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THE PERIODIC CYCLIC HOMOLOGY OF CROSSED PRODUCTS OF FINITE TYPE ALGEBRAS

JACEK BRODZKI, SHANTANU DAVE, AND VICTOR NISTOR

Abstract. We study the periodic cyclic homology groups of the cross-product of a finite type algebra A by a discrete group Γ. In case A is commutative and Γ is finite, our results are complete and given in terms of the singular cohomology of the sets of fixed points. These groups identify our cyclic homology groups with the "orbifold cohomology" of the underlying (algebraic) orbifold. The proof is based on a careful study of localization at fixed points and of the resulting Koszul complexes. We provide examples of Azumaya algebras for which this identification is, however, no longer valid. As an example, we discuss some affine Weyl groups.

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Introduction

Let A be an algebra and let Γ be a (discrete) group acting on it by a morphism $\alpha : \Gamma \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ denotes the group of automorphisms of A. Then, to this action of Γ on A, we can associate the crossed product algebra $A \rtimes \Gamma$ consisting of finite sums of elements of the form $a\gamma$, $a \in A$, $\gamma \in \Gamma$, subject to the relations

$$\gamma a = \alpha_\gamma(a)\gamma\,. \tag{1}$$

Crossed products appear often in algebra and analysis and can be used to model several geometric structures. They play a fundamental role in non-commutative

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geometry [21]. For instance, cross-products can be used to recover equivariant K-theory [5, 6, 23, 33, 44].

Cyclic homology is a homological theory for algebras that can be used to recover the de-Rham cohomology of a smooth, compact manifold M as the periodic cyclic homology of the algebra $\mathcal{C}^\infty(M)$ of smooth functions on M [20] (see also [21, 46, 51, 52, 56, 78]). In this paper, we are interested in the algebraic counterpart of this result [33] (see also [17, 30, 31, 32, 53]), in view of its connections with the representation theory of reductive p-adic groups (see Section 3 for references and further comments), and we provide some more general results as well.

Let us assume that our algebra A is an algebra over a ring $\mathbf{k}$. So $1 \in \mathbf{k}$, but A is not required to have a unit. Nevertheless, for simplicity, in this paper, we shall restrict to the case when A has a unit. Let us assume that the group Γ acts in a compatible way on both the algebra A and the ring $\mathbf{k}$. Using α to denote the action on $\mathbf{k}$ as well, this means that

$$\alpha_\gamma(fa) \,=\, \alpha_\gamma(f)\alpha_\gamma(a)\,, \quad \text{for all} \;\; f \in \mathbf{k} \;\; \text{and} \;\; a \in A\,.$$

In this paper we study the Hochschild, cyclic, and periodic cyclic homology groups of $A \rtimes \Gamma$ using the additional information provided by the action of Γ on $\mathbf{k}$. We begin with some general results and then particularize, first, to the case when A is a finite type algebra over $\mathbf{k}$ and then, further, to the case when $A = \mathbf{k}$. (Recall [48] that A is a finite type algebra over $\mathbf{k}$ if A is a $\mathbf{k}$-algebra, $\mathbf{k}$ is a quotient of a polynomial ring, and A is finitely generated as a $\mathbf{k}$-module.)

For $\gamma \in \Gamma$, let us denote by $\langle\gamma\rangle$ the conjugacy class of $\gamma \in \Gamma$ and by $\langle\Gamma\rangle$ the set of conjugacy classes of of Γ. Our first step is to use the decomposition

$$\mathrm{HH}_q(A \rtimes \Gamma) \,\simeq\, \bigoplus_{\langle\gamma\rangle\in\langle\Gamma\rangle} \mathrm{HH}_q(A \rtimes \Gamma)_\gamma \tag{2}$$

of the Hochschild homology groups of $A \rtimes \Gamma$ [5, 6, 13, 33, 46, 62]. Let $\mathrm{Prim}(\mathbf{k})$ be the maximal ideal spectrum of $\mathbf{k}$ and denote by $\mathrm{supp}(A) \subset \mathrm{Prim}(\mathbf{k})$ the support of A. A first observation is that, if $\gamma \in \Gamma$ is such γ has no fixed points on $\mathrm{supp}(A)$, then the component corresponding to γ in the direct sum decomposition, Equation 2, vanishes, that is $\mathrm{HH}_q(A \rtimes \Gamma)_\gamma = 0$ for all q. Consequently, we also have $\mathrm{HC}_q(A \rtimes \Gamma)_\gamma = \mathrm{HP}_q(A \rtimes \Gamma)_\gamma = 0$ for all q. More generally, this allows us to show that the groups $\mathrm{HH}_q(A \rtimes \Gamma)_\gamma$, $\mathrm{HC}_q(A \rtimes \Gamma)_\gamma$, and $\mathrm{HP}_q(A \rtimes \Gamma)_\gamma$ are supported at the ideals fixed by γ.

We are especially interested in the case $A = \mathcal{O}[V]$, the algebra of regular functions on an affine algebraic variety V, in which we obtain complete results, identifying the periodic cyclic homology groups with the corresponding orbifold homology groups [19]. More precisely, we obtain the following result.

Theorem 0.1. *Let A be a quotient of the ring of polynomials $\mathbb{C}[X_1, X_2, \ldots, X_n]$ and let Γ be a finite group acting on A by automorphisms. Let $\{\gamma_1, \ldots, \gamma_\ell\}$ be a list of representatives of conjugacy classes of Γ, let C_j be the centralizer of γ_j in Γ, and let $V_j \subset V$ be the set of fixed points of γ_j acting on the subset $V \subset \mathbb{C}^n$ corresponding to set of maximal ideals of A. Then*

$$\mathrm{HP}_q(A \rtimes \Gamma) \;\cong\; \bigoplus_{j=1}^{\ell} \bigoplus_{k\in\mathbb{Z}} H^{q-2k}(V_j;\mathbb{C})^{C_j}\,.$$

We refer to [4, 28, 58, 66] for the definition of orbifold cohomology groups in the category of smooth manifolds and to its connections with cyclic homology. We are interested in the case $A = \mathcal{O}[V]$ due to its connections with affine Weyl groups.

For simplicity, we shall consider from now on only complex algebras. Thus $\mathbf{k}$ and A are complex vector spaces in a compatible way. The paper is organized as follows. In the first section we recall the definitions of Hochschild and cyclic homologies and of their twisted versions with respect to the action of an endomorphism. Then we specialize these groups to cross products and discuss the connection between the Hochschild and cyclic homologies of a crossed product and their twisted versions. Our calculations are based on completions with respect to ideals, so we discuss them in the last subsection of the first section. The main results are contained in the second section. They are based on a thorough study of the twisted Hochschild homology of a finitely generated commutative complex algebra using Koszul complexes. As an example, in the last section, we recover the cyclic homology of certain group algebras of certain affine Weyl groups.

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1. Basic definitions

We review in this section several constructions and results needed in what follows. Thus we introduce the operators b and B needed to define our various version of the Hochschild and cyclic homologies that we will use. We also recall results on the topological (or I-adic) versions of these groups, which are less common and reduce to the usual definitions when $I = 0$.

1.1. **Hochschild and cyclic homology.** Let A be a complex algebra with unit and denote by A^{op} the algebra A with the opposite multiplication and $A^e := A \otimes A^{\mathrm{op}}$, so that A becomes a left A^e–module with $(a_0 \otimes a_1) \cdot a := a_0 a a_1$.

Let g be an endomorphism of A and define for $x = a_0 \otimes a_1 \otimes \ldots \otimes a_n \in A^{\otimes n+1}$

$$\begin{aligned} b'_g(x) :=\ & a_0 g(a_1) \otimes a_2 \otimes \ldots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i a_0 \otimes \ldots \otimes a_i a_{i+1} \otimes \ldots \otimes a_n \\ \text{and} \quad b_g(x) :=\ & b'_g(x) + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}\,. \end{aligned} \tag{3}$$

Let us denote by $\mathcal{B}'_q(A) := A \otimes (A/\mathbb{C}1)^{\otimes q} \otimes A$ and $\mathcal{B}_q(A) := A \otimes (A/\mathbb{C}1)^{\otimes q}$, $q \geq 0$. Then b'_g and b_g descend to maps $b'_g : \mathcal{B}'_q(A) \to \mathcal{B}'_{q-1}(A)$ and $b_g : \mathcal{B}_q(A) \to \mathcal{B}_{q-1}(A)$. As usual, when g is the identity, we write simply b instead of b_g and b' instead of b'_g. We then define

Definition 1.1. The *g-twisted Hochschild homology groups* $\mathrm{HH}_*(A, g)$ of A are the homology groups of the complex $(\mathcal{B}_*(A), b_g) = (\mathcal{B}_q(A), b_g)_{q \geq 0}$.

Thus, when g is the identity, and hence $b_g = b$, we obtain that $\mathrm{HH}_*(A, g)$ are simply the usual Hochshild homology $\mathrm{HH}_*(A)$ groups of A.

Let A_g be the A^e *right-module* $a(a_1 \otimes a_2) := a_2 a g(a_1)$. By tensoring the complex $(\mathcal{B}'_q(A), b')_{q \geq 1}$ with A_g over A^e we obtain the complex $(\mathcal{B}_q(A), b_g)_{q \geq 0}$. Since the complex $(\mathcal{B}'_q(A), b')$ is acyclic–and thus provides a projective resolution of A with $A^e := A \otimes A^{\mathrm{op}}$ modules–we obtain [13, 33, 51, 62, 77]

Proposition 1.2. *For every $q \geq 0$, we have a natural isomorphism*

$$\mathrm{HH}_q(A, g) \simeq \mathrm{Tor}_q^{A^e}(A_g, A) . \tag{4}$$

See [9, 34] for some generalizations. Let us assume now that A is an algebra over a (commutative) ring $\mathbf{k}$. Then we see by a direct calculation that the differential b_g is $\mathbf{k}$–linear, and hence each $\mathrm{HH}_q(A, g)$ is naturally a $\mathbf{k}$–module. Moreover, both A and A_g are $\mathbf{k} \otimes \mathbf{k}$-modules:

$$(z_1 \otimes z_2)a := g(z_1)z_2 a , \quad \text{for } z_1, z_2 \in \mathbf{k} \text{ and } a \in A_g . \tag{5}$$

Consequently, $\mathrm{Tor}_q^{A^e}(A_g, A)$ is naturally a $\mathbf{k}\otimes\mathbf{k}$-module, with the action of $z_1 \otimes z_2 \in \mathbf{k} \otimes \mathbf{k}$ on either component of the Tor-group being the same. Examining the proof of Proposition 1.2, we see that the isomorphism $\mathrm{HH}_q(A, g) \simeq \mathrm{Tor}_q^{A^e}(A_g, A)$ of that proposition is compatible with these module structures. More related information is contained in the following results. We first need to recall some basic definitions.

Let R be a ring (all our rings have units). We assume $\mathbb{C} \subset R$, for simplicity (that's the only case we need anyway). We denote by $\widehat{R}$ the set of maximal ideals $\mathfrak{m}$ of R with the Jacobson topology. We assume that $R/\mathfrak{m} \simeq \mathbb{C}$ as $\mathbb{C}$-algebras for any maximal ideal $\mathfrak{m}$. Recall that if M is a module over a ring R (all our rings have units), then the *support* of M is the set of maximal ideals $\mathfrak{m} \in \widehat{R}$ such that the localization $M_\mathfrak{m} := A_\mathfrak{m} M$ is nonzero. Thus we have that $M_\mathfrak{m} = 0$ if, and only if, for every $m \in M$, there exists $x \notin \mathfrak{m}$ such that $xm = 0$.

