Proposition 2.3 and Theorem 6.14 imply:

Proposition 6.17. *Let $\chi = \chi_{a, \vec{\alpha}} \in \delta(\mathcal{E}(X))$. Then there is a unique continuous linear map $\tau_{a, \vec{\alpha}} : \mathcal{E}(X) \rightarrow \mathcal{E}(X)$ such that $\tau_{a, \vec{\alpha}}(u(q)v(p)) = (\tau_{a, \vec{\alpha}}u)(q)v(p)$ for all $u \in \mathcal{E}(X)$ and $v \in \mathcal{C}_0(X^*)$. This map is a morphism and a linear projection of $\mathcal{E}(X)$ onto its subalgebra $\mathcal{E}(X/[\vec{\alpha}]) \rtimes X$.*

Now we shall use the special form of the morphisms $\tau_{a, \vec{\alpha}}$ in order to improve the compactness criterion of Theorem 2.4.

Theorem 6.18. *Let $A \in \mathcal{E}(X)$. Then for each $\alpha \in \mathbb{S}_X$ and $a \in \alpha$ the limit $\tau_\alpha(A) \equiv \alpha.A := \text{s-lim}_{r \rightarrow +\infty} T_{ra}^* A T_{ra}$ exists and is independent of the choice of a . The map τ_α is a morphism and a linear projection of $\mathcal{E}(X)$ onto its subalgebra $\mathcal{E}(X/[\alpha]) \rtimes X$. The operator A is compact if, and only if, $\tau_\alpha(A) = 0$ for all $\alpha \in \mathbb{S}_X$.*

Proof. The first assertion follows from the preceding results, but it is easier to prove it directly. Indeed, it suffices to consider A of the form $A = u(q)v(p)$ with $u \in \mathcal{E}(X)$ and $v \in \mathcal{C}_0(X^*)$. Then $T_{ra}^* A T_{ra} = \tau_{ra}(A) = \tau_{ra}(u(q))v(p)$ which converges to $(\alpha.u)(q)v(p)$ by Proposition 6.2 (or see Remark 6.4). The properties of the endomorphism τ_α are consequences of the same proposition. Everything follows also by using general properties of crossed products and the fact that at the abelian level $\tau_\alpha : \mathcal{E}(X) \rightarrow \mathcal{E}(X/[\alpha])$ is a covariant morphism. To prove the compactness assertion, note first that $\tau_\alpha(A) = 0$ if A is compact because $T_{ra} \rightarrow 0$ weakly as $r \rightarrow \infty$. Then if $A \in \mathcal{E}(X)$ and $\tau_\alpha(A) = 0$ for all $\alpha \in \mathbb{S}_X$ then it is clear by (6.10) that $\tau_{\alpha, \vec{\alpha}}(A) = 0$ if $\vec{\alpha} \neq \emptyset$ hence $\tau_\chi(A) = 0$ for all $\chi \in \delta(\mathcal{E}(X))$, and so A is compact by Theorem 2.4. \square

Remark 6.19. If Y is a linear subspace of X then the algebras $\mathcal{E}(X/Y)$ and $\mathcal{E}(X/Y)$ are a priori defined by our formalism as algebras of operators on $L^2(X/Y)$. In Section 6.1 we defined $\mathcal{E}(X/Y)$ as a subalgebra of $\mathcal{C}_b^u(X)$ and by relation (6.8) this is natural because of our general convention to identify subalgebras of $\mathcal{C}_b^u(X/Y)$ with subalgebras of $\mathcal{C}_b^u(X)$. On the other hand, we note that the algebras $\mathcal{E}(X/Y) = \mathcal{E}(X/Y) \rtimes (X/Y)$ and $\mathcal{E}(X/Y) \rtimes X$ are quite different objects: indeed

$$\mathcal{E}(X/Y) \rtimes X \simeq (\mathcal{E}(X/Y)) \otimes C^*(Y) \quad (6.15)$$

by a general fact from the theory of crossed products, namely

$$(\mathcal{A} \otimes \mathcal{B}) \rtimes (G \times H) \simeq (\mathcal{A} \rtimes G) \otimes (\mathcal{B} \rtimes H) \quad (6.16)$$

if (\mathcal{A}, G) and (\mathcal{B}, H) are amenable C^* dynamical systems. In particular:

$$\mathcal{E}(X/[\alpha]) \rtimes X \simeq (\mathcal{E}(X/[\alpha])) \otimes C^*([\alpha]) \quad (6.17)$$

