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# EXHAUSTING FAMILIES OF REPRESENTATIONS AND SPECTRA OF PSEUDODIFFERENTIAL OPERATORS

VICTOR NISTOR AND NICOLAS PRUDHON

ABSTRACT. A powerful tool in the spectral theory and the study of Fredholm conditions for (pseudo)differential operators is provided by families of representations of a naturally associated algebra of bounded operators. Motivated by this approach, we define the concept of an *strictly norming family* of representations of a  $C^*$ -algebra  $A$ . Let  $\mathcal{F}$  be a strictly norming family of representations of  $A$ . We have then that an abstract differential operator  $D$  affiliated to  $A$  is invertible if, and only if,  $\phi(D)$  is invertible for all  $\phi \in \mathcal{F}$ . This property characterizes strictly norming families of representations. We provide necessary and sufficient conditions for a family of representations to be strictly norming. If  $A$  is a separable  $C^*$ -algebra, we show that a family  $\mathcal{F}$  of representations is strictly norming if, and only if, every irreducible representation of  $A$  is weakly contained in a representation  $\phi \in \mathcal{F}$ . However, this result is not true, in general, for non-separable  $C^*$ -algebras. A typical application of our results is to parametric families of differential operators arising in the analysis on manifolds with corners, in which case we recover the fact that a parametric operator  $P$  is invertible if, and only if, its Mellin transform  $\hat{P}(\tau)$  is invertible, for all  $\tau \in \mathbb{R}^n$ . The paper is written to be accessible to non-specialists in  $C^*$ -algebras.

## CONTENTS

Introduction	1
1. $C^*$ -algebras and their primitive ideal spectrum	3
2. Faithful families	5
3. Full and strictly norming families	7
4. Topology on the spectrum and strictly norming families	10
5. Unbounded operators	14
5.1. Abstract affiliated operators	14
5.2. The case of true operators	16
6. Parametric pseudodifferential operators	17
References	19

## INTRODUCTION

Let us begin by motivating the present work using spectral theory and the related Fredholm conditions for pseudodifferential operators. A typical result in spectral

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theory of  $N$ -body Hamiltonians [9, 15, 16, 20] associates to the Laplacian  $H$  a family of other operators  $H_\phi$ ,  $\phi \in \mathcal{F}$ , such that the essential spectrum  $\text{Spec}_{ess}(H)$  of  $H$  is obtained in terms of the usual spectra  $\text{Spec}(H_\phi)$  of  $H_\phi$  as the closure of the union of the later:

$$(1) \quad \text{Spec}_{ess}(H) = \overline{\bigcup_{\phi \in \mathcal{F}} \text{Spec}(H_\phi)}.$$

It was noticed that sometimes the closure is not necessary, and one of the motivations of our paper is to clarify this issue. It is well known that the operators  $H_\phi$  are obtained as homomorphic images of the operator  $H$ , that is  $H_\phi = \phi(H)$ , where the morphisms  $\phi \in \mathcal{F}$  are defined on a certain  $C^*$ -algebra associated to  $H$ . This justifies the study of families of representations. See for example [16] for results in this direction.

Another motivation comes from the characterization of Fredholm operators (Fredholm conditions) for (pseudo)differential operators [21]. More precisely, for suitable manifolds  $M$  and for differential operators  $D$  on  $M$  compatible with the geometry, there was devised a procedure to associate to  $M$  the following data:

- (i) spaces  $Z_\alpha$ ,  $\alpha \in I$ ;
- (ii) groups  $G_\alpha$ ,  $\alpha \in I$ ; and
- (iii)  $G_\alpha$ -invariant differential operators  $D_\alpha$  acting on  $Z_\alpha \times G_\alpha$ .

This data can be used to characterize the Fredholm property of  $D$  as follows. Let  $m$  be the order of  $D$ , then

$$(2) \quad D : H^s(M) \rightarrow H^{s-m}(M) \text{ is Fredholm} \Leftrightarrow \begin{aligned} &D \text{ is elliptic and} \\ &D_\alpha \text{ is invertible for all } \alpha \in I. \end{aligned}$$

Moreover, the spaces  $Z_\alpha$  and the groups  $G_\alpha$  are independent of  $D$ . If  $M$  is compact (without boundary), then the index  $I$  is empty (so there are no  $D_\alpha$ s). In general, for non-compact manifolds, the conditions on the operators  $D_\alpha$  are, however, necessary. The non-compact geometries to which this characterization of Fredholm operators applies include: asymptotically euclidean manifolds, asymptotically hyperbolic manifolds, manifolds with poly-cylindrical ends, and many others. Again, the operators  $P_\alpha$  are homomorphic images of the operator  $P$ , which motivates the study of families of representations.

The results in [15, 16, 21] mentioned above are the main motivation for this work, which is a purely theoretical one on the representation theory of  $C^*$ -algebras, even though the applications are to spectral theory and (pseudo)differential operators. We thus define the concept of a *strictly norming family*  $\mathcal{F}$  of representations of a  $C^*$ -algebra  $A$  as having the property that for any  $a \in A$ , there exists  $\phi \in \mathcal{F}$  such that  $\|a\| = \|\phi(a)\|$ . We have learned from G. Skandalis that he has also considered this condition (private communication). The family  $\mathcal{F}$  does not have to consist of irreducible representations. Let  $\mathcal{F}$  be a strictly norming family of representations of  $A$ , we show then that an abstract differential operator  $D$  affiliated to  $A$  is invertible if, and only if,  $\phi(D)$  is invertible for all  $\phi \in \mathcal{F}$ . This property characterizes strictly norming families of representations. We provide a necessary and sufficient conditions for a family of representations to be strictly norming in terms of the topology on the Jacobson primitive ideal spectrum  $\text{Prim}(A)$  of  $A$ . If  $A$  is a separable  $C^*$ -algebras, we show that a family  $\mathcal{F}$  of representations is strictly norming if, and only if, every irreducible representation of  $A$  is weakly contained in a representation  $\phi \in \mathcal{F}$ . Some related results were obtained by Exel in [14] and

by Roch in [30]. These results are concerned with sufficient families of irreducible representations and are discussed below. See also [17, 31].