Returning to our setting, let $\mathfrak{m}, \mathfrak{n} \in \mathrm{Spec}(\mathbf{k})$ be maximal ideals and $\chi_\mathfrak{m} : \mathbf{k}/\mathfrak{m} \to \mathbb{C}$ and $\chi_\mathfrak{n} : \mathbf{k}/\mathfrak{n} \to \mathbb{C}$ be the canonical isomorphisms. Then $(\mathfrak{m}, \mathfrak{n}) \in \mathrm{Spec}(\mathbf{k}) \times \mathrm{Spec}(\mathbf{k})$ corresponds to the morphism $\chi_\mathfrak{m} \otimes \chi_\mathfrak{n} \to \mathbf{k} \otimes \mathbf{k} \to \mathbb{C}$. This identifies the maximal ideal spectrum of $\mathbf{k} \otimes \mathbf{k}$ with $\mathrm{Spec}(\mathbf{k}) \times \mathrm{Spec}(\mathbf{k})$.

Lemma 1.3. *The support of A_g in $\mathrm{Spec}(\mathbf{k}) \times \mathrm{Spec}(\mathbf{k})$ is contained in the set*

$$\{ (g^{-1}(\mathfrak{m}),\ \mathfrak{m}),\ \mathfrak{m} \in \widehat{\mathbf{k}} \} .$$

Proof. Assume that $(\mathfrak{m}, \mathfrak{n}) \in \mathrm{Spec}(\mathbf{k}) \times \mathrm{Spec}(\mathbf{k})$ is such that $g^{-1}(\mathfrak{n}) \neq \mathfrak{m}$. Since $g^{-1}(\mathfrak{n}) \neq \mathfrak{m}$, we have that $\chi_\mathfrak{m} \neq \chi_\mathfrak{n} \circ g$. Then there exists $z \in \mathbf{k}$ such that $\chi_\mathfrak{m}(z) \neq \chi_\mathfrak{n}(g(z))$. Hence $w := z \otimes 1 - 1 \otimes g(z)$ satisfies $\chi_\mathfrak{m} \otimes \chi_\mathfrak{n}(w) = \chi_\mathfrak{m}(z) - \chi_\mathfrak{n}(g(z)) \neq 0$ is not in $(\mathfrak{m}, \mathfrak{n})$. However, if $a \in A_g$, then $wa = g(z)a - g(z)a = 0$. By definition, this means that the localization of A_g at $(\mathfrak{m}, \mathfrak{n})$ is zero, and hence $(\mathfrak{m}, \mathfrak{n})$ is not in the support of A_g. □

Recall [14, 48] that if $S \subset \mathbf{k} \otimes \mathbf{k}$ is a multiplicative subset and M and N are two bimodules (or A^e left-modules), then

$$S^{-1}\,\mathrm{Tor}_q^{A^e}(M, N) \simeq \mathrm{Tor}_q^{A^e}(S^{-1}M, S^{-1}N) \simeq \mathrm{Tor}_q^{S^{-1}A^e}(S^{-1}M, S^{-1}N) . \tag{6}$$

This shows that the support of $\mathrm{Tor}_q^{A^e}(M, N)$ is contained in the intersection of the supports of M and N.

Let us assume for the rest of this section that the given endomorphism g of A has a counterpart in an endomorphism of $\mathbf{k}$–also denoted g–such that $g(za) = g(z)g(a)$, for all $z \in \mathbf{k}$ and $a \in A$.

Corollary 1.4. *Let $S \subset \mathbf{k}$ be a g-invariant multiplicative subset. We have*

$$\mathrm{HH}_q(S^{-1}A, g) \simeq S^{-1}\,\mathrm{HH}_q(A, g) .$$

Hence the support of $\mathrm{HH}_*(A,g)$ *is contained in the set*

$$\{\,(\mathfrak{m},\mathfrak{m}),\ \mathfrak{m}\in\mathrm{Spec}(\mathbf{k}),\ g^{-1}(\mathfrak{m})=\mathfrak{m}\,\}\,.$$

Proof. The first part of the proof follows from Equation (6) for the multiplicative subset $S\otimes S\subset\mathbf{k}\otimes\mathbf{k}$. (A direct proof can be obtained as in [48] for example). The support of A is contained in the diagonal $\{(\mathfrak{m},\mathfrak{m}),\ \mathfrak{m}\in\mathrm{Spec}(\mathbf{k})\}$. The support of A_g is contained in the set $\{(g^{-1}(\mathfrak{m}),\mathfrak{m}),\ x\in\mathrm{Spec}(\mathbf{k})\}$. Since $\mathrm{HH}_q(A,g)\simeq\mathrm{Tor}_q^{A^e}(A,A_g)$ and

$$\{(\mathfrak{m},\mathfrak{m})\}\cap\{(g^{-1}(\mathfrak{m}),\mathfrak{m})\}\ =\ \{(\mathfrak{m},\mathfrak{m}),\ g^{-1}(\mathfrak{m})=\mathfrak{m}\}\ \subset\ \mathrm{Spec}(\mathbf{k})\times\mathrm{Spec}(\mathbf{k})\,,$$

the result follows. □

Of the two $\mathbf{k}$-module structures on A and A_g, we shall now retain just the one given by multiplication to the left, that is, the usual $\mathbf{k}$-module structure. We also have the following.

Proposition 1.5. *The support of* $\mathrm{HH}_q(A,g)$ *as a* $\mathbf{k}$*-module is*

$$\{\,\mathfrak{m}\in\mathrm{Spec}(\mathbf{k}),\ g^{-1}(\mathfrak{m})=\mathfrak{m}\,\}.$$

Proof. Let $\mathbf{k}_1$ and $\mathbf{k}_2$ be finitely generated commutative complex algebras with unit and let M be a $\mathbf{k}_1\otimes\mathbf{k}_2$-module with support $K\subset\mathrm{Spec}(\mathbf{k}_1\otimes\mathbf{k}_2)=\mathrm{Spec}(\mathbf{k}_1)\times\mathrm{Spec}(\mathbf{k}_2)$. Then the support of M as a $\mathbf{k}_1$ module is the projection of K onto the first component. Combining this fact with Corollary 1.4 yields the desired result. □

This proposition then gives the following standard consequences

Corollary 1.6. *If* $\mathfrak{n}\in\mathrm{Spec}(\mathbf{k})$ *is such that* $g^{-1}(\mathfrak{n})\neq\mathfrak{n}$*, then* $\mathrm{HH}_q(A,g)_{\mathfrak{n}}=0$*. In particular, if* $g^{-1}(\mathfrak{m})\neq\mathfrak{m}$ *for all maximal ideals* $\mathfrak{m}$ *of* $\mathbf{k}$*, then* $\mathrm{HH}_q(A,g)=0$*.*

Proof. Since the support of $\mathrm{HH}_q(A,g)$ is the set $\{\mathfrak{m}\in\mathrm{Spec}(\mathbf{k}),g^{-1}(\mathfrak{m})=\mathfrak{m}\}$, we obtain that $\mathfrak{n}$ is not in the support of $\mathrm{HH}_q(A,g)$, that is, $\mathrm{HH}_q(A,g)_{\mathfrak{n}}=0$. This proves the first part. The second part follows from the fact that a $\mathbf{k}$-module with empty support is zero. □

In order to introduce the closely related cyclic homology groups, let us first recall the following standard operators acting on $A^{\otimes n}$, $n\geq 0$, [20]

$$\begin{aligned} s(a_0\otimes a_1\otimes\ldots\otimes a_n)\ &:=\ 1\otimes a_0\otimes a_1\otimes\ldots\otimes a_n\,,\\ t_g(a_0\otimes a_1\otimes\ldots\otimes a_n)\ &:=\ (-1)^n g(a_n)\otimes a_0\otimes\ldots\otimes a_{n-1}\,,\ \text{and}\\ B_g(a_0\otimes a_1\otimes\ldots\otimes a_n)\ &:=\ s\sum_{k=0}^{n}t_g^k(a_0\otimes a_1\otimes\ldots\otimes a_n)\,. \end{aligned}\tag{7}$$

Let γ act diagonally on $\mathcal{B}_q(A)$: $g(a_0\otimes a_1\otimes\ldots\otimes a_n)=g(a_0)\otimes g(a_1)\otimes\ldots\otimes g(a_n)$. Then B_g descends to differential

$$B_g : X_q(A) := \mathcal{B}_q(A)/(g-1)\mathcal{B}_q(A)\to X_{q+1}(A)$$

satisfying $B_g^2=0$ and $B_gb_g+b_gB_g=0$. Therefore $(X_q(A),b_g,B_g)$ is a mixed complex [43], where b_g is the g-twisted Hochschild homology boundary map defined in Equation (3). Recall how its cyclic homology are computed. Let

$$\mathcal{C}_n(A)\ :=\ \bigoplus_{q\geq 0}X_{n-2q}(A)\,.$$

Then the *g-twisted cyclic homology* groups of the unital algebra A, denoted $\mathrm{HC}_n(A, g)$, are the homology groups of the cyclic complex $(\mathcal{C}_*(A), b_g + B_g)$ [20]. The homology groups of the 2-periodic complex

$$b_g + B_g \,:\, \prod_{q\in\mathbb{Z}} X_{i+2q}(A) \to \prod_{q\in\mathbb{Z}} X_{i+1+2q}(A)\,, \quad i \in \mathbb{Z}/2\mathbb{Z}\,,$$

are the *g-twisted periodic cyclic homology* groups of the unital algebra A and are denoted $\mathrm{HP}_n(A, g)$.

1.2. **Crossed-products.** Let in this subsection A be an arbitrary complex algebra and Γ be a group acting on A by automorphisms: $\alpha : \Gamma \to \mathrm{Aut}(A)$. We do not assume any topology on Γ, that is, Γ is *discrete*. Then $A \rtimes \Gamma$ is generated by $a\gamma$, $a \in A$, $\gamma \in \Gamma$, subject to the relation $a\gamma b\gamma' = a\alpha_\gamma(b)\gamma\gamma'$.

We want to study the cyclic homology of $A \rtimes \Gamma$. Both the cyclic and Hochschild complexes of $A \rtimes \Gamma$ decompose in direct sums of complexes indexed by the conjugacy classes $\langle\gamma\rangle$ of Γ. Explicitly, to $\langle\gamma\rangle := \{g\gamma g^{-1}, g \in \Gamma\}$ there is associated the subcomplex of the Hochschild complex $(\mathcal{B}_q(A \rtimes \Gamma), b)$ generated linearly by tensors of the form $a_0\gamma_0 \otimes a_1\gamma_1 \otimes \ldots \otimes a_k\gamma_k$ with $\gamma_0\gamma_1 \ldots \gamma_k \in \langle\gamma\rangle$. The homology groups of this subcomplex of the Hochschild complex $(\mathcal{B}_q(A \rtimes \Gamma), b)$ associated to $\langle\gamma\rangle$ will be denoted by $\mathrm{HH}_*(A \rtimes \Gamma)_\gamma$. We similarly define $\mathrm{HC}_*(A \rtimes \Gamma)_\gamma$ and $\mathrm{HP}_*(A \rtimes \Gamma)_\gamma$, see [33, 46, 62] for instance. This yields the decomposition of Equation (2). Similarly,

$$\mathrm{HC}_q(A \rtimes \Gamma) \,\simeq\, \oplus_{\langle\gamma\rangle\in\langle\Gamma\rangle} \mathrm{HC}_q(A \rtimes \Gamma)_\gamma\,. \tag{8}$$

If Γ has finitely many conjugacy classes, a similar relation holds also for periodic cyclic homology. Such decompositions hold more generally for group-graded algebras [53].

Let us assume now that Γ is finite. Let, for any $g \in \Gamma$, C_g denote the centralizer of g in Γ, that is, $C_g = \{\gamma \in \Gamma, g\gamma = \gamma g\}$. Then we have the following result [8, 13, 15, 32, 33, 62, 53, 61, 72].