Corollary 6.20. *The map $\tau(A) = (\tau_\alpha(A))_{\alpha \in \mathbb{S}_X}$ induces an injective morphism*

$$\mathcal{E}(X)/\mathcal{K}(X) \hookrightarrow \prod_{\alpha \in \mathbb{S}_X} \mathcal{E}(X/[\alpha]) \rtimes X. \quad (6.18)$$

The following theorem is an immediate consequence of the preceding corollary.

Theorem 6.21. *Let H be a self-adjoint operator on $L^2(X)$ affiliated to $\mathcal{E}(X)$. Then for each $\alpha \in \mathbb{S}_X$ and $a \in \alpha$ the limit $\tau_\alpha(H) \equiv \alpha.H = \text{s-lim}_{r \rightarrow +\infty} T_{ra}^* H T_{ra}$ exists and is independent of the choice of a . We have $\sigma_{\text{ess}}(H) = \overline{\bigcup_{\alpha \in \mathbb{S}_X} \sigma(\alpha.H)}$.*

The question whether the union $\bigcup_{\alpha \in \mathbb{S}_X} \sigma(\alpha.H)$ is closed or not will not be treated in this paper (see [31] for related results). That the union is closed if $\mathcal{E}(X)$ is replaced by the standard N -body algebra $\mathcal{E}_{\mathcal{C}_0}(X)$ is shown in [18, Theorem 6.27] and is a consequence of the fact that $\{\tau_\alpha(A), \alpha \in \mathbb{S}_X\}$ is a compact subset of $\mathcal{E}_{\mathcal{C}_0}(X)$ for each $A \in \mathcal{E}_{\mathcal{C}_0}(X)$. Unfortunately this is not true in the present case.

Lemma 6.22. *If $A \in \mathcal{E}(X)$, then $\{\tau_\alpha(A), \alpha \in \mathbb{S}_X\}$ is a relatively compact subset of $\mathcal{E}(X)$, but is not compact in general.*

Proof. We first show that $\{\tau_\alpha(A), \alpha \in \mathbb{S}_X\}$ is a relatively compact set in $\mathcal{E}(X)$. Since the product and the sum of two relatively compact subsets is relatively compact, it suffices to prove this for A in a generating subset of the algebra $\mathcal{E}(X)$, so we may assume that $A = u(q)v(p)$ with $u \in \mathcal{C}(\overline{X/Y})$ and $v \in \mathcal{C}_0(X^*)$ for some subspace Y . Then $\tau_\alpha(A) = A$ if $\alpha \subset Y$ and $\tau_\alpha(A) = \tau_\alpha(u)v(p)$ if $\alpha \not\subset Y$. In the second case we have $\tau_\alpha(u) \in \mathbb{C}$ and $|\tau_\alpha(u)| \leq \|u\|$, so it is clear that the set of the $\tau_\alpha(A)$ is relatively compact.