A typical application is to parametric families of differential operators arising in the analysis on manifolds with corners (more precisely, in the case of manifolds with polycylindrical ends). In that case, we recover the fact that an operator compatible with the geometry is invertible if, and only if, its Mellin transform is invertible. We discuss also several other examples. Due to the nature of the main applications to other areas than the study of  $C^*$ -algebras, we write the paper with an eye towards the non-specialist in  $C^*$ -algebras.

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## 1. $C^*$ -ALGEBRAS AND THEIR PRIMITIVE IDEAL SPECTRUM

We begin with a review of some needed general  $C^*$ -algebra results. We recall [12] that a  $C^*$ -algebra is a complex algebra  $A$  together with a conjugate linear involution  $*$  and a complete norm  $\| \cdot \|$  such that  $(ab)^* = b^*a^*$ ,  $\|ab\| \leq \|a\|\|b\|$ , and  $\|a^*a\| = \|a\|^2$ , for all  $a, b \in A$ . (The fact that  $*$  is an involution means that  $a^{**} = a$ .) In particular, a  $C^*$ -algebra is also a Banach algebra. Let  $\mathcal{H}$  be a Hilbert space and denote by  $\mathcal{L}(\mathcal{H})$  the space of linear, bounded operators on  $\mathcal{H}$ . One of the main reasons why  $C^*$ -algebras are important is that every norm-closed subalgebra  $A \subset \mathcal{L}(\mathcal{H})$  that is also closed under taking Hilbert space adjoints is a  $C^*$ -algebra. Abstract  $C^*$ -algebras have many non-trivial properties that can then be used to study the concretely given algebra  $A$ . Conversely, every abstract  $C^*$ -algebra is isometrically isomorphic to a norm closed subalgebra of  $\mathcal{L}(\mathcal{H})$  (the Gelfand-Naimark theorem, see [12, theorem 2.6.1]).

Let  $A$  denote a generic  $C^*$ -algebra throughout this paper. A *representation* of  $A$  on the Hilbert space  $\mathcal{H}_\pi$  is a  $*$ -morphism  $\pi : A \rightarrow \mathcal{L}(\mathcal{H}_\pi)$  to the algebra of bounded operators on  $\mathcal{H}_\pi$ . We shall use the fact that every morphism  $\phi$  of  $C^*$ -algebras (and hence any representation of a  $C^*$ -algebra) has norm  $\|\phi\| \leq 1$ . Consequently, every bijective morphism of  $C^*$ -algebras is an isometric isomorphism, and, in particular

$$(3) \quad \|\phi(a)\| = \|a + \ker(\phi)\|_{A/\ker(\phi)}.$$

It follows that if  $\phi_1$  and  $\phi_2$  are representations of  $A$  with the same kernel, then  $\|\phi_1(a)\| = \|\phi_2(a)\|$  for all  $a \in A$ . A two-sided ideal  $I \subset A$  is called *primitive* if it is the kernel of an irreducible representation. We shall denote by  $\text{Prim}(A)$  the set of primitive ideals of  $A$ . For any two-sided ideal  $J \subset A$ , we have that its primitive ideal spectrum  $\text{Prim}(J)$  identifies with the set of all the primitive ideals of  $A$  not containing the two-sided ideal  $J \subset A$ . It turns out then that the sets of the form  $\text{Prim}(J)$ , where  $J$  ranges through the set of two-sided ideals  $J \subset A$ , define a topology on  $\text{Prim}(A)$ , called the *Jacobson topology* on  $\text{Prim}(A)$ .

By  $\phi : A \rightarrow \mathcal{L}(\mathcal{H}_\phi)$  we shall denote generic representations of  $A$ . For any representation  $\phi$  of  $A$ , we define its support,  $\text{supp}(\phi) \subset \text{Prim}(A)$  as the complement

of  $\text{Prim}(\ker(\phi))$ , that is,  $\text{supp}(\phi) := \text{Prim}(A) \setminus \text{Prim}(\ker(\phi))$  is the set of primitive ideals of  $A$  containing  $\ker(\phi)$ .

*Remark 1.1.* The irreducible representations of  $A$  do not form a set (there are too many of them). The *unitary equivalence classes* of irreducible representations of  $A$  do form a set however, which we shall denote by  $\hat{A}$ . By  $\pi : A \rightarrow \mathcal{L}(\mathcal{H}_\pi)$  we shall denote an arbitrary *irreducible* representation of  $A$ . There exists then by definition a surjective map

$$(4) \quad \text{can} : \hat{A} \rightarrow \text{Prim}(A)$$

that associates to (the class of) each irreducible representation  $\pi \in \hat{A}$  its kernel  $\ker(\pi)$ . For each  $a \in A$  and each irreducible representation  $\pi$  of  $A$ , the algebraic properties of  $\pi(a)$  depend only on the kernel of  $\pi$ . That yields a well defined function

$$(5) \quad \text{can} : \hat{A} \ni \pi \rightarrow \|\pi(a)\| \in [0, \|a\|],$$

which descends to a well defined function

$$(6) \quad n_a : \text{Prim}(A) \ni \pi \rightarrow \|\pi(a)\| \in [0, \|a\|], \quad n_a(\ker(\pi)) = \|\pi(a)\|.$$