Proposition 1.7. *Let Γ be a finite group acting on A and $\gamma \in \Gamma$. Then we have natural isomorphisms $\mathrm{HH}_q(A \rtimes \Gamma)_\gamma \simeq \mathrm{HH}_q(A, \gamma)^{C_\gamma}$, $\mathrm{HC}_q(A \rtimes \Gamma)_\gamma \simeq \mathrm{HC}_q(A, \gamma)^{C_\gamma}$, and $\mathrm{HP}_q(A \rtimes \Gamma)_\gamma \,\simeq\, \mathrm{HP}_q(A, \gamma)^{C_\gamma}$.*

An approach to this result (as well as to the spectral sequence for the case when Γ is not necessarily finite), using cyclic objects and their generalizations, is contained in [62]. This proposition explains why we are interested in the twisted Hochschild homology groups. The same proof will apply in the case of algebras endowed with filtrations. A proof of this proposition in the more general case of filtered algebras is provided in Proposition 1.13. Algebras with these properties are called *topological algebras* in [41], but this terminology conflicts with the terminology used in other papers in the field.

1.3. **Finite type algebras and completions.** We continue to assume that A is a $\mathbf{k}$–algebra with unit, for some commutative ring $\mathbf{k}$, $\mathbb{C} \subset \mathbf{k}$. We shall need the following definition from [48]

Definition 1.8. A $\mathbf{k}$-algebra A is called a *finite type* $\mathbf{k}$*-algebra* if $\mathbf{k}$ is a finitely generated ring and A is a finitely generated $\mathbf{k}$-module.

We shall need to consider completions of our algebras and complexes with respect to the topology defined by the powers of an ideal. Let us consider then a vector space V endowed with a decreasing filtration

$$V = F_0V \supset F_1V \supset \ldots \supset F_nV \supset \ldots .$$

Then its *completion* is defined by

$$\widehat{V} = \varprojlim V/F_jV . \tag{9}$$

If the natural map $V \to \widehat{V}$ is an isomorphism, we shall say that V is *complete*. A linear map $\phi : V \to W$ between two filtered vector spaces is called *continuous* if for every integer $n \geq 0$ there is an integer $k \geq 0$ such that $\phi(F_kV) \subset F_nW$. It is called *bounded* if there is an integer k such that $\phi(F_nV) \subset F_{n-k}W$ for all n. Clearly, every bounded map is continuous. A bounded map $\phi : V \to W$ of filtered vector spaces extends to a linear map $\widehat{\phi} : \widehat{V} \to \widehat{W}$ between their completions. If V and W are filtered vector spaces, the completed tensor product is defined as

$$V\widehat{\otimes}W := \varprojlim V/F_jV \otimes W/F_jW . \tag{10}$$

We shall need the following standard lemma

Lemma 1.9. *Let $f_* : (V_*, d) \to (W_*, d)$, $* \geq 0$, be a morphism of filtered complexes with $f_q(F_nV_q) \subset F_nW_q$ for all $n, q \geq 0$. We assume that all groups V_q and W_q, $q \geq 0$, are complete and that f_* induces isomorphisms $H_q(F_nV/F_{n+1}V) \to H_q(F_nW/F_{n+1}W)$ for all $n \geq 0$ and $q \geq 0$. Then f is a quasi-isomrphism.*

Proof. A spectral sequence argument (really, just the "Five Lemma") shows that f_* defines a quasi-isomorphism $(V_*/F_nV_*, d) \to (W_*/F_nW_*, d)$, for all $n \geq 0$. The assumption that the modules V_q and W_q are complete and an application of the $\varprojlim^1$-exact sequence for the homology of a projective limit of complexes then give the result. □

Typically, the filtered vector spaces V that we will consider will come from A-modules endowed with the I-adic filtration corresponding to a two-sided ideal I of A. More precisely, $F_nV := I^nV$, for some fixed two-sided ideal I of A. Most of the time, the ideal I will be of the form $I = I_0A$, where $I_0 \subset \mathbf{k}$ is an ideal. Consequently, if M and N are a right, respectively a left, A–module endowed with the I–adic filtration, then we define

$$M\widehat{\otimes}_A N := \varprojlim M/I^nM \otimes_A N/I^nN . \tag{11}$$

Basic results of homological algebra extend to A–modules endowed with filtrations if one is careful to use only *admissible* resolutions, that is resolutions that have *bounded* $\mathbb{C}$–linear contractions, in the spirit of relative homological algebra. The completion of every admissible resolution by finitely generated free modules is admissible, and this is essential for our argument.

Let A be a filtered algebra by $F_qA \subset A$, where $F_qAF_pA \subset F_{p+q}A$. We then define $A^{\widehat{\otimes}q} := \varprojlim (A/F_nA)^{\otimes q}$ be the topological tensor product in the category of complete modules. In the case of the I-adic filtration, we have $F_pA := I^pA$ and $A^{\widehat{\otimes}q} := \varprojlim (A/I^nA)^{\otimes q}$. Let g be an endomorphism $g : A \to A$ as before, and assume also that g preserves the filtration and that it induces an endomorphism of $\mathbf{k}$ as well. (So g is an endomorphism of A as a $\mathbb{C}$-algebra, not as a $\mathbf{k}$-algebra.)

Then the *topological, twisted Hochschild complex* [71] of A is denoted $(\widehat{\mathcal{B}}_*(\widehat{A}), b_g)$ and is the completion of the usual Hochschild complex $(\mathcal{B}_*(A), b_g)$ with repect to the filtration topology topology. See also [41]. Explicitly,

$$\widehat{A} \xleftarrow{b_g} A\widehat{\otimes}A \xleftarrow{b_g} A^{\widehat{\otimes}3} \xleftarrow{b_g} \dots \xleftarrow{b_g} A^{\widehat{\otimes}n+1} \xleftarrow{b_g} \dots \tag{12}$$

We shall denote by $\mathrm{HH}_*^{\mathrm{top}}(\widehat{A}, g)$ the homology of the topological, twisted Hochschild complex. Note that $\mathrm{HH}_q(\widehat{A}, g)$ is naturally a $\mathbf{k}$-module since b_g is $\mathbf{k}$-linear. When $I = 0$, we recover, of course, the usual Hochschild homology groups $\mathrm{HH}_*(A, g)$ and we have natural, $\mathbf{k}$-linear maps $\mathrm{HH}_q(A, g) \to \mathrm{HH}_*^{\mathrm{top}}(\widehat{A}, g)$. The topological, b'–complex $(\widehat{\mathcal{B}}_*(\widehat{A}), b')$ is defined analogously. We then see that s is a bounded contraction for $(\widehat{\mathcal{B}}_*(\widehat{A}), b')$, where, we recall $s(a_0 \otimes a_1 \otimes \dots \otimes a_n) = 1 \otimes a_0 \otimes a_1 \otimes \dots \otimes a_n$. In particular, the complex so $(\widehat{\mathcal{B}}_*(\widehat{A}), b')$ is still acyclic.

From now on, we shall fix an ideal I of A and consider only the I-adic filtration given by $F_pA = I^pA$. We will see than that both the Hochschild homology and the periodic cyclic homology groups behave very well under completions for the I-adic topologies. We need however to introduce some notation and auxiliary material for the following theorem.

We shall need the following notation. Let $I = I_0A$, where I_0 is an ideal of $\mathbf{k}$. We denote by $\widehat{\mathbf{k}}$ the completion of $\mathbf{k}$ with respect to the topology defined by the powers of I_0, that is, $\widehat{\mathbf{k}} := \varprojlim \mathbf{k}/I_0^n$. Then $\widehat{A} \simeq A \otimes_{\mathbf{k}} \widehat{\mathbf{k}}$, by standard homological algebra [2, 10]. Similarly, we shall denote by $\widehat{A}^e := \widehat{A}\widehat{\otimes}\widehat{A}^{\mathrm{op}}$, the completion of A^e with respect to the topology defined by the ideal $I_2 := I_0 \otimes \mathbf{k} + \mathbf{k} \otimes I_0$. Let g be a $\mathbf{k}$-algebra endomorphism of A and let A_g be the A^e right-module with the action $a(a' \otimes a'') = a''ag(a')$.

The following result was proved for $g = 1$ (identity automorphism) in [48].

Theorem 1.10. *Let A be a finite-type $\mathbf{k}$-algebra. Using the notation just introduced, we have that $\mathrm{HH}_q(A, g)$ is a finitely generated $\mathbf{k}$-module for each q. If we endow A with the filtration defined by the powers of I, then the natural map $\mathrm{HH}_*(A, g) \to \mathrm{HH}_*^{\mathrm{top}}(\widehat{A}, g)$ and the $\mathbf{k}$–module structure on $\mathrm{HH}_*(A)$ define a $\mathbf{k}$-module isomorphism*

$$\mathrm{HH}_*(A, g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}} \simeq \mathrm{HH}_*^{\mathrm{top}}(\widehat{A}, g).$$

Proof. We consider a resolution of A by finitely-generated, free, left A^e modules say $(A^e \otimes V_i, d_i)$, which always exist since A^e is Noetherian. By tensoring this resolution with A_g over A^e, we obtain that the homology groups $\mathrm{HH}_q(A, g) \simeq \mathrm{Tor}_q^{A^e}(A_g, A)$ are all finitely generated $\mathbf{k}$-modules. Since completion over Noetherian rings is exact in the category of finitely generated $\mathbf{k}$-modules, the above complex completed by $I_2 := I_0 \otimes \mathbf{k} + \mathbf{k} \otimes I_0$ provides an admissible resolution of $\widehat{A}$ namely $(\widehat{A}^e \otimes V_i, \widehat{d}_i)$. (Recall that by the admissibility of a complex we mean the existence of a bounded contraction, this property in this case is provided by [48].) Now $\mathrm{Tor}_*^{\widehat{A}^e}(\widehat{A}_g, \widehat{A})$ is the homology of the complex,

$$\widehat{A}_g \otimes_{\widehat{A}^e} (\widehat{A}^e \otimes V_i, \widehat{d}_i) = (\widehat{A}_g \otimes V_i, 1 \otimes \widehat{d}_i).$$

The right-hand side is the completion with respect to the ideal I of the complex $(A_g \otimes V_i, 1 \otimes d_i)$, a complex whose homology is $\mathrm{HH}_*(A, g)$. Hence

$$\mathrm{Tor}_*^{\widehat{A}^e}(\widehat{A}_g, \widehat{A}) \,=\, \mathrm{HH}_*(A, g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}\,,$$

Since $\mathrm{HH}_*(A,g)$ is a finitely generated $\mathbf{k}$-module. On the hand, since the I-adic bar-complex $(\widehat{\mathcal{B}}_*(\widehat{A}),b')$ has an obvious contraction that makes it admissible, there exist morphisms

$$(\widehat{A^e}\otimes V_i,\widehat{d}_i)\xrightarrow{\phi}(\widehat{\mathcal{B}}_*(\widehat{A}),b')\xrightarrow{\psi}(\widehat{A^e}\otimes V_i,\widehat{d}_i),$$

with $\phi\circ\psi$, respectively $\psi\circ\phi$, homotopic to identity on $(\widehat{\mathcal{B}}_*(\widehat{A}),b')$, respectively $(\widehat{A^e}\otimes V_i,\widehat{d}_i)$, thus after tensoring with $\widehat{A}_g$ over $\widehat{A}^e$, we obtain that $\mathrm{Tor}_*^{\widehat{A}^e}(\widehat{A}_g,\widehat{A})=\mathrm{HH}_*^{top}(\widehat{A},g)$. □

Using the same method, one can prove also that

$$\mathrm{HH}_*^{\mathrm{top}}(\widehat{A},g)\simeq\mathrm{Tor}_q^{\widehat{A}^e}(\widehat{A}_g,\widehat{A})\,, \tag{13}$$

where the right hand side denotes the derived functors of the tensor product of $\widehat{A}^e$ filtered modules. This result that was proved by Hübl in the commutative case, [41]. We now recall some properties of periodic cyclic homology with repect to completions. We have the following result of Goodwille [36].