We shall give now an example when this set is not closed. Let $X = \mathbb{R}^2$, $Y = \{0\} \times \mathbb{R}$, and let us identify $X/Y = \mathbb{R} \times \{0\}$. The operator A will be of the form $A = u(q)v(p)$ so that $\tau_\alpha(A) = \tau_\alpha(u)(q)v(p)$ with $u = u_0 + u_Y$ for some $u_0 \in \mathcal{C}(\overline{X})$ and $u_Y \in \mathcal{C}(\overline{X/Y})$. We have $\overline{X/Y} = \mathbb{R} \times \{0\} \equiv [-\infty, +\infty]$ hence $\mathbb{S}_{\overline{X/Y}}$ consists of two points $\pm\infty$. If $\alpha \in \mathbb{S}_X$ then $\tau_\alpha(u) = u_0(\alpha) + \tau_\alpha(u_Y)$ where $\tau_\alpha(u_Y) = u_Y$ if $\alpha \subset Y$ and $\tau_\alpha(u_Y) = u_Y(\pi_Y(\alpha))$ if $\alpha \not\subset Y$. In the last case we have only two possibilities: $\tau_\alpha(u_Y) = u_Y(+\infty)$ if α is in the open right half-plane and $\tau_\alpha(u_Y) = u_Y(-\infty)$ if α is in the open left half-plane.

Let β be the upper half-axis, i.e. $\beta = \{(0, y), y > 0\}$, and let us choose u_0 such that $u_0(\gamma) \neq u_0(\beta)$ for all $\gamma \in \mathbb{S}_X, \gamma \neq \beta$. Then choose u_Y such that $u_Y(+\infty) - u_Y(-\infty)$ be strictly larger than $u_0(\gamma) - u_0(\beta)$ for all $\gamma \in \mathbb{S}_X$. Then $\{\tau_\alpha(u), \alpha \in \mathbb{S}_X\}$ consists of the following elements: $u_0(\beta) + u_Y, u_0(-\beta) + u_Y, u_0(\alpha) + u_Y(+\infty)$ if α is in the open right half-plane, and $u_0(\alpha) + u_Y(-\infty)$ if α is in the open left half-plane. We shall prove that this set is not closed. Let $\{\alpha_n\}$ be a sequence of rays in the open right half-plane which converge to β . Then $\tau_{\alpha_n}(u) = u_0(\alpha_n) + u_Y(+\infty)$ is a sequence of complex numbers which converges to $u_0(\beta) + u_Y(+\infty)$. This number cannot be of the form $\tau_\gamma(u)$ for some $\gamma \in \mathbb{S}_X$ because if $\gamma \subset Y$ then $\tau_\gamma(u) = u_0(\gamma) + u_Y$ is not a number, if γ is in the open right half-plane then $\tau_\gamma(u) = u_0(\gamma) + u_Y(+\infty)$ which cannot be equal to $u_0(\beta) + u_Y(+\infty)$ because $u_0(\gamma) \neq u_0(\beta)$, and if γ is in the open left half-plane then $\tau_\gamma(u) = u_0(\gamma) + u_Y(-\infty)$ which cannot be equal to $u_0(\beta) + u_Y(+\infty)$ because $u_0(\gamma) - u_0(\beta) < u_Y(+\infty) - u_Y(-\infty)$. \square

Remark 6.23. It is important to notice that finding good compactifications of X related to the N -body problem is useful for the problem of approximating numerically the eigenvalues and eigenfunctions of N -body Hamiltonians [1, 15, 14, 13, 16, 37]. In particular, this gives a further justification for trying to find the structure of the character space of $\mathcal{E}(X)$.

6.4. Self-adjoint operators affiliated to $\mathcal{E}(X)$. Our purpose here is to show that the class of self-adjoint operators affiliated to $\mathcal{E}(X)$ is quite large. For this we need a more explicit description of the algebras $\mathcal{C}(\overline{X/Y}) \rtimes X$. Observe first that if Z is a supplement of Y in X , so Z is a linear subspace of X such that $Y \cap Z = \{0\}$ and $Y + Z = X$, then:

$$\mathcal{C}(\overline{X/Y}) \rtimes X = C^*(Y) \otimes \mathcal{S}(Z) \quad \text{relatively to } L^2(X) = L^2(Y) \otimes L^2(Z). \quad (6.19)$$

Indeed, $\mathcal{C}(\overline{X/Y}) \rtimes X$ is the norm closed subspace generated by the operators of the form $u(q)v(p)$ with $u \in \mathcal{C}(\overline{X/Y})$ and $v(p) \in C^*(X)$. But once Z is chosen, we may identify $\mathcal{C}(\overline{X/Y}) = 1 \otimes \mathcal{C}(\overline{Z})$ and $C^*(X) = C^*(Y) \otimes C^*(Z)$, hence (6.19). Of course, this is a particular case of the relation (6.16) from Remark 6.19.