A  $C^*$ -algebra is called *type I* if, and only if, the surjection  $\text{can} : \hat{A} \rightarrow \text{Prim}(A)$  of Equation (4) is, in fact, a bijection. Then the discussion of Remark 1.1 becomes unnecessary and several arguments below will be (slightly) simplified since we will not have to make distinction between equivalence classes of irreducible representations and their kernels. Fortunately, many (if not all) of the  $C^*$ -algebras that arise in the study of pseudodifferential operators and of other practical questions are type I  $C^*$ -algebras. In spite of this, it seems unnatural at this time to restrict our study to type I  $C^*$ -algebras. Therefore, we will not assume that  $A$  is a type I  $C^*$ -algebra, unless this assumption is really needed. Nevertheless, when  $A$  is a type I  $C^*$ -algebra, we will identify  $\hat{A}$  and  $\text{Prim}(A)$ .

We shall need the following simple (and well known) lemma.

**Lemma 1.2.** *The map  $n_a : \text{Prim}(A) \ni \pi \rightarrow \|\pi(a)\| \in [0, \|a\|]$  is lower semi-continuous, that is, the set  $\{I \in \text{Prim}(A), \|a + I\|_{A/I} > t\}$  is open for any  $t \in \mathbb{R}$ .*

We include the simple proof for the benefit of the non-specialist.

*Proof.* Let us fix  $t \in \mathbb{R}$ . Since  $n_a$  takes on non-negative values, we may assume  $t \geq 0$ . Let then  $\chi : [0, \infty) \rightarrow [0, 1]$  be a continuous function that is zero on  $[0, t^2]$  but is  $> 0$  on  $(t^2, \infty)$  and let  $b = \chi(a^*a)$ , which is defined using the functional calculus with continuous functions. If  $\phi : A \rightarrow \mathcal{L}(\mathcal{H}_\phi)$  is a representation of  $A$ , then we have that  $\|\phi(a)\|^2 = \|\phi(a^*a)\| \leq t^2$  if, and only if,

$$\chi(\phi(a^*a)) = \phi(\chi(a^*a)) = \phi(b) = 0.$$

Let then  $J$  be the (closed) two sided ideal generated by  $b$ , that is,  $J := \overline{AbA}$ . Then

$$\begin{aligned} \{I \in \text{Prim}(A), \|a + I\|_{A/I} \leq t\} &= \{I \in \text{Prim}(A), b \in I\} \\ &= \{I \in \text{Prim}(A), J \subset I\} = \text{Prim}(A) \setminus \text{Prim}(J), \end{aligned}$$

is hence a closed set. Consequently,  $\{I \in \text{Prim} A, \|a + I\|_{A/I} > t\}$  is open, as claimed.  $\square$

## 2. FAITHFUL FAMILIES

Let  $\mathcal{F}$  be a set of representations of  $A$ . We say that the family  $\mathcal{F}$  is *faithful* if the direct sum representation  $\rho := \bigoplus_{\phi \in \mathcal{F}} \phi$  is injective. Faithful families of irreducible representations of a  $C^*$ -algebra  $A$  were called *weakly sufficient* in [30]. The results of this subsection are for the most part very well-known, see for instance [30], but we include them for the purpose of later reference and in order to compare them with the properties of strictly norming families. We have the following well known result that will serve us as a model for characterization of “strictly norming families” of representations in the next subsection.

**Proposition 2.1.** *Let  $\mathcal{F}$  be a family of representations of the  $C^*$ -algebra  $A$ . The following are equivalent:*

- (i) *The family  $\mathcal{F}$  is faithful.*
- (ii) *The union  $\bigcup_{\phi \in \mathcal{F}} \text{supp}(\phi)$  is dense in  $\text{Prim}(A)$ .*
- (iii)  *$\|a\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\|$  for all  $a \in A$ .*

*Proof.* (i) $\Rightarrow$ (ii). We proceed by contradiction. Let us assume that (i) is true, but that (ii) is not true. That is, we assume that  $\bigcup_{\phi \in \mathcal{F}} \text{supp}(\phi)$  is not dense in  $\text{Prim}(A)$ . Then there exists a non empty open set  $\text{Prim}(J) \subset \text{Prim}(A)$  that does not intersect  $\bigcup_{\phi \in \mathcal{F}} \text{supp}(\phi)$ , where  $J \subset A$  is a non-trivial two-sided ideal. Then  $J \neq 0$  is contained in the kernel of  $\bigoplus_{\phi \in \mathcal{F}} \phi$  and hence  $\mathcal{F}$  is not faithful. This is a contradiction, and hence (ii) must be true if (i) is true.