Theorem 1.11 (Goodwillie)**.** *If $J\subset B$ is a nilpotent ideal of an algebra B, then the quotient morphism $B\to B/J$ induces an isomorphism $\mathrm{HP}_*(B)\to\mathrm{HP}_*(B/J)$.*

This gives the following result of [71] (Application 1.15(2)).

Theorem 1.12 (Seibt)**.** *Let J be a two–sided ideal of an algebra B. Then the quotient morphism $B\to B/J$ induces an isomorphism $\mathrm{HP}_*^{\mathrm{top}}(\widehat{B})\to\mathrm{HP}_*(B/J)$, where the completion is with respect to the powers of J.*

Let us assume that $I\subset A$ is a two-sided ideal invariant for the action of the finite group Γ, so that $\widehat{A}\rtimes\Gamma$ is defined. Then the Hochschild, cyclic, and periodic cyclic complexes of the completion of $\widehat{A}\rtimes\Gamma$ still decompose according to the conjugacy classes of Γ. Recall that C_γ denotes the centralizer of $\gamma\in\Gamma$ in Γ (that is, the set of elements in Γ commuting with γ) and denote $J:=I\rtimes\Gamma=I\otimes\mathbb{C}[\Gamma]$.

Proposition 1.13. *Let Γ be a finite group, then $\widehat{A}\rtimes\Gamma$ is naturally isomorphic to the $J:=I\rtimes\Gamma$-adic completion of $A\rtimes\Gamma$ and we have natural isomorphisms $\mathrm{HH}_q^{\mathrm{top}}(\widehat{A}\rtimes\Gamma)_\gamma\simeq\mathrm{HH}_q^{\mathrm{top}}(\widehat{A},\gamma)^{C_\gamma}$, $\mathrm{HC}_q^{\mathrm{top}}(\widehat{A}\rtimes\Gamma)_\gamma\simeq\mathrm{HC}_q^{\mathrm{top}}(\widehat{A},\gamma)^{C_\gamma}$, and, similarly, $\mathrm{HP}_q^{\mathrm{top}}(\widehat{A}\rtimes\Gamma)_\gamma\simeq\mathrm{HP}_q^{\mathrm{top}}(\widehat{A},\gamma)^{C_\gamma}$.*

Proof. The first statement is a direct consequence of the definitions. Observe that $\widehat{\mathcal{B}}_n(\widehat{A\rtimes\Gamma})=(A\rtimes\Gamma)^{\widehat{\otimes}n+1}$ is the same as $A^{\widehat{\otimes}n+1}\rtimes\Gamma^{n+1}$ and this space admits a known decomposition into conjugacy classes with respect to the action of Γ, namely,

$$\widehat{\mathcal{B}}_q(\widehat{A\rtimes\Gamma})\;=\;\bigoplus_{\langle\gamma\rangle\in\langle\Gamma\rangle}\widehat{\mathcal{B}}_q(\widehat{A\rtimes\Gamma})_\gamma,$$

where $(a_0,a_1,\dots,a_n)(g_0,g_1,\dots,g_n)\in\widehat{\mathcal{B}}_*(\widehat{A\rtimes\Gamma})_\gamma$ exactly when $g_0g_1\dots g_n\in\langle\gamma\rangle$. As in Proposition 1.7, the chain map,

$$(a_0,a_1,\dots,a_n)(g_0,g_1,\dots,g_n)\;\mapsto\;\frac{1}{|C_\gamma|}\sum_{h\in C_\gamma}(\gamma hg_0^{-1}(a_0),h(a_1),\dots,hg_1g_2\cdots g_{n-1}(a_n))$$

now provides an quasi-isomorphism between $\widehat{\mathcal{B}}_*(\widehat{A\rtimes\Gamma})_\gamma$ and $\widehat{\mathcal{B}}_*(\widehat{A},b_\gamma)^{C_\gamma}$. □

We have the following corollary.

Corollary 1.14. *In the framework of Proposition 1.13, let us assume that* $\gamma \in \Gamma$ *acts freely on* $\mathrm{Spec}(\mathbf{k})$. *Then*

$$\mathrm{HH}_q^{\mathrm{top}}(\widehat{A} \rtimes \Gamma)_\gamma \,=\, \mathrm{HH}_q^{\mathrm{top}}(\widehat{A} \rtimes \Gamma)_\gamma \,=\, \mathrm{HH}_q^{\mathrm{top}}(\widehat{A} \rtimes \Gamma)_\gamma \,=\, 0\,.$$

Proof. Corollary 1.6 gives that $\mathrm{HH}_q(A, \gamma) = 0$ for all q. Proposition 1.7 then gives that $\mathrm{HH}_q(A \rtimes \Gamma)_\gamma = 0$ for all q. Connes' SBI-long exact sequence relating the cyclic and Hochschild homologies then gives that $\mathrm{HC}_q(A \rtimes \Gamma)_\gamma = 0$ for all q. This in turn implies the vanishing of the periodic cyclic homology. □

2. Crossed products of regular functions

Our focus from now on will be on the case when A is a finitely generated commutative algebra and Γ is finite. Goodwillie's theorem then allows us to reduce to the case when $A = \mathcal{O}[V]$, where V is a complex, affine algebraic variety and where $\mathcal{O}[V]$ is the ring of regular functions on V. We will thus concentrate on crossed products of the form $\mathcal{O}[V] \rtimes \Gamma$, where Γ is a discrete group acting by automorphisms on $\mathcal{O}[V]$. We begin with a discussion of the case when there is, in fact, no group.

2.1. **The commutative case.** In this section we recall some known constructions and results, see [4, 13, 31, 32, 33, 45, 51, 52] and the references therein. We mostly follow (and expand) the method in [15], where rather the case of $\mathcal{C}^\infty$-functions was considered. We begin by fixing some notation. As usual, we shall denote by $\Omega^q(X)$ the space of algebraic q–forms on a smooth, algebraic variety X. If $Y \subset X$ is a smooth subvariety, we denote by $\omega_{|Y} \in \Omega^q(Y)$ the restriction of $\omega \in \Omega^q(X)$ to Y.

In this subsection, we will consider affine, complex algebraic varieties X and V, where X is assumed to be smooth, $V \subset X$, and $I \subset \mathcal{O}[X]$ is the ideal defining V in X. Recall that $\mathcal{O}[X]$ denotes the ring of regular functions on X, it is the quotient of a polynomial ring by the ideal defining X. Let $g : \mathcal{O}[X] \to \mathcal{O}[X]$ be an endomorphism such that $g^{-1}(I) = I$. We shall denote by X^g the set of fixed points of $g : X \to X$ and assume that it is also a smooth variety. We then let χ_g be the *twisted* Connes-Hochschild-Kostant-Rosenberg map $\chi_g : \mathcal{O}[X]^{\otimes n+1} \to \Omega^n(X^g)$,

$$\chi_g(a_0 \otimes \ldots \otimes a_n) \to \frac{1}{n!} a_0 da_1 \ldots da_n|_{X^g}\,, \tag{14}$$

where the restriction to X^g is defined since X^g is smooth. The map χ_g descends to a map

$$\chi_g : \mathrm{HH}_n(\mathcal{O}[X], g) \,\to\, \Omega^n(X^g)\,, \quad n \geq 0\,, \tag{15}$$

for any fixed n. When $g = 1$, the identity automorphism, the map χ_g is known to be an isomorphism for each n [52]. This fundamental result has been proved in the smooth, compact manifold case in [20]. See also [40, 41, 47, 51].

We want to identify the groups $\mathrm{HH}_*(A, g)$, $A = \mathcal{O}[X]$, and their completions with respect to the powers of I^n. The idea is to first use localization to reduce to the case when X is a vector space, $X = \mathbb{C}^N$, and g acts linearly on $\mathbb{C}^N$. In that case, the calculation is achieved by constructing a more appropriate free resolution of the module A_g with free A^e modules using Koszul complexes (here $A = \mathcal{O}[X]$).

While the proof is rather lenghty, it is standard and follows a well understood path in algebraic geometry and commutative algebra. However, this path is not so well known and understood when it comes to cyclic homology calculations, so we include all the details for the benefit of the reader. The reader with more

experience (or less patience), can skip directly to Proposition 2.16. Our first step is to construct the relevant Koszul complex.

Definition 2.1. Let R be a commutative complex algebra, M be an R-module, E be a finite dimensional complex vector space, and $f : E \to R$ be a linear map. Then to this data we associate the Koszul complex $(\mathbb{K}_*, \partial) = (\mathbb{K}_*(M, E, f), \partial)$, with differential $\partial : \mathbb{K}_j \to \mathbb{K}_{j-1}$, defined by

$$\mathbb{K}_j = \mathbb{K}_j(M, E, f) := M \otimes \wedge^j E \quad \text{and}$$

$$\partial(m \otimes e_{i_1} \wedge \ldots \wedge e_{i_j}) = \sum_{k=1}^{j} (-1)^{k-1} f(e_{i_k}) m \otimes e_{i_1} \wedge \ldots \wedge \widehat{e}_{i_k} \wedge \ldots \wedge e_{i_j} ,$$

where $m \in M$ and $e_1, \ldots, e_n$ is an arbitrary basis of E.

In the rest of this subsection, by E and F, sometimes decorated with indices, we shall denote *finite-dimensional, complex vector spaces.* The definition above is correct as explained in the following remark.

Remark 2.2. The definition of the differential in the Koszul comples is, in fact, independent of the choice of a basis of E. Indeed, if we denote by $i_\xi : \wedge^j E \to \wedge^{j-1} E$ the contraction with $\xi \in E^*$, then $\partial = \sum_j f(e_j) \otimes i_{e_j^*}$, where e_j^* is the dual basis of e_j, and this definition is immediately seen to be independent of the choice of the basis (e_j) of E.

We need to look first at the simplest instances of Koszul complexes.

Example 2.3. Let $E = \mathbb{C}$. Then $f : E \to R$ is completely determined by $r := f(1) \in R$. Then, up to an isomorphism, $\mathbb{K}_*(R, \mathbb{C}, f)$ is isomorphic to the complex

$$0 \longleftarrow M \xleftarrow{r} M \longleftarrow 0 \longleftarrow 0 \ \ldots$$

that has only two non-zero modules: $\mathbb{K}_0 \simeq \mathbb{K}_1 \simeq M$. We are, in fact most interested in the case $M = R = \mathcal{O}[\mathbb{C}] \simeq \mathbb{C}[x]$ (we identify the ring of polynomial functions on $\mathbb{C}$ with the ring of polynomials in one indeterminate X) and $r = f(1) = x$, in which case the Koszul complex $\mathbb{K}_*(\mathbb{C}[x], \mathbb{C}, x)$ provides a free resolution of the $\mathbb{C}[x]$ module $\mathbb{C} \simeq \mathbb{C}[x]/x\mathbb{C}[x]$ (thus on $\mathbb{C}$ we consider the $\mathbb{C}[x]$-module structure given by "evaluation at 0", that is, $P \cdot \lambda = P(0)\lambda$ for $P \in \mathbb{C}[x]$ and $\lambda \in \mathbb{C}$). The other case in which we are interested is the complex $\mathbb{K}_*(\mathbb{C}[x], \mathbb{C}, 0)$, in which case, the homology coincides with $\mathbb{K}_*(\mathbb{C}[x], \mathbb{C}, x)$ itself, because the differential is zero.

We treat the other Koszul complexes that we need by combining the two basic examples considered in Example 2.3 above, using also the functoriality of the Koszul complex. In all our applications, we will have $M = R$, so we assume from now on that this is the case. The next remark deals with functoriality.