It is useful to express (6.19) in an intrinsic way, independent of the choice of Z . This is in fact an extension of Theorem 4.2 to the present setting.

Observe first that if A is a bounded operator on $L^2(X)$ and $[A, T_y] = 0$ for all $y \in Y$, then $T_x^* A T_x$ depends only on the class $z = \pi_Y(x)$ of x in X/Y . Thus we have an action τ of X/Y on the set of operators A in the commutant of $\{T_y\}_{y \in Y}$ such that $\tau_z(A) = T_x^* A T_x$ if $\pi_Y(x) = z$. Later on we shall keep the notation $\tau_\alpha(A) = T_a^* A T_a$ for $a \in X/Y$ since the correct interpretation should be clear from the context.

Theorem 6.24. $\mathcal{C}(\overline{X/Y}) \rtimes X$ is the set of $A \in \mathcal{B}(X)$ which have the position-momentum limit property and are such that

- (i) $[A, T_y] = 0$ for all $y \in Y$,
- (ii) for each $\alpha \in \mathbb{S}_{X/Y}$ the limit $s\text{-}\lim \tau_\alpha(A)^{(*)}$ with $a \rightarrow \alpha$ in X/Y exists.

Proof. Let $\tilde{\alpha} = \pi_Y^{-1}(\tilde{\alpha})$ be the inverse image of the filter $\tilde{\alpha}$ through the map π_Y , i.e. the set of subsets of X of the form $\pi_Y^{-1}(F)$ with $F \in \tilde{\alpha}$. This is a translation invariant

filter of subsets of X and if f is a function defined on X/Y with values in a topological space \mathcal{B} then $\lim_{z \rightarrow \alpha} f(z) = b$ if and only if $\lim_{x \rightarrow \tilde{\alpha}} f \circ \pi_Y(x) = b$. It is then clear that the condition (ii) above is equivalent to the fact that $s\text{-}\lim_{x \rightarrow \tilde{\alpha}} T_x^* A T_x$ exists for each $\alpha \in \mathbb{S}_{X/Y}$. Now the proof is essentially a repetition of the proof of Theorem 4.2, the filter $\tilde{\alpha}$ on X/Y being replaced by the translation invariant filter $\tilde{\alpha}$ on X . \square

Now we describe concrete classes of self-adjoint operators affiliated to $\mathcal{E}(X)$. There is no simple analogue of Theorem 5.2 in the present context, but one can extend Proposition 4.7 and Theorem 5.3. For this we shall use the following particular case of Theorem 5.7.

Proposition 6.25. *Let H_0 be a positive operator strictly affiliated to a C^* -algebra of operators \mathcal{C} on a Hilbert space \mathcal{H} . Let V be a continuous sesquilinear form on $D(H_0^{1/2})$ such that $V \geq -\mu H_0 - \nu$ for some numbers μ, ν with $\mu < 1$. If $(H_0 + 1)^{-1} V (H_0 + 1)^{-1/2} \in \mathcal{C}$, then the form sum $H = H_0 + V$ is a self-adjoint operator strictly affiliated to \mathcal{C} .*

Now we give two affiliation criteria similar to those pointed out in the case of $\mathcal{S}(X)$.