(ii) $\Rightarrow$ (iii). For a given  $a \in A$ , the map sending the kernel  $\ker \pi$  of an irreducible representation  $\pi$  to  $\|\pi(a)\|$  is a lower semi-continuous function  $\text{Prim}(A) \rightarrow [0, \infty)$ , by Lemma 1.2. Moreover, for any  $a \in A$  there exists an irreducible representation  $\pi_a$  such that  $\|\pi_a(a)\| = \|a\|$ . Hence, for every  $\epsilon > 0$ ,  $\{\pi \in \text{Prim}(A), \|\pi(a)\| > \|a\| - \epsilon\}$  is a non empty open set (it contains  $\ker \pi_a$ ) and then it contains some  $\pi \in \bigcup_{\phi \in \mathcal{F}} \text{supp}(\phi)$ , since the later set was assumed to be dense in  $\text{Prim}(A)$ . Let  $\phi \in \mathcal{F}$  be such that  $\ker(\pi) \supset \ker(\phi)$ . Then

$$\|a\| \geq \|\phi(a)\| \geq \|\pi(a)\| > \|a\| - \epsilon,$$

where the first inequality is due to the general fact that representations of  $C^*$ -algebras have norm  $\leq 1$  and the second one is due to the fact that

$$\|\phi(a)\| = \|a + \ker(\phi)\|_{A/\ker(\phi)} \geq \|a + \ker(\pi)\|_{A/\ker(\pi)} = \|\pi(a)\|,$$

by Equation (3). Consequently,  $\|a\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\|$ , as desired.

(iii) $\Rightarrow$ (i). Let  $\rho := \bigoplus_{\phi \in \mathcal{F}} \phi : A \rightarrow \bigoplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi)$ . We need to show that  $\rho$  is injective. The norm on  $\bigoplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi)$  is the sup norm, that is,  $\|(T_\phi)_{\phi \in \mathcal{F}}\| = \sup_{\phi \in \mathcal{F}} \|T_\phi\|$ . Therefore  $\|\rho(a)\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\| = \|a\|$ , since we are assuming (iii). Consequently,  $\rho$  is isometric, and hence it is injective.  $\square$

In the next proposition we shall need to assume that  $A$  is unital (that is, that it has a unit  $1 \in A$ ). This assumption is not very restrictive since, given any non-unital  $C^*$ -algebra  $A_0$ , the algebra with adjoint unit  $A = A_0^+ := A_0 \oplus \mathbb{C}$  has a unique  $C^*$ -algebra norm.

For any unital  $C^*$ -algebra  $A$  and any  $a \in A$ , we denote by  $\text{Spec}_A(a)$  the *spectrum* of  $a$  in  $A$ , defined by

$$\text{Spec}_A(a) := \{ \lambda \in \mathbb{C}, \lambda - a \text{ is not invertible in } A \}.$$

It is known classically that  $\text{Spec}_A(a)$  is compact and non-empty, unlike in the case of unbounded operators [12]. For  $A$  non-unital,  $\text{Spec}(a) := \text{Spec}_{A^+}(a)$ . We shall need the following general property of  $C^*$ -algebras.

**Lemma 2.2.** *Let  $A_1 \subset B$  be two  $C^*$ -algebras and  $a \in A_1$  be such that it has an inverse in  $B$ , denoted  $a^{-1}$ . Then  $a^{-1} \in A_1$ . In particular, the spectrum of  $a$  is independent of the  $C^*$ -algebra in which we compute it:*

$$(7) \quad \text{Spec}_{A_1}(a) = \text{Spec}_B(a) =: \text{Spec}(a).$$

If  $a \in A$  for some non-unital  $C^*$ -algebra, then we define  $\text{Spec}(a) := \text{Spec}_{A^+}(a)$ , where  $A^+ := A \oplus \mathbb{C}$ , so  $\text{Spec}(a)$  is independent of the  $C^*$ -algebra containing  $a$  also in the non-unital case.

**Proposition 2.3.** *Let  $\mathcal{F}$  be a faithful family of representations of a unital  $C^*$ -algebra  $A$ . An element  $a \in A$  is invertible if, and only if,  $\phi(a)$  is invertible in  $\mathcal{L}(\mathcal{H}_\phi)$  for all  $\phi \in \mathcal{F}$  and the set  $\{\|\phi(a)^{-1}\|, \phi \in \mathcal{F}\}$  is bounded.*

*Proof.* If  $a$  is invertible,  $\phi(a)$  also is invertible and  $\|\phi(a)^{-1}\| = \|\phi(a^{-1})\| \leq \|a^{-1}\|$  is hence bounded. Conversely, let  $\rho$  be the direct sum of all the representations  $\phi \in \mathcal{F}$ , that is,

$$(8) \quad \rho := \bigoplus_{\phi \in \mathcal{F}} \phi : A \longrightarrow \bigoplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi).$$

If  $\|\phi(a)\|$  is invertible for all  $\phi \in \mathcal{F}$  and there exists  $M$  independent of  $\phi$  such that  $\|\phi(a)^{-1}\| \leq M$ , then  $b := (\phi(a)^{-1})_{\phi \in \mathcal{F}}$  is a well defined element in  $B := \bigoplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi)$  and  $b$  is an inverse for  $\rho(a)$  in  $B$ . Let  $A_1 := \rho(A)$ . Then  $\rho(a) \in A_1$  is invertible in  $B$ . Therefore by the Lemma 2.2  $\rho(a)$  is invertible in  $A_1$ . Then observe that since  $\rho$  is continuous, injective, and surjective morphism of  $C^*$ -algebras, it defines an isomorphism of algebras  $A \rightarrow A_1$ . We then conclude that  $a$  is invertible in  $A$  as well.  $\square$

The following is a converse of the above proposition. Recall that  $a \in A$  is called *normal* if  $aa^* = a^*a$ .

**Proposition 2.4.** *Let  $\mathcal{F}$  be a family of representations of a unital  $C^*$ -algebra  $A$  with the following property:*

“If  $a \in A$  is such that  $\phi(a)$  is invertible in  $\mathcal{L}(\mathcal{H}_\phi)$  for all  $\phi \in \mathcal{F}$  and the set  $\{\|\phi(a)^{-1}\|, \phi \in \mathcal{F}\}$  is bounded, then  $a$  is invertible in  $A$ .”