Remark 2.4. First, a ring morphism $\phi : R \to R_1$ induces a morphism of complexes

$$\phi_* \ : \ \mathbb{K}_*(R, E, f) \ \to \ \mathbb{K}_*(R_1, E, \phi \circ f)$$

given by the formula $\phi_*(r \otimes e_{i_1} \wedge \ldots \wedge e_{i_j}) = \phi(r) \otimes e_{i_1} \wedge \ldots \wedge e_{i_j}$. Then we notice also that a linear map $g : E_1 \to E$ induces a natural morphism of complexes

$$g_* : \mathbb{K}_*(R, E_1, f \circ g) \to \mathbb{K}_*(R, E, f)$$

given by the formula $g_*(r \otimes e_{i_1} \wedge \ldots \wedge e_{i_j}) = r \otimes g(e_{i_1}) \wedge \ldots \wedge g(e_{i_j})$. If g is an isomorphism, then g_* is an isomorphism as well.

Let us denote by $\dot{\otimes}$ the graded tensor product of complexes. Koszul complexes can be simplified using the following well known lemma whose proof is a direct calculation.

Lemma 2.5. *Assume that $R = R_1 \otimes R_2$, $E = E_1 \oplus E_2$, $f_i : E_i \to R_i$ linear, and $f = f_1 \otimes 1 + 1 \otimes f_2 := (f_1 \otimes 1, 1 \otimes f_2)$. Then*

$$\mathbb{K}_*(R, E, f) \simeq \mathbb{K}_*(R_1, E_1, f_1) \dot{\otimes} \mathbb{K}_*(R_2, E_2, f_2)\,.$$

Let E be a vector space and let E^* be its dual. We denote by $\mathcal{O}[E]$ the ring of regular functions on E, as usual. In this case, $\mathcal{O}[E]$ is isomorphic to the ring of polynomial functions in $\dim(E)$ variables. Then there is a natural inclusion

$$i : E^* \to \mathcal{O}[E]\,. \tag{16}$$

Although we shall not use that, let us mention, in order to provide some intuition, that the differential of the resulting Koszul complex $\mathbb{K}_*(\mathcal{O}[E], E^*, i)$ is the Fourier transform of the standard de Rham differential. Recall that in this subsection E and F (sometimes with indices) denote finite-dimensional, complex vector spaces. Its homology is given by the following lemma.

Lemma 2.6. *Let $i : E^* \to \mathcal{O}[E]$ be the canonical inclusion. Then the resulting complex $\mathbb{K}_*(\mathcal{O}[E], E^*, i)$ has homology*

$$H_q\mathbb{K}_*(\mathcal{O}[E], E^*, i) \simeq \begin{cases} 0 & \text{if } q > 0 \\ \mathbb{C} & \text{if } q = 0\,, \end{cases} \tag{17}$$

with the isomorphism for $q = 0$ being given by the the evaluation at $0 \in E$ morphism $\mathbb{K}_0(\mathcal{O}[E], E^*, i) = \mathcal{O}[E] \to \mathbb{C}$.

Proof. We can assume that $E = \mathbb{C}^n$, by the functoriality of the Koszul complexes (Remark 2.4). If $n = 1$, we may identify $\mathcal{O}[E] = \mathbb{C}[x]$, in which case the result has already been proved (see Example 2.3). In general, we can proceed by induction by writting $\mathbb{C}^n = \mathbb{C}^{n-1} \oplus \mathbb{C}$ and using the fact that the Koszul complex associated to $E = \mathbb{C}^n$ is the tensor product of the Koszul complexes associated to $\mathbb{C}^{n-1}$ and $\mathbb{C}$ and then invoquing Lemma 2.5. The calculation of the homology groups is completed using the Künneth formula, which gives that the homology of a tensor product of complexes (over $\mathbb{C}$) is the tensor product of the individual homologies [54]. See also [35]. □

It is useful to state explicitly the following direct consequence of the above lemma and of the functoriality of the Koszul complexes (Remark 2.4).

Corollary 2.7. *Let $R = \mathcal{O}[F]$ and let $g : E \to F^*$ be an isomorphism. Let R_1 be a commutative complex algebra. Denote by $f := i \circ g : E \to \mathcal{O}[F]$, then*

$$H_q\mathbb{K}_*(R \otimes R_1, E, f) \simeq \begin{cases} 0 & \text{if } q > 0 \\ R_1 & \text{if } q = 0\,, \end{cases} \tag{18}$$

with the isomorphism for $q = 0$ being induced by the the evaluation at $0 \in F$ morphism $\mathbb{K}_0(R \otimes R_1, E, f) = R \otimes R_1 = \mathcal{O}[F] \otimes R_1 \to \mathbb{C} \otimes R_1 = R_1$.

Most of the time, the above corollary will be used for $R_1 = \mathbb{C}$. We now prove the result that we need for Künneth complexes.

Proposition 2.8. *Let $h : E_1 \to E^*$ be a linear map and $f := i \circ h : E_1 \to \mathcal{O}[E]$, where $i : E^* \to \mathcal{O}[E]$ is the canonical inclusion. Denote by $F \subset E$ the annihilator of $h(E_1) \subset E^*$ and by $F_1 \subset E_1$ the kernel of h. Then the restriction morphism $\mathcal{O}[E] \to \mathcal{O}[F]$ and the choice of a projection $E_1 \to F_1$ define isomorphisms*

$$res \,:\, H_q\mathbb{K}_*(\mathcal{O}[E], E_1, f) \,\simeq\, H_q\mathbb{K}_*(\mathcal{O}[F], F_1, 0) \,=\, \mathcal{O}[F] \otimes \wedge^q F_1 \,.$$

These isomorphisms are independent of the choice of the projection $E_1 \to F_1$.

Proof. Let F_1' be the kernel of the chosen projection $E_1 \to F_1$, which thus gives a direct sum decomposition $E_1 = F_1 \oplus F_1'$. Similarly, let us chose a complement F' to F in E, which also gives a direct sum decomposition $E = F \oplus F'$. Then we obtain that $\mathcal{O}[E] = \mathcal{O}[F] \otimes \mathcal{O}[F']$ and that the map f splits as $f_1 \otimes 1 + 1 \otimes f_2$, $f_1 : F_1 \to \mathcal{O}[F]$, $f_2 : F_1' \to \mathcal{O}[F']$, with $f_1 = 0$ and with f_2 obtained from a linear isomorphism $F_1' \to (F')^*$. Then Lemma 2.5 gives that $\mathbb{K}_*(\mathcal{O}[E], E_1, f) \simeq \mathbb{K}_*(\mathcal{O}[F], F_1, 0) \dot{\otimes} \mathbb{K}_*(\mathcal{O}[F'], F_1', f_2)$. The homology of the last complex is given by Corollary 2.7 with $R_1 = \mathbb{C}$ and is seen to be $\mathbb{C}$ in dimension zero and zero in the other dimensions. This gives the desired result since $\mathbb{K}_*(\mathcal{O}[F], F_1, 0)$ has vanishing differentials. □

Corollary 2.9. *Let $h : E^* \to E^*$ be a linear map, such that h induces an injective map $E^*/\ker(h) \to E^*/\ker(h)$ Let $f := i \circ h : E^* \to \mathcal{O}[E]$, where $i : E^* \to \mathcal{O}[E]$ is the canonical inclusion. Denote by $F \subset E$ the annihilator of $h(E^*) \subset E^*$. Then the restriction morphisms $\mathcal{O}[E] \to \mathcal{O}[F]$ and $E^* \to F^*$ define isomorphisms*

$$res_{\mathbb{K}} \,:\, H_q\mathbb{K}_*(\mathcal{O}[E], E^*, f) \,\simeq\, H_q\mathbb{K}_*(\mathcal{O}[F], F^*, 0) \,=\, \mathcal{O}[F] \otimes \wedge^q F^* \,=\, \Omega^p[F] \,.$$

Proof. This follows from Proposition 2.8 for $E_1 = E^*$ as follows. First of all, by assumption, we have that $F_1 := \ker(h : E^* \to E^*)$ has $h(E^*)$ as complement. We shall therefore choose the projection $E^* \to F_1$ to vanish on $h(E^*)$. But $E^*/h(E^*) \simeq F^*$ naturally, by restricting linear forms on E to F, since F was defined as the anihilator of $h(E^*)$. Thus $F_1 \cong F^*$ and the projection $E^* \to F_1$ becomes simply the restriction $E^* \to F^*$. □

Koszul complexes are relevant for Hochschild homology through the following lemma that is also well known [52]. We denote as usual by S_q the group of permutations of the set $\{1, \ldots, q\}$, $q \geq 1$.

Lemma 2.10. *Let $i : E^* \to \mathcal{O}[E]$ be the canonical inclusion and $\delta : E^* \to R^e := R \otimes R$ be given by the formula $\delta(\xi) = i(\xi) \otimes 1 - 1 \otimes i(\xi)$. Then the Koszul complex $\mathbb{K}_*(R^e, E^*, \delta)$ is a resolution of R by projective R^e-modules. Fix a basis $e_1, \ldots, e_n$ of E^* and define $\kappa_0 : \mathbb{K}_*(R^e, E^*, \delta) \to (\mathcal{B}'_*(R), b')$ by the formula*

$$\kappa_0(a_1 \otimes a_2 \otimes e_{j_1} \wedge \ldots \wedge e_{j_p}) \,:=\, \sum_{\sigma \in S_p} \mathrm{sign}(\sigma)\, a_1 \otimes i(e_{j_{\sigma(1)}}) \otimes \ldots \otimes i(e_{j_{\sigma(p)}}) \otimes a_2 \,.$$

Then κ_0 is a chain map, that is, $\kappa_0 \partial = b' \kappa_0$.

We provide a proof in our setting, for the benefit of the reader.

Proof. Recall that R is commutative, so $R^{\mathrm{op}} = R$, and hence the notation $R^e = R \otimes R$ is justified. The Koszul complex $\mathbb{K}_*(R^e, E^*, \delta)$ is a resolution of R by projective R^e-modules by Corollary 2.7 which is used for $R_1 = \mathcal{O}[E]$ and the decomposition $R^e \simeq R_2 \otimes R_1$, where R_2 is the polynomial ring generated by the image of δ, R_1 is the polynomial ring generated by vectors of the form $i(\xi) \otimes 1 + 1 \otimes i(\xi)$, $\xi \in E$, and $R_1 \simeq R_2 \simeq R$. The fact that κ_0 is a chain map is a direct calculation. □

We then obtain

Corollary 2.11. *Let $g : E \to E$ be linear and $f := i \circ (g^* - 1) : E^* \to E^* \subset \mathcal{O}[E]$. Let us denote also by g the endomorphism of $\mathcal{O}[E]$ induced by g. Using the notation of Lemma 2.10, we have that $\kappa_E : (\mathbb{K}_*(R, E^*, f), \partial) \to (\mathcal{B}_*(R), b_g)$*

$$\kappa_E(a \otimes e_{j_1} \wedge \ldots \wedge e_{j_p}) := \sum_{\sigma \in S_p} \operatorname{sign}(\sigma)\, a \otimes i(e_{j_{\sigma(1)}}) \otimes i(e_{j_{\sigma(2)}}) \otimes \ldots \otimes i(e_{j_{\sigma(p)}})$$

is a quasi-isomorphism (that is, $\kappa_E \partial = b_g \kappa_E$, and it induces an isomorphism of homology groups).

Proof. Denote $R := \mathcal{O}[E]$, as above. The groups $\mathrm{HH}_q(R, g)$ can be obtained from any projective resolution of R as a R^e module by tensoring this projective resolution with R_g over R^e, by Proposition 1.2. We shall use to this end the resolution of Lemma 2.10. Thus $\mathrm{HH}_q(R, g)$ are the homology groups of the complex

$$(\mathbb{K}_*(R^e, E^*, \delta) \otimes_{R \otimes R} R_g, \partial \otimes 1)\,.$$

By the functoriality of the Koszul complex (Remark 2.4), this tensor product is nothing else but the Koszul complex $\mathbb{K}_*(R, E^*, f)$. Let κ_0 be as in the statement of Lemma 2.10. We have then that $\kappa_0 \otimes_{R \otimes R} 1_{R_g} = \kappa_E$. Since κ_0 is a morphism of projective resolutions by Lemma 2.10, it follows that κ_E is a quasi-isomorphism by standard homological algebra. □

We shall need the following lemma, which is a particular case of Proposition 2.16. Some closely related results with a Hopf algebra flavor can be found in [32] and some applications of it to deformation of algebras are given in [72].