Proposition 6.26. *Let $H = h(p) + V$ where $h : X^* \rightarrow [0, \infty)$ is a continuous, proper function and $V = \sum_Y V_Y$ where the V_Y are bounded symmetric linear operators on $L^2(X)$ that are equal to zero but for a finite number of Y and satisfy:*

- (i) $\lim_{k \rightarrow 0} \|[M_k, V_Y]\| = 0$,
- (ii) $[T_y, V_Y] = 0$ for all $y \in Y$,
- (iii) $s\text{-}\lim_{a \in X/Y, a \rightarrow \alpha} T_a^* V_Y T_a$ exists for each $\alpha \in \mathbb{S}_{X/Y}$.

Then H is affiliated to $\mathcal{E}(X)$.

Example 6.27. *For example, let V_Y be the operator of multiplication by $v_Y \circ \pi_Y$ where $v_Y : X/Y \rightarrow \mathbb{R}$ is bounded, Borel, and $\lim_{z \rightarrow \alpha} v_Y(z)$ exists for each $\alpha \in \mathbb{S}_{X/Y}$.*

We denote $|\cdot|$ a quadratic norm on X^* .

Theorem 6.28. *Let $h : X^* \rightarrow [0, \infty)$ be locally Lipschitz with derivative h' such that for some real numbers $c, s > 0$ and all $k \in X^*$ with $|k| > 1$ one has:*

$$c^{-1}|k|^{2s} \leq h(k) \leq c|k|^{2s} \quad \text{and} \quad |h'(k)| \leq c(1 + |k|^{2s}). \quad (6.20)$$

Let $V = \sum V_Y$ with $V_Y = 0$ but for a finite number of Y and $V_Y : \mathcal{H}^s \rightarrow \mathcal{H}^{-s}$ symmetric operators satisfying the following conditions:

- (i) for each $\mu > 0$ there is a real number ν that $V_Y \geq -\mu h(p) - \nu$
- (ii) $\lim_{k \rightarrow 0} \|[M_k, V_Y]\|_{\mathcal{H}^s \rightarrow \mathcal{H}^{-s}} = 0$,
- (iii) $[T_y, V_Y] = 0$ for all $y \in Y$,
- (iv) $s\text{-}\lim_{a \in \mathbb{S}_{X/Y}, a \rightarrow \alpha} T_a^* V_Y T_a$ exists in $B(\mathcal{H}^s, \mathcal{H}^{-s})$ for all $\alpha \in \mathbb{S}_{X/Y}$.

Then $h(p) + V$ is a symmetric operators $\mathcal{H}^s \rightarrow \mathcal{H}^{-s}$ which induces a self-adjoint operator H in $L^2(X)$ affiliated to $\mathcal{E}(X)$.

Both Proposition 6.26 and Theorem 6.28 follow from Proposition 6.25. Indeed, let us set $\langle p \rangle = (1 + |p|^2)^{1/2}$. Since we have $1 + h(p) \sim \langle p \rangle^{2s}$, it suffices to prove that for each Y the operator $\langle p \rangle^{-2s} V_Y \langle p \rangle^{-s}$ is in $\mathcal{C}(\overline{X/Y}) \rtimes X$. This clearly follows from Theorem 6.24.

Example 6.29. Theorem 6.28 covers uniformly elliptic operators $H = \sum_{|j|, |k| \leq s} p^j a_{jk} p^k$ (with $s \geq 1$ integer and j, k multi-indices) whose coefficients a_{jk} are finite sums of functions of the form $v_Y \circ \pi_Y$ with $v_Y : X/Y \rightarrow \mathbb{R}$ bounded measurable and such that $\lim_{z \rightarrow \alpha} v_Y(z)$ exists for each $\alpha \in \mathbb{S}_{X/Y}$. Note that we may allow the a_{jk} to be irregular (i.e. only bounded measurable) in the principal part (i.e. $|j| = |k| = s$) of the operator because above we have the power $\langle p \rangle^{-2s}$ and not $\langle p \rangle^{-s}$ on the left of V_Y . Of course, the coefficients of the lower order terms are allowed to be unbounded, cf. Theorem 6.28.