*Then the family  $\mathcal{F}$  is faithful.*

*Proof.* Clearly, the family  $\mathcal{F}$  is not empty, since otherwise all elements of  $A$  would be invertible, which is not possible. Let us assume, by contradiction, that the family  $\mathcal{F}$  is *not* faithful. Then, by Proposition 2.1(ii), there exists a non-empty open set  $V \subset \text{Prim}(A)$  that does not intersect  $\bigcup_{\phi \in \mathcal{F}} \text{supp}(\phi)$ . Let  $J \subset A$ ,  $J \neq 0$ , be the (closed) two-sided ideal corresponding to  $V$ , that is,  $V = \text{Prim}(J)$ . Since  $\mathcal{F}$  is non-empty, we have  $J \neq \text{Prim}(A)$ . Then every  $\phi \in \mathcal{F}$  is such that  $\phi = 0$  on  $J$ . Let  $a \in J$ ,  $a \neq 0$ . By replacing  $a$  with  $a^*a \in J$ , we can assume  $a \geq 0$ . Let  $\lambda \in \text{Spec}(a)$ ,  $\lambda \neq 0$ . Such a  $\lambda$  exists since  $a$  is normal and non-zero. Let  $c := \lambda - a$ . Then, for any  $\phi \in \mathcal{F}$ ,  $\phi(c) = \lambda \in \mathbb{C}$  is invertible and  $\|\phi(c)^{-1}\| = \lambda^{-1}$  is bounded. However,  $c$  is not invertible (in any  $C^*$ -algebra containing it) since it belongs to the non-trivial ideal  $J$ .  $\square$

Recall that  $\mathcal{C}_0(X)$  is the set of continuous functions on  $X$  that have vanishing limit at infinity. Then  $\mathcal{C}_0(X)$  is a commutative  $C^*$ -algebra, and all commutative  $C^*$ -algebras are of this form.

*Example 2.5.* Let  $\mu_\alpha, \alpha \in I$ , be a family of positive, regular Borel measures on a locally compact space  $X$ . Let  $\phi_\alpha$  be the corresponding multiplication representation of the  $C^*$ -algebra  $\mathcal{C}_0(X) \rightarrow \mathcal{L}(L^2(X, \mu_\alpha))$ . We have  $\text{supp}(\phi_\alpha) = \text{supp}(\mu_\alpha)$  and the family  $\mathcal{F} := \{\phi_\alpha, \alpha \in I\}$  is faithful if, and only if,  $\cup_{\alpha \in I} \text{supp}(\mu_\alpha)$  is dense in  $X$ . In particular, if each  $\mu_\alpha$  is the Dirac measure concentrated at some  $x_\alpha \in X$ , then  $\phi_\alpha(f) = f(x_\alpha) =: \text{ev}_{x_\alpha}(f) \in \mathbb{C}$  and  $\text{supp}(\mu_\alpha) = \{x_\alpha\}$ . We shall henceforth identify  $x_\alpha \in X$  with the corresponding evaluation irreducible representation  $\text{ev}_{x_\alpha}$ . Then we have that

$$\mathcal{F} = \{\text{ev}_{x_\alpha}, \alpha \in I\} \text{ is faithful} \Leftrightarrow \{x_\alpha, \alpha \in I\} \text{ is dense in } X.$$

This example extends right away to  $C^*$ -algebras of the form  $\mathcal{C}_0(X; \mathcal{K})$  of functions with values compact operators on some given Hilbert space.

We conclude our discussion of faithful families with the following result. We denote by  $\overline{\cup S_\alpha} := \overline{\cup_\alpha S_\alpha}$  the closure of the union of the family of sets  $S_\alpha$ .

**Proposition 2.6.** *Let  $\mathcal{F}$  be a family of representations of a unital  $C^*$ -algebra  $A$ . Then  $\mathcal{F}$  is faithful if, and only if, for any normal  $a \in A$ ,*

$$(9) \quad \text{Spec}(a) = \overline{\cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))}.$$

*Proof.* Let us assume first that the family  $\mathcal{F}$  is faithful and that  $a$  is normal. Since we have that  $\text{Spec}(\phi_0(a)) \subset \text{Spec}(a)$  for any representation  $\phi_0$  of  $A$ , it is enough to show that  $\text{Spec}(a) \subset \overline{\cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))}$ . Let us assume the contrary and let  $\lambda \in \text{Spec}(a) \setminus \overline{\cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))}$ . By replacing  $a$  with  $a - \lambda$ , we can assume that  $\lambda = 0$ . We thus have that  $\phi(a)$  is invertible for all  $\phi \in \mathcal{F}$ , but  $a$  is not invertible (in  $A$ ). Moreover,  $\|\phi(a)^{-1}\| \leq \delta^{-1}$ , where  $\delta$  is the distance from  $\lambda = 0$  to the spectrum of  $\phi(a)$ , by the properties of the functional calculus for normal operators. This is however a contradiction by Proposition 2.3, which implies that  $a$  must be invertible in  $A$  as well.