Lemma 2.12. *We use the notation of Corollary 2.11 and assume that $g - 1 : E/\ker(g-1) \to E/\ker(g-1)$ injective. Let $E^g := \ker(g-1)$. Then the restriction $\mathcal{O}[E] \to \mathcal{O}[E^g]$ defines isomorphisms*

$$res_{\mathrm{HH}} : \mathrm{HH}_q(\mathcal{O}[E], g) \;\to\; \mathrm{HH}_q(\mathcal{O}[E^g])\,,$$

and hence the twisted Connes-Hochschild-Kostant-Rosenberg map χ_g of Equation (14) *gives isomorphisms*

$$\chi_g : \mathrm{HH}_q(\mathcal{O}[E], g) \;\cong\; \Omega^q(E^g)\,.$$

Proof. Let $R = \mathcal{O}[E]$, as before and $f := g^* - 1$. Corollary 2.11 gives that the groups $\mathrm{HH}_q(\mathcal{O}[E], g)$ are isomorphic to the homology groups of the Koszul complex $(\mathbb{K}_*(R, E^*, f), \partial)$. Since the annihilator of $(g^* - 1)(E^*)$ is $\ker(g-1) =: E^g$, we have by Corollary 2.9 that the homology groups of this Koszul complex are the same as those of the Koszul complex $(\mathbb{K}_*(\mathcal{O}[E^g], (E^g)^*, 0)$, with the isomorphism being given by restriction from E to E^g, a map that we will denote $res_{\mathbb{K}} : \mathbb{K}_*(\mathcal{O}[E], E^*, f) \to \mathbb{K}_*(\mathcal{O}[E^g], (E^g)^*, 0)$.

Let us denote by $res_{\mathcal{B}} : (\mathcal{B}(\mathcal{O}[E]), b_g) \to (\mathcal{B}(\mathcal{O}[E^g]), b)$ the natural map given by restriction. Then we have that the restriction maps $res_{\mathcal{B}}$ and the restriction map $res_{\mathbb{K}}$ just defined satisfy $res_{\mathcal{B}} \circ \kappa_E = \kappa_{E^g} \circ res_{\mathbb{K}}$, with the maps κ as in Corollary 2.11. That same corollary gives that κ_E are κ_{E^g} quasi-isomorphisms. Since $res_{\mathbb{K}}$ is also a quasi-isomophisms, we obtain that $res_{\mathcal{B}}$ is a quasi-isomorphism as well. Recall that res_{HH} is the map induced at the level of homology by $res_{\mathcal{B}}$. That means that res_{HH} is an isomorphism and hence proves the first half of our statement.

To prove the last part of our statement, we first notice that the Connes-Hochschild-Kostant-Rosenberg map is an isomorphism $\chi : \mathrm{HH}_q(\mathcal{O}[E^g]) \to \Omega^q(E^g)$. Hence so is χ_g, as the composition $\chi_g := \chi \circ res_{\mathcal{B}}$. $\square$

2.2. **Completion at a maximal ideal.** For the rest of this paper, we shall use completions extensively. Let us fix a maximal ideal $\mathfrak{m}$ of our base ring $\mathbf{k}$. We denote by $\widehat{\mathbf{k}}_{\mathfrak{m}}$ the completion of $\mathbf{k}$ with respect to $\mathfrak{m}$: $\widehat{\mathbf{k}}_{\mathfrak{m}} := \varprojlim \mathbf{k}/\mathfrak{m}^n\mathbf{k}$. For the rest of this subsection, all completions will be with respect to $\mathfrak{m}$, and this will be stressed by including $\mathfrak{m}$ as an index. For any $\mathbf{k}$-module M, we shall denote by

$$\widehat{M}_{\mathfrak{m}} := \varprojlim M/\mathfrak{m}^n M$$

the completion of M with respect to the topology defined by an ideal $\mathfrak{m} \subset \mathbf{k}$. We notice that the completion $\widehat{M}_{\mathfrak{m}}$ is naturally a $\widehat{\mathbf{k}}_{\mathfrak{m}}$-module, and hence we obtain a natural map $comp : M \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} \to \widehat{M}_{\mathfrak{m}}$. In case M is finitely generated as a $\mathbf{k}$-module (always the case in what follows), then $comp : M \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} \to \widehat{M}_{\mathfrak{m}}$ is an isomorphism [2, 10], which will be used as an identification from now on. Thus, in order not to overburden the notation, we shall drop *comp* from the notation and simply identify $\widehat{M}_{\mathfrak{m}}$ and $M \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}}$.

Remark 2.13. We know that every element $x \in \mathbf{k} \smallsetminus \mathfrak{m}$ is invertible in $\widehat{\mathbf{k}}_{\mathfrak{m}}$ by writing a convergent Neumann series for the inverse. Therefore, for every $\mathbf{k}$-module M, the canonical map $M \to \widehat{M}_{\mathfrak{m}}$ factors through $M_{\mathfrak{m}} := (\mathbf{k} \smallsetminus \mathfrak{m})^{-1}M$, that is, it is the composition of the canonical maps $M \to M_{\mathfrak{m}} \to \widehat{M}_{\mathfrak{m}}$. In particular, if $M_{\mathfrak{m}} = 0$, then $\widehat{M}_{\mathfrak{m}} = 0$, since the image of $M_{\mathfrak{m}}$ in $\widehat{M}_{\mathfrak{m}}$ is dense. Moreover, we see that completion with respect to $\mathfrak{m}$ and localization at $\mathfrak{m}$ commute, so there is no danger of confusion in the notation $\widehat{M}_{\mathfrak{m}}$: that is, $(\widehat{M})_{\mathfrak{m}} = \widehat{(M_{\mathfrak{m}})}$.

Remark 2.14. If $f : M \to N$ is a morphism of $\mathbf{k}$-modules, we shall denote by $\widehat{f}_{\mathfrak{m}} : \widehat{M}_{\mathfrak{m}} \to \widehat{N}_{\mathfrak{m}}$ the induced morphism of the corresponding completions with respect to the topology defined by the powers of $\mathfrak{m}$. If M and N are finitely generated, we shall make no difference between $\widehat{f}_{\mathfrak{m}} : \widehat{M}_{\mathfrak{m}} \to \widehat{N}_{\mathfrak{m}}$ and $f \otimes_{\mathbf{k}} 1 : M \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} \to N \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}}$ Assume M and N are finitely generated $\mathbf{k}$-modules. Then it is a standard result in commutative algebra that f is an isomorphism if, and only if, $\widehat{f}_{\mathfrak{m}}$ is an isomorphism for all maximal ideals $\mathfrak{m}$ of A [2, 10].

Let A be a finite type $\mathbf{k}$ algebra. We let

$$can \,:\, \mathrm{HH}_*(A, g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} \;\cong\; \mathrm{HH}_*^{\mathrm{top}}(\widehat{A}_{\mathfrak{m}}, g) \tag{19}$$

denote the canonical isomorphism of Theorem 1.10. It is obtained from the fact that $\mathrm{HH}_*^{\mathrm{top}}(\widehat{A}_{\mathfrak{m}}, g)$ is a $\widehat{\mathbf{k}}_{\mathfrak{m}}$-module using also the natural map $\mathrm{HH}_*(A, g) \to \mathrm{HH}_*^{\mathrm{top}}(\widehat{A}_{\mathfrak{m}}, g)$ induced by the inclusion $A \to \widehat{A}_{\mathfrak{m}}$. We shall use the map *can* in the following corollary in the following setting: E will be a finite dimensional, complex vector space, $\mathbf{k} := \mathcal{O}[E]$, and $\mathfrak{m}$ will be the maximal ideal of $\mathbf{k} := \mathcal{O}[E]$ corresponding to functions vanishing at 0. Then we shall consider, as explained, the filtrations defined by $\mathfrak{m}$ and denote by $\widehat{\mathcal{O}}[E]_{\mathfrak{m}}$ and $\widehat{\mathcal{O}}[E^g]_{\mathfrak{m}}$ the completions of $\mathcal{O}[E]$ and $\mathcal{O}[E^g]$ with respect to the filtrations defined by the powers of $\mathfrak{m}$.

The following corollary is an analog of Lemma 2.12 for completed algebras.

Corollary 2.15. *We use the notation and assumptions of Lemma 2.12, in particular, g is a linear endomorphism of E such that $g-1 : E/\ker(g-1) \to E/\ker(g-1)$*

is injective. Let $\mathfrak{m}$ *be the maximal ideal of* $\mathbf{k} := \mathcal{O}[E]$ *corresponding to functions vanishing at 0. Then the restriction* $\widehat{\mathcal{O}}[E]_{\mathfrak{m}} \to \widehat{\mathcal{O}}[E^g]_{\mathfrak{m}}$ *defines an isomorphism*

$$\widehat{res}_{\mathrm{HH}} \,:\, \mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}}[E]_{\mathfrak{m}}, g) \;\to\; \mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}}[E^g]_{\mathfrak{m}})\,.$$

Define $\widehat{\chi}_g$ *by* $\chi_g \circ can = \chi_g \otimes_{\mathbf{k}} 1$. *Then we have an isomorphism*

$$\widehat{\chi}_g : \mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}}[E]_{\mathfrak{m}}, g) \;\cong\; \widehat{\Omega}^q(E^g)_{\mathfrak{m}} \;:=\; \varprojlim \Omega^q(E^g)/\mathfrak{m}^n\Omega^q(E^g)\,.$$

As the notation suggests, $\widehat{\chi}_g$ is, up to canonical identifications, nothing but the extension by continuity (or completion) of the usual Connes-Hochschild-Kostant-Rosenberg map.

Proof. For the first part of the proof, let us denote for any $\mathbf{k}$-algebra A by *nat* the natural map $\mathrm{HH}_*(A, g) \to \mathrm{HH}_*^{\mathrm{top}}(\widehat{A}, g)$. Hence $can = nat \otimes_{\mathbf{k}} 1$, see Equation (19). Let us consider then the diagram

$$\begin{array}{ccc} \mathrm{HH}_q(\mathcal{O}[E], g) & \xrightarrow{res_{\mathrm{HH}}} & \mathrm{HH}_q(\mathcal{O}[E^g]) \\ {\scriptstyle nat}\big\downarrow & & {\scriptstyle nat}\big\downarrow \\ \mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}[E]}_{\mathfrak{m}}, g) & \xrightarrow{\widehat{res}_{\mathrm{HH}}} & \mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}[E^g]}_{\mathfrak{m}}) \end{array} \tag{20}$$

whose arrows are given by the natural morphisms of the corresponding algebras and hence is commutative. That is, we have the relation $nat \circ res_{\mathrm{HH}} = \widehat{res}_{\mathrm{HH}} \circ nat$, which gives then the relation $can \circ (res_{\mathrm{HH}} \otimes_{\mathbf{k}} 1) = \widehat{res}_{\mathrm{HH}} \circ can$, that is, that the diagram

$$\begin{array}{ccc} \mathrm{HH}_q(\mathcal{O}[E], g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} & \xrightarrow{res_{\mathrm{HH}} \otimes_{\mathbf{k}} 1} & \mathrm{HH}_q(\mathcal{O}[E^g]) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} \\ {\scriptstyle can}\big\downarrow & & {\scriptstyle can}\big\downarrow \\ \mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}[E]}_{\mathfrak{m}}, g) & \xrightarrow{\widehat{res}_{\mathrm{HH}}} & \mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}[E^g]}_{\mathfrak{m}}) \end{array} \tag{21}$$

is commutative. The map $res_{\mathrm{HH}} : \mathrm{HH}_q(\mathcal{O}[E], g) \to \mathrm{HH}_q(\mathcal{O}[E^g])$ is an isomorphism by Lemma 2.12. Therefore $res_{\mathrm{HH}} \otimes_{\mathbf{k}} 1 : \mathrm{HH}_q(\mathcal{O}[E], g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} \to \mathrm{HH}_q(\mathcal{O}[E^g]) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}}$ is an isomorphism as well. Since the vertical arrows (that is, the maps *can*) are isomorphisms, we obtain that $\widehat{res}_{\mathrm{HH}}$ is an isomorphism as well.