Example 6.30. We give more explicit conditions on the lower order terms in the case of non-relativistic Schrödinger operators. Then X is an Euclidean space (so we may identify $X/Y = Y^\perp$) and $H_0 := \Delta = p^2$ is the (positive) Laplace operator, hence $s = 1$. The total Hamiltonian is of the form $H = \Delta + \sum_Y V_Y$ where the sum is finite and $V_Y = 1 \otimes V_Y^\circ$ where $V_Y^\circ : \mathcal{H}^1(Y^\perp) \rightarrow \mathcal{H}^{-1}(Y^\perp)$ is a symmetric linear operator whose relative form bound with respect to the Laplace operator on Y^\perp is zero. Then assume $M_k V_Y^\circ = V_Y^\circ M_k$ for all $k \in Y^\perp$. For example, V_Y° could be the operator of multiplication by a function $v_Y : Y^\perp \rightarrow \mathbb{R}$ of Kato class $K_{n(Y)}$ with $n(Y) = \dim(Y^\perp)$ (see Section 1.2 in [8], especially assertion (2) page 8) but it could also be a distribution of non zero order. Indeed, we may take as v_Y the divergence of a vector field on Y^\perp whose components have squares of Kato class (e.g. are bounded functions): this covers highly oscillating perturbations of potentials which have radial limits at infinity. Note that the Kato class is convenient because $v_Y \circ \pi_Y$ is of class $K_{\dim(X)}$, see [8, p. 8]. To get (iv) of Theorem 6.28 it suffices to assume $\lim_{a \rightarrow \alpha} v_Y(\cdot + a)$ exists strongly in $B(\mathcal{H}^1(Y^\perp), \mathcal{H}^{-1}(Y^\perp))$ for each $\alpha \in \mathbb{S}_{Y^\perp}$.

The perturbations V considered above are all of the N -body type, i.e. they are sums of components V_Y indexed by subspaces Y of X which formally look like $V_Y = v_Y \circ \pi_Y$ with $v_Y : X/Y \rightarrow \mathbb{R}$. In the usual N -body problem this is perfectly natural because the corresponding C^* -algebra is graded, hence a product $V_Y V_Z$ is of the form $V_{Y \cap Z}$ (with a careful interpretation of the product if the potentials are unbounded). But this is not the case in the present context so we shall also give some examples where V contains products of some V_Y and V_Z .

The next proposition is an easy consequence of the definitions. Note that the multiplier algebra $\mathcal{M}\mathcal{E}(X)$ of $\mathcal{E}(X)$ can be identified with the set of operators $M \in \mathcal{B}(X)$ such that MA and AM belong to $\mathcal{E}(X)$ for all $A \in \mathcal{E}(X)$. For example $\mathcal{E}(X) \cup \mathcal{E}(X) \subset \mathcal{M}\mathcal{E}(X)$.

Proposition 6.31. *Let H_0 be a self-adjoint operator (strictly) affiliated to $\mathcal{E}(X)$. If V is a symmetric operator in $\mathcal{M}\mathcal{E}(X)$ then $H = H_0 + V$ is (strictly) affiliated to $\mathcal{E}(X)$.*

One may extend this to irregular or unbounded V by a regularization procedure. For example, assume that we are in the context of Theorem 6.28 and let V be a locally integrable function such the convolutions $\theta * V$ with C_c^∞ functions belong to $\mathcal{E}(X)$. If for any functions ξ, ξ_2 of class C_c^∞ on X^* the operators $\xi_1(p)(\theta * V)(q)\xi_2(p)$ converge in norm when $\theta \rightarrow \delta$, then $h(p) + V$ is affiliated to $\mathcal{E}(X)$. This holds if V is a bounded Borel function. One may extend this idea to operators $V : \mathcal{H}^s \rightarrow \mathcal{H}^{-s}$ even in the non local case by using an abstract version of the regularization by convolutions, namely $\int_X T_x^* V T_x \eta(x) dx$.

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