To prove the converse, let us assume that  $\text{Spec}(a) \subset \overline{\cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))}$ , for all normal elements  $a \in A$ . Let  $J$  be a non-trivial (closed *selfadjoint*) two-sided ideal on which all the representations  $\phi \in \mathcal{F}$  vanish. We have to show that  $J = 0$ , which would imply that  $\mathcal{F}$  is faithful. Let  $a \in J$  be a normal element. Then  $\text{Spec}(a) \subset \overline{\cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))} = \{0\}$ . Since  $a$  is normal we deduce  $a = 0$  and hence  $J$  has no normal element other than 0. Then, for any  $a \in J$ , we can write  $a = 1/2(a+a^*) + 1/2(a-a^*)$ , the sum of two normal elements in  $J$  because  $J$  is selfadjoint. Therefore  $1/2(a+a^*) = 1/2(a-a^*) = 0$ , and hence  $a = 0$  and  $J = 0$ .  $\square$

*Example 2.7.* Let  $\mathcal{G}$  be an amenable groupoid with units  $\mathcal{G}^{(0)}$  and  $\mathcal{R} := \{\lambda_x, x \in \mathcal{G}^{(0)}\}$  be the set of regular representations of  $C^*(\mathcal{G})$ . Then  $\mathcal{R}$  is a faithful family of representations of  $C_r^*(\mathcal{G})$ , the reduced  $C^*$ -algebra associated to  $\mathcal{G}$  [2, 6].

### 3. FULL AND STRICTLY NORMING FAMILIES

Let us notice that Example 2.5 shows that the ‘sup’ in the relation  $\|a\| = \sup_{\phi \in \mathcal{F}} \|\phi(a)\|$  (Proposition 2.1) may not be attained. It also shows that the closure of the union in Equation (9) is needed. Sometimes, in applications, one does obtain however the stronger version of these results (that is, the sup is attained and



the closure is not needed), see [9, 16], for example. Moreover, the condition that the norms of  $\phi(a)^{-1}$  be uniformly bounded (in  $\phi$ ) for any fixed  $a \in A$  is inconvenient and often not needed in applications. For this reason, we introduce now a more restrictive class of families of representations of  $A$ .

Recall that  $\text{supp}(\phi)$  is the set of primitive ideals of  $A$  that contain  $\ker(\phi)$ . We shall denote by  $A' := A$  if  $A$  has a unit and  $A' := A \oplus \mathbb{C}$  the algebra of adjoint unit if  $A$  does not have a unit. The definitions of “strictly norming” and “sufficient” families below are not new [14, 30].

**Definition 3.1.** Let  $\mathcal{F}$  be a set of representations of the  $C^*$ -algebra  $A$ .

- (i) We shall say that  $\mathcal{F}$  is *exhausting* if  $\text{Prim}(A) = \bigcup_{\phi \in \mathcal{F}} \text{supp}(\phi)$ .
- (ii) We shall say that  $\mathcal{F}$  is *strictly norming* if, for any  $a \in A$ , there exists  $\phi \in \mathcal{F}$  such that  $\|\phi(a)\| = \|a\|$ .
- (iii) We shall say that  $\mathcal{F}$  is *sufficient* if an element  $1 + a \in A'$ ,  $a \in A$ , is invertible if, and only if,  $1 + \pi(a)$  is invertible for any  $\pi \in \mathcal{F}$ .

*Example 3.2.* By classical results, the set of all irreducible representations of a  $C^*$ -algebra is strictly norming. A proof of this well-known fact is contained in [14]. See also Theorem 3.7.

*Remark 3.3.* Let  $\mathcal{F}_i$ ,  $i = 1, 2$ , be two families of representations of  $A$ . Let denote by  $\mathcal{I}_i := \{\ker(\phi), \phi \in \mathcal{F}_i\}$ . We assume that  $\mathcal{I}_1 = \mathcal{I}_2$ . Then the families  $\mathcal{F}_i$  are at the same time exhausting or not. The same is true for the properties of being strictly norming, or sufficient. So these properties are really properties of a family of ideals of  $A$  rather than of families of representations of  $A$ . Nevertheless, it is customary to work with families of representation rather than families of ideals.

Let us record the following simple facts, for further use.

**Proposition 3.4.** *Let  $\mathcal{F}$  be a set of representations of a  $C^*$ -algebra. If  $\mathcal{F}$  is exhausting, then  $\mathcal{F}$  is strictly norming. If  $\mathcal{F}$  is strictly norming, then it is also faithful.*

*Proof.* Let  $A$  be the given  $C^*$ -algebra. Let us prove first that any exhausting family  $\mathcal{F}$  is strictly norming. Indeed, let  $a \in A$ . Then there exists an irreducible representation  $\pi$  of  $A$  such that  $\|\pi(a)\| = \|a\|$ . Let  $\phi \in \mathcal{F}$  be such that  $\pi \in \text{supp}(\phi)$ , then, as in the proof of (ii) $\Rightarrow$ (iii) in Proposition 2.1, we have that  $\|a\| = \|\pi(a)\| \leq \|\phi(a)\| \leq \|a\|$ . Hence  $\|\phi(a)\| = \|a\|$ .

Let us prove first that any strictly norming family  $\mathcal{F}$  is faithful. Indeed, let us consider the representation  $\rho := \bigoplus_{\phi \in \mathcal{F}} \phi : A \rightarrow \bigoplus_{\phi \in \mathcal{F}} \mathcal{L}(H_\phi)$ . By the definition of a strictly norming family of representations, the representation  $\rho$  is isometric. Therefore it is injective and consequently  $\mathcal{F}$  is faithful.  $\square$

We summarize the above Proposition in

$$\mathcal{F} \text{ exhausting} \Rightarrow \mathcal{F} \text{ strictly norming} \Rightarrow \mathcal{F} \text{ faithful.}$$

In the next two examples we will see that there exist faithful families that are not strictly norming and strictly norming families that are not exhausting.