The fact that $\widehat{\chi}_g$ is an isomorphism follows from the commutative diagram

$$\begin{array}{ccc} \mathrm{HH}_q(\mathcal{O}[E], g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} & \xrightarrow{\chi_g \otimes_{\mathbf{k}} 1} & \Omega^q(E^g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} \\ {\scriptstyle can}\big\downarrow & & \big\| \\ \mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}[E]}_{\mathfrak{m}}, g) & \xrightarrow{\widehat{\chi}_g} & \widehat{\Omega}^q(E^g)_{\mathfrak{m}} \end{array} \tag{22}$$

(that is, $\widehat{\chi}_g \circ can = \chi_g \otimes_{\mathbf{k}} 1$) and the fact that *can* and χ_g are isomorphisms. □

In plain terms, one has that $\widehat{\chi}_g$ is an isomorphism since it is the completion of an isomorphism.

We now come back to the case of a general smooth, complex algebraic variety X. We obtain the following result (due to Brylinski in the case of the algebra of smooth functions [13]). See also [28]

Proposition 2.16. *Let X be a smooth, complex, affine algebraic variety and g an endomorphism of $\mathcal{O}[X]$ such that X^g is also a smooth algebraic variety and such that, for any fixed point $x \in X^g$, T_xX^g is the kernel of $g_* - 1$ acting on T_xX and $g_* - 1$ induces an injective endomorphism of T_xX/T_xX^g. Then the twisted Connes-Hochschild-Kostant-Rosenberg map χ_g induces isomorphisms*

$$\chi_g \,:\, \mathrm{HH}_q(\mathcal{O}[X], g) \;\cong\; \Omega^q(X^g)\ .$$

Proof. Recall that, for a finitely generated $\mathbf{k}$-module M, we identify $\widehat{M}_{\mathfrak{m}}$ with $M \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}}$. However, in this proof, it will be convenient to work with the tensor product $M \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}}$ rather than with the completion $\widehat{M}_{\mathfrak{m}}$, for notational simplicity. Denote $\mathbf{k} = \mathcal{O}[X]$ and let $\mathfrak{m}$ be an arbitrary maximal ideal of $\mathbf{k}$. For the clarity of the presentation, we shall write in this proof χ_g^X instead of simply χ_g. As explained in Remark 2.14, it is enough to check that the map

$$\chi_g^X \otimes_{\mathbf{k}} 1 \,:\, \mathrm{HH}_q(\mathcal{O}[X], g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} \;\to\; \Omega^q(X^g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}}\ . \tag{23}$$

is an isomorphism (since the maximal ideal $\mathfrak{m}$ of A was chosen arbitrarily). We shall now check that this property is satisfied.

Indeed, if $\mathfrak{m}$ is not fixed by g (recall that the maximal ideals $\mathfrak{m}$ of $\mathcal{O}[X]$ and the points of X are in one-to-one correspondence), then $\chi_g^X \otimes_{\mathbf{k}} 1$ is an isomorphism since both groups in Equation (23) are zero, by Corollary 1.6 and Remark 2.13.

Let us therefore assume that $\mathfrak{m}$ is fixed by g and denote by $E := (\mathfrak{m}/\mathfrak{m}^2)^*$ the tangent space to X at $\mathfrak{m}$. The assumption that X is smooth at $\mathfrak{m}$ means, by definition [38], that we have a natural isomorphism $tan : \widehat{\mathcal{O}}[X]_{\mathfrak{m}} \simeq \widehat{\mathcal{O}}[E]_{\mathfrak{m}}$. Similarly, we have a natural isomorphism

$$\Omega^q(X^g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} \;\simeq\; \widehat{\Omega}^q(X^g)_{\mathfrak{m}} \;\simeq\; \widehat{\Omega}^q(E^g)_{\mathfrak{m}} \;\simeq\; \Omega^q(E^g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}}\ ,$$

which we shall also denote by tan.

Recall that the canonical map $can : \mathrm{HH}_q(\mathcal{O}[X], g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} \to \mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}[X]}_{\mathfrak{m}}, g)$ is an isomorphism (by Theorem 1.10) and consider then the commutative diagram

$$\begin{array}{ccc}
\mathrm{HH}_q(\mathcal{O}[X], g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} & \xrightarrow{\chi_g^X \otimes_{\mathbf{k}} 1} & \Omega^q(X^g) \otimes_{\mathbf{k}} \widehat{\mathbf{k}}_{\mathfrak{m}} \\
{\scriptstyle can}\big\downarrow & & \big\| \\
\mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}[X]}_{\mathfrak{m}}, g) & \xrightarrow{\widehat{\chi}_g^X} & \widehat{\Omega}^q(X^g)_{\mathfrak{m}} \\
{\scriptstyle tan}\big\downarrow & & {\scriptstyle tan}\big\downarrow \\
\mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}[E]}_{\mathfrak{m}}, g) & \xrightarrow{\widehat{\chi}_g^E} & \widehat{\Omega}^q(E^g)_{\mathfrak{m}}
\end{array} \tag{24}$$

in which the vertical arrows are isomorphisms as explained.

In order to complete the proof, it is enough to check that the bottom horizontal arrow is also an isomorphism. The problem is that g is not a linear map. Let us denote by $Dg : T_{\mathfrak{m}}E \to T_{\mathfrak{m}}E$ the differential of the map g at $\mathfrak{m}$ and consider the chain maps

$$\begin{array}{c}
\chi_g^E : (\mathcal{B}_*(\mathcal{O}[E]), b_g) \;\to\; (\Omega^*(E^g), 0) \quad \text{and} \\
\chi_{Dg}^E : (\mathcal{B}_*(\mathcal{O}[E]), b_{Dg}) \;\to\; (\Omega^*(E^g), 0)
\end{array} \tag{25}$$

of filtered complexes (with the filtration defined by the powers of $\mathfrak{m}$). The second map is a quasi-isomorphism by Corollary 2.15. Recall that each of the three filtered

complexes in Equation (25) gives rise to a spectral sequence with E^1 term given as the homology of the subquotient complexes. We have that the corresponding E^1-terms and maps between these E^1-terms depend only on Tg. Since χ^E_{Dg} induces a quasi-isomorphism, the result follows from Lemma 1.9. □

We note that the assumptions of our proposition are satisfied for g a finite order automorphism.

We have the following analog of a result of [41], where from now on the completions are with respect to an ideal I, and hence the subscript $\mathfrak{m}$ will be dropped from the notation.

Theorem 2.17. *Let $A = \mathcal{O}[X]$ with X a complex, smooth, affine algebraic variety, $V \subset X$ a subvariety with ideal I and g finite order automorphism of $\mathcal{O}[X]$ that leaves V invariant. Then the twisted Connes-Hochschild-Kostant-Rosenberg map χ_g induces an isomorphism*

$$\chi_g \,:\, \mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}[X]}, g) \;\cong\; \widehat{\Omega}^q(V^g) \,:=\, \varprojlim \Omega^q(X^g)/I^n\Omega^q(X^g)\,,$$

where the completions are taken with respect to the powers of I.

Proof. This follows by taking the I-adic completion of the map χ_g of Proposition 2.16 and then using Theorem 1.10. □

2.3. **Crossed products.** We now return to cross products. The cyclic homology of crossed products was extensively studied due to its connections with noncommutative geometry [21]. See [1, 18, 42, 55, 63, 79, 81, 82] for a few recent results. The homology of cross-product algebras was studied recently by Dave in relation to residues [27] and by Manin and Marcolli in relation to arithmetic geometry [57]. In this section, we compute the homology of $\mathcal{O}[V] \rtimes \Gamma$, for V an algebraic variety (not necessarily smooth) and Γ a finite group. The idea is to reduce to the case when V is the affine space $\mathbb{C}^n$ acted upon linearly by Γ. In this case the Hochschild and cyclic homology groups of $\mathcal{O}[V] \rtimes \Gamma$ were computed in [32, 72], but we need a slight enhacement of those results, which are provided by the results of the previous subsection.

Let us therefore fix an affine, complex algebraic variety V and let Γ be a finite group acting by automorphisms on $\mathcal{O}[V]$. Let us choose a Γ-equivariant embedding $V \to X$, where X is a smooth, affine algebraic variety on which Γ acts by regular maps. Propositions 1.7 and 2.16 then give the following result (see also [28, 32, 72]).

Proposition 2.18. *Let Γ be a finite group acting on a smooth, complex, affine algebraic variety X and $\{\gamma_1, \ldots, \gamma_\ell\}$ be a list of representatives of its conjugacy classes. We denote by C_j the centralizer of γ_j and by X_j be the set of fixed points of γ_j. Then*

$$\mathrm{HH}_q(\mathcal{O}[X] \rtimes \Gamma) \;\cong\; \bigoplus_{j=1}^{\ell} \Omega^q(X_j)^{C_j},$$

$$\mathrm{HC}_q(\mathcal{O}[X] \rtimes \Gamma) \;\cong\; \bigoplus_{j=1}^{\ell} \left(\Omega^q(X_j)^{C_j}/d\Omega^{q-1}(X_j)^{C_j} \,\oplus\, \bigoplus_{k\geq 1} H^{q-2k}(X_j)^{C_j} \right),$$

and

$$\mathrm{HP}_q(\mathcal{O}[X] \rtimes \Gamma) \;\cong\; \bigoplus_{j=1}^{\ell} \bigoplus_{k\in\mathbb{Z}} H^{q-2k}(X_j)^{C_j}\,.$$

Proof. The result for Hochschild homology follows from Propositions 1.7 and 2.16. Let $g = \gamma_j$, for some j. The result for Hochschild homology, together with the equation $\chi_g B_g = d\chi_g$, with d the de Rham differential, then gives the other isomorphisms using a standard argument based on mixed complexes. Mixed complexes were introduced in [43]. They are also reviewed in [48, 70].

Let us recall now this standard argument based on mixed complexes. For any $\gamma \in \Gamma$ with centralizer C_γ, the twisted Connes-Hochschild-Kostant-Rosenberg map χ_γ defines a map of mixed complexes

$$\chi_\gamma \,:\, \big(\mathcal{B}_*(\mathcal{O}[X])^{C_\gamma}, b_\gamma, B_\gamma\big) \to (\Omega^*(X^\gamma)^{C_\gamma}, 0, d)\,,$$

which is an isomorphism in Hochschild homology by Proposition 2.16. It follows that χ_γ induces an isomorphism of the corresponding cyclic and periodic cyclic homology groups. Since the cohomology groups of the de Rham complex $\Omega^*(X)$ identify with $H^*(X;\mathbb{C})$, the singular cohomology groups of X with complex coefficients, [30, 31, 37, 41] and these groups vanish for $*$ large. □

Similarly, we obtain

Theorem 2.19. *Let Γ be a finite group acting on a smooth, complex, affine algebraic variety X and $V \subset X$ be an invariant subvariety. Let $\{\gamma_1, \ldots, \gamma_\ell\}$ be a list of representatives of conjugacy classes of Γ, let C_j be the centralizer of γ_j, and let X_j and V_j be the set of fixed points of γ_j. We complete with respect to the powers of the ideal $I \subset \mathcal{O}[X]$ defining V in X. Then we have*

$$\mathrm{HH}_q^{\mathrm{top}}(\widehat{\mathcal{O}[X]} \rtimes \Gamma) \;\cong\; \bigoplus_{j=1}^{\ell} \widehat{\Omega}^q(X_j)^{C_j}$$

$$\mathrm{HC}_q^{\mathrm{top}}(\widehat{\mathcal{O}[X]} \rtimes \Gamma) \;\cong\; \bigoplus_{j=1}^{\ell} \left(\widehat{\Omega}^q(X_j)^{C_j}/d\widehat{\Omega}^{q-1}(X_j)^{C_j} \;\oplus\; \bigoplus_{k\geq 1} H^{q-2k}(V^j;\mathbb{C})^{C_j} \right)$$

$$\mathrm{HP}_q^{\mathrm{top}}(\widehat{\mathcal{O}[X]} \rtimes \Gamma) \;\cong\; \bigoplus_{j=1}^{\ell} \bigoplus_{k\in\mathbb{Z}} H^{q-2k}(V^j;\mathbb{C})^{C_j}\,.$$

Proof. The result for Hochschild homology follows from Proposition 1.13 and Theorem 2.17. The rest for is very similar, except that one has to use also the fact that the de Rham complex $\widehat{\Omega}^*(V) := \varprojlim \Omega(X)/I^k\Omega(X)$ endowed with the de Rham differential has cohomology $H^*_{\mathrm{inf}}(V) \simeq H^*(V;\mathbb{C})$ [30, 31, 33, 37, 41]. □

This result was obtained in the case of smooth functions in [15]. We now come to our main result. Note that in the following theorem we do not assume the variety V to be smooth and that we recover the orbifold homology groups. The case when Γ is trivial is due to Feigin and Tsygan [33] (see also Emmanouil's papers [30, 31] and the references therein) for the case when V is smooth, see also the paper by Dolgushev and Etinghof [28]. See also [39].

Theorem 2.20. *Let Γ be a finite group acting on a complex, affine algebraic variety V. We do not assume V to be smooth. Let $\{\gamma_1, \ldots, \gamma_\ell\}$ be a list of representatives of conjugacy classes of Γ, let C_j be the centralizer of γ_j, and let $V_j \subset V$ be the set*

of fixed points of γ_j. Then

$$\mathrm{HP}_q(\mathcal{O}[V] \rtimes \Gamma) \;\cong\; \bigoplus_{j=1}^{\ell} \bigoplus_{k\in\mathbb{Z}} H^{q-2k}(V_j;\mathbb{C})^{C_j}\,.$$

Proof. Let us choose a Γ-equivariant embedding $V \subset E$ into an affine space. This can be done by first choosing a finite system of generators $I := \{a_i\}$ of $\mathcal{O}[V]$ and then replacing it with $I_\Gamma := \{\gamma a_i\}$, for all $\gamma \in \Gamma$. We let E to be the vector space with basis (γ, a_i). Let $I \subset \mathcal{O}[E]$ be the ideal defining V in E.

We let $\mathbf{k} = \mathcal{O}[E]^\Gamma = \mathcal{O}[E/\Gamma]$, then $\mathbf{k}$ is a finitely generated complex algebra by Hilbert's finiteness theorem [49]. Moreover, $\mathcal{O}[E]$ is a finitely generated $\mathbf{k}$-module, and hence $\mathcal{O}[E] \rtimes \Gamma$ is also a finite type $\mathbf{k}$-algebra. Let $I_0 := \mathbf{k} \cap I$. We also notice that there exists j such that $I^j \subset I_0\mathcal{O}[E] \subset I$. Indeed, this is due to the fact that if a character $\phi : \mathcal{O}[E] \to \mathbb{C}$ vanishes on I_0, then it vanishes on I as well, so I and $I_0\mathcal{O}[E]$ have the same nilradical. To prove this, let us consider $\phi : \mathcal{O}[E] \to \mathbb{C}$ that vanishes on I_0 and let $a \in I$. Since $a \in I$ and I is Γ-invariant, the polynomial $P(X) := \prod_{\gamma\in\Gamma}(X - \gamma(a))$ has coefficients in I that are invariant with respect to γ, so they are in I_0. This gives $\phi(P(X)) = X^n = \prod_{\gamma\in\Gamma}(X - \phi(\gamma(a)))$, with n the number of elements in Γ. Therefore $\phi(\gamma(a)) = 0$ for all γ. In particular, $\phi(a) = 0$.

The fact that there exists j such that $I^j \subset I_0\mathcal{O}[E] \subset I$ gives that completing with respect to $I \rtimes \Gamma$ or with respect to I_0 has the same effect. Then $\mathcal{O}[E] \rtimes \Gamma/I \rtimes \Gamma \simeq \mathcal{O}[V]\rtimes\Gamma$. Therefore $\mathrm{HP}_q^{\mathrm{top}}(\mathcal{O}[E]\rtimes\Gamma) \simeq \mathrm{HP}_q(\mathcal{O}[V]\rtimes\Gamma)$, by Seibt's theorem (Theorem 1.12). The result then follows from Theorem 2.19. □

This theorem extends the calculations for the cross-products of the form $\mathcal{C}^\infty(X)\rtimes\Gamma$, see [8, 15, 13, 29, 33, 60, 62, 69]. A similar result exists for orbifolds in the $\mathcal{C}^\infty$-setting (often the resulting groups are called "orbifold cohomology" groups). It would be interesting to extend our result to "algebraic orbifolds". Theorem 0.1 now follows right away. Since for a finite type algebra A and Γ finite, $A \rtimes \Gamma$ is again a finite type algebra, it would be interesting to compare the result of Theorem 2.20 with the spectral sequence of [48], which are based on the excision exact sequence in cyclic homology [25, 26, 59]. Cyclic cohomology for various algebras (among which cross-products play a central role) has played a role in index theory [6, 21, 23, 22, 50, 65, 67, 68].

We can now complete the proof of one of our main theorems, Theorem 0.1.

Proof. (of Theorem 0.1). Let I be the nilradical of A. Then $A/I \simeq \mathcal{O}[V]$ and I is nilpotent. The result then follows from Goodwillie's result, Theorem 1.11 and Theorem 2.20. □

Remark 2.21. The above result is no longer true if we replace $A = \mathcal{O}[V]$ with an Azumaya algebra. Indeed, let $A = M_2(\mathbb{C})$ and let $\Gamma := (\mathbb{Z}/2\mathbb{Z})^2$ act on A by the inner automorphisms induced by the matrices

$$u_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad u_2 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then $A \rtimes \Gamma \simeq M_4(\mathbb{C})$ by the morphism defined by

$$M_2(\mathbb{C}) \ni a \to a \otimes 1 \in M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \simeq M_4(\mathbb{C}) \tag{26}$$

and by $u_i \to u_i \otimes u_i$. Therefore the periodic cyclic homology of $A\rtimes\Gamma$ is concentrated at the identity.

This remark is related to a result of P. Green. See the beginning of the introduction to [80] for a statement of this result in the form we need it (but with $\mathbb{Z}_2$ replaced by the unit circle group $S^1 = \mathbb{T}$).

3. Affine Weyl groups

We now use the general results developed in previous sections to determine the periodic cyclic cohomology of the group algebras of (extended) affine Weyl groups. These results continue the results in [7], where the corresponding Hecke algebras were studied. In fact, cyclic and Hochschild homologies behave remarkably well for the algebras associated to reductive p-adic groups [3, 7, 24, 48, 73, 74, 75, 76]. Some connections with the Langlands program were pointed out in [64].

An (extended) affine Weyl group is the crossed product $W = X \rtimes W_0$, where W_0 is a finite Weyl groups and X is a sublattice of the lattice of weights of a complex algebraic group with Weyl group W_0. Then $\mathbb{C}[W]$, the group algebra of W, satisfies $\mathbb{C}[W] \cong \mathcal{O}(X^*) \rtimes W_0$, that is, it is isomorphic to the crossed product algebra of the ring of regular functions on $X^* := \mathrm{Hom}(X, \mathbb{C}^*)$, the dual torus of X, by the action of W_0. The cyclic homology groups of $\mathbb{C}[W] \cong \mathcal{O}(X^*) \rtimes W_0$ thus can be computed in two, dual ways, either using the results of [16, 20, 47] on the periodic cyclic homology of group algebras or using our determination in Theorem 2.20.

As a concrete example, let us compute the periodic cyclic cohomology of the group algebra $\mathbb{C}[W]$ of the extended Weyl group $W := \mathbb{Z}^n \rtimes S_n$, the symmetric Weyl group S_n acting by permutation on the components of $\mathbb{Z}^n$. Denote by

$$\Pi(n) := \{(\lambda_1, \lambda_2, \ldots, \lambda_r),\ \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0,\ \sum \lambda_j = n,\ \lambda_i \in \mathbb{Z}\}$$

the set of partitions of n. The set $\Pi(n)$ is in bijective correspondence with the set of conjugacy classes of S_n. If $S_\lambda \simeq S_{\lambda_1} \times S_{\lambda_2} \times \ldots \times S_{\lambda_r}$ denotes the Young (or parabolic) subgroup of S_n consisting of permutations leaving the first $\sum_{j=1}^k \lambda_j$ elements invariant, for all k, then to the partition λ there corresponds a conjugacy class in S_n represented by a permutation $\sigma_\lambda \in S_\lambda$, which in each factor is the cyclic permutation $(l, l+1, \ldots, m)$ of maximum length, for suitable integers $l < m$ (more precisely $l = \lambda_1 + \lambda_2 + \ldots + \lambda_k + 1$ and $m = \lambda_1 + \lambda_2 + \ldots + \lambda_{k+1}$, for a suitable k). Let σ_λ be that element. Then the set of fixed points of σ_λ acting on $(\mathbb{C}^*)^n$ is a tours naturally identified with $(\mathbb{C}^*)^r$.

The centralizer $(S_n)_{\sigma_\lambda}$ is isomorphic to a semi-direct product of a permutation group, denoted Q_λ, which permutes the equal length cycles of σ_λ, and the commutative group generated by the cycles of σ_λ. The action of the centralizer $(S_n)_{\sigma_\lambda}$ on the set $(X^*)^{\sigma_\lambda} \simeq (\mathbb{C}^*)^r$ of fixed points of σ_λ descends to an action of Q_λ on that set. The group Q_λ permutes the equal-length cycles of σ_λ and acts accordingly by permutations on $(\mathbb{C}^*)^r$. Let $t(\lambda)$ be the number of permutation subgroups appearing as factors of Q_λ. Thus, $t(\lambda)$ is the set of distinct values in the sequence $\{\lambda_1, \lambda_2, \ldots, \lambda_r\}$. Then $(\mathbb{C}^*)^r/Q_\lambda$ is the product of $(\mathbb{C}^*)^{t(\lambda)}$ and a euclidean space, so $\mathrm{H}^*((\mathbb{C}^*)^r/Q_\lambda) \cong \mathrm{H}^*((\mathbb{C}^*)^{t(\lambda)}) \cong \mathrm{H}^*(\mathbb{T}^{t(\lambda)})$, with $\mathbb{T} \simeq S^1$ the unit circle..

Theorem 3.1. *Let $A = \mathbb{C}[W]$ be the group algebra of the extended affine Weyl group $W = \mathbb{Z}^n \rtimes S_n$. Then*

$$\mathrm{HP}_q(A) \cong \bigoplus_{\lambda \in \Pi(n)} \bigoplus_{k=0}^{[t(\lambda)/2]} \mathrm{H}^{2k+q}\left(\mathbb{T}^{t(\lambda)}\right), \quad q = 0, 1\,.$$

See also [12, 11] for some related results. Theorem 3.1 and Lemma 3.2 in [12] extends some of these results to Hecke algebras.

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