*Example 3.5.* We consider again the framework of Example 2.5 and consider only families of *irreducible representations*. Thus  $A = \mathcal{C}_0(X)$ , for a locally compact space  $X$ . The irreducible representations of  $A$  then identify with the points of  $X$ , since  $X \simeq \text{Prim}(A) = \hat{A}$ . A family  $\mathcal{F}$  of *irreducible* representations of  $A$  thus identifies with a subset  $\mathcal{F} \subset X$ . We then have that a family  $\mathcal{F} \subset X$  of irreducible

representations of  $A = \mathcal{C}_0(X)$  is faithful if, and only if,  $\mathcal{F}$  is *dense* in  $X$ . On the other hand, a family of irreducible representations of  $A = \mathcal{C}_0(X)$  is exhausting if, and only if,  $\mathcal{F} = X$ .

The relation between exhausting and strictly norming families is not so simple. We begin with the following remark on the above example.

*Remark 3.6.* If  $X$  is moreover *metrisable*, then every strictly norming family  $\mathcal{F} \subset X$  is also exhausting, because for any  $x \in X$ , there exists a compactly supported, continuous function  $\psi_x : X \rightarrow [0, 1]$  such that  $\psi_x(x) = 1$  and  $\psi_x(y) < 1$  for  $y \neq x$  (we can do that by arranging that  $\psi_x(y) = 1 - d(x, y)$ , for  $d(x, y)$  small, and use the Tietze extension theorem. In general, however, it is not true that any strictly norming family is exhausting. Indeed, let  $I$  be an uncountable set and  $X = [0, 1]^I$ . Let  $x \in X$  be arbitrary, then the family  $\mathcal{F} := X \setminus \{x\}$  is strictly norming but is not exhausting. See also Propositions 4.4.

We now explain how the concepts of exhausting and strictly norming sets of representations are useful for invertibility questions. Let us first notice that if an element  $a \in A$  is not invertible, then either  $a^*a$  or  $aa^*$  are not invertible. This is seen for example by assuming that  $A \subset \mathcal{L}(\mathcal{H})$ . Assume the contrary. Then there exist  $y, z$  such that  $ya^*a = 1$  and  $aa^*z = 1$ . So  $b := ya^*a = ya^*aa^*z = a^*z$  satisfies  $ab = ba = 1$ , and hence  $a$  is invertible. The following result was proved in [30]. Since the proof in that paper is for families of irreducible representations, we also include a (different) proof.

**Theorem 3.7.** *Let  $\mathcal{F}$  be a set of representations of a unital  $C^*$ -algebra  $A$ . Then  $\mathcal{F}$  is strictly norming if, and only if, it is sufficient.*

*Proof.* Let us assume (i) and let  $a \in A$  be such that  $\phi(a)$  is invertible for all  $\phi \in \mathcal{F}$ . We want to show that  $a$  is invertible as well. Let us assume, by contradiction, that it is not invertible. Then either  $a^*a$  or  $aa^*$  is not invertible. By replacing  $a$  with  $a^*$  (which is also not invertible), we can assume that  $a^*a$  is not invertible. Then  $0 \in \text{Spec}(a^*a)$  and hence the element  $b := \|a\|^2 - a^*a$  has norm  $\|b\| = \|a\|^2$ . Therefore there exists  $\phi \in \mathcal{F}$  such that  $\|\phi(b)\| = \|b\|$ , since we have assumed that  $\mathcal{F}$  is strictly norming. Therefore  $\|a\|^2 - \phi(a)^*\phi(a) = \phi(b)$  has norm  $\|b\| = \|a\|^2$ , and hence  $0$  is in the spectrum of  $\phi(a)^*\phi(a)$ , which is then not invertible. Therefore  $\phi(a)$  is not invertible. We have thus obtained a contradiction.

Conversely, let us assume (ii) and let  $a \in A$ . We want to show that there exists  $\phi \in \mathcal{F}$  such that  $\|\phi(a)\| = \|a\|$ . Let us consider again  $b := \|a\|^2 - a^*a$ , which is not invertible in  $A$ , by the properties of functional calculus. Therefore, there exists  $\phi \in \mathcal{F}$  such that  $\phi(b) = \|a\|^2 - \phi(a)^*\phi(a)$  is not invertible. Since  $\|\phi(a)\| \leq \|a\|$ , this is possible only if  $\|\phi(a)\| = \|a\|$ .  $\square$

The following characterisation of Fredholm operators is a consequence of the above theorem and is sometimes useful in applications.

**Corollary 3.8.** *Let  $1 \in A \subset \mathcal{L}(\mathcal{H})$  be a sub- $C^*$ -algebra of bounded operators on the Hilbert space  $\mathcal{H}$  containing the algebra of compact operators on  $\mathcal{H}$ ,  $\mathcal{K} = \mathcal{K}(\mathcal{H})$ . Let  $\mathcal{F}$  be a faithful family of representations of  $A/\mathcal{K}$ . We then have the following characterisation of Fredholm operators  $a \in A$ :*

*$a \in A$  is Fredholm if, and only if,  $\phi(a)$  is invertible in for all  $\phi \in \mathcal{F}$ .*

The following proposition is the analog of Proposition 2.6 in the framework of strictly norming families.

**Theorem 3.9.** *Let  $\mathcal{F}$  be a family of representations of a unital  $C^*$ -algebra  $A$ . Then  $\mathcal{F}$  is strictly norming if, and only if, for any  $a \in A$ ,*

$$(10) \quad \text{Spec}(a) = \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a)).$$

*Proof.* Let us assume first that the family  $\mathcal{F}$  is strictly norming. We proceed in analogy with the proof of Proposition 2.6. Since we have that  $\text{Spec}(\phi_0(a)) \subset \text{Spec}(a)$  for any representation  $\phi_0$  of  $A$ , it is enough to show that  $\text{Spec}(a) \subset \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$ . Let us assume the contrary and let  $\lambda \in \text{Spec}(a) \setminus \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$ . By replacing  $a$  with  $a - \lambda$ , we can assume that  $\lambda = 0$ . We thus have that  $\phi(a)$  is invertible for all  $\phi \in \mathcal{F}$ , but  $a$  is not invertible (in  $A$ ). This is however a contradiction by Theorem 3.7, which implies that  $a$  must be invertible in  $A$  as well.

To prove the converse, let us assume that  $\text{Spec}(a) \subset \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$  for all  $a \in A$ . We shall use Theorem 3.7. Let us assume that  $a \in A$  and that  $\phi(a)$  is invertible for all  $\phi \in \mathcal{F}$ . Then  $0 \notin \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$ . Since  $\text{Spec}(a) \subset \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(a))$ , we have that  $0 \notin \text{Spec}(a)$ , and hence  $a$  is invertible. Theorem 3.7 then shows that the family  $\mathcal{F}$  is strictly norming.  $\square$

*Example 3.10.* If  $\mathcal{G}$  is an étale, Hausdorff, second countable, amenable groupoid, then the family  $\mathcal{R} := \{\lambda_x, x \in \mathcal{G}^{(0)}\}$  of regular representations introduced in Example 2.7 is strictly norming for  $C_r^*(\mathcal{G})$  [14].

#### 4. TOPOLOGY ON THE SPECTRUM AND STRICTLY NORMING FAMILIES

Let us discuss now in more detail the relation between the concept of strictly norming family and the simpler (to check) concept of an exhausting family. The following theorem describes the class of  $C^*$ -algebras for which every strictly norming family is also exhausting. It explains Example 3.5 and Remark 3.6.

**Lemma 4.1.** *Let  $A$  be a  $C^*$ -algebra,  $J$  a two-sided ideal, and  $\pi$  a representation of  $A$  such that  $\pi|_J$  is non-degenerate. Also let  $a \in A$ ,  $0 \leq a \leq 1$  such that  $\|\pi(a)\| = 1$  and  $\eta > 0$ . Then there exists  $c \in J$ ,  $c \geq 0$ ,  $\|c\| \leq \eta$  such that  $\|\pi(a + c)\| \geq 1 + \eta/2$ .*

*Proof.* For any fixed  $\varepsilon > 0$  there exists a unit vector  $\xi$  such that  $\langle \pi(a)\xi, \xi \rangle \geq 1 - \varepsilon$ . Then the positive form  $\varphi(b) := \langle \pi(b)\xi, \xi \rangle$  has norm  $\|\varphi\| \leq \|\xi\|^2 = 1$  and satisfies  $\varphi(a) \geq 1 - \varepsilon$ . But if  $(u_\lambda)$  is an approximate unit in  $J$ , then

$$\|\varphi\| \geq \|\varphi|_J\| = \lim \varphi(u_\lambda) = \|\xi\| = 1.$$

So  $\|\varphi|_J\| = \|\varphi\| = 1$ . Hence there exists  $c_0 \in J$ ,  $c_0 \geq 0$ ,  $\|c_0\| = 1$ , such that  $\varphi(c_0) \geq 1 - \varepsilon$ . We then set  $c = \eta c_0$  and indeed, for  $\varepsilon$  small enough

$$\|a + c\| \geq \varphi(a + c) \geq 1 - \varepsilon + \eta(1 - \varepsilon) \geq 1 + \eta/2.$$

This completes the proof.  $\square$

We shall use the above lemma in the form of the following corollary.

**Corollary 4.2.** *Let  $I_0 = \ker(\pi_0) \in \text{Prim}(A)$  be a primitive ideal and  $V_0 \supset \dots \supset V_n \supset V_{n+1} \dots$  be a decreasing sequence of open neighbourhoods of  $I_0$  in  $\text{Prim}(A)$ . Then there exists  $a \in A$  such that  $\|a\| = \|\pi_0(a)\| = 1$  and  $\|\pi(a)\| \leq 1 - 2^k$  for any irreducible representation  $\pi$  such that  $\ker(\pi) \notin V_k$ .*

*Proof.* To construct  $a \in A$  with the desired properties, let us consider the ideals  $J_n$  defining the sets  $V_n$ , that is,  $V_n = \text{Prim}(J_n)$ . The element  $a$  we are looking for will be the limit of a sequence  $a_n \in A$ , where  $a_n$  are defined by induction to satisfy the following properties:

- (i)  $0 \leq a_n \leq 1$ ;
- (ii)  $\|\pi_0(a_n)\| = 1$ ;
- (iii)  $\|\pi(a_n)\| \leq 1 - 2^{-k}$  for all irreducible representations  $\pi$  such that  $\ker(\pi) \in \text{Prim}(A) \setminus \text{Prim}(J_k)$  for  $k = 0, 1, \dots, n$ ;
- (iv)  $\|a_n - a_{n-1}\| \leq 2^{2-n}$  for  $n \geq 1$ .

We define  $a_0$  as follows. We first choose  $b_0 \in J_0$  such that  $0 \leq b_0$ , and  $\pi_0(b_0) \neq 0$ . By rescaling  $b_0$  with a positive factor, we can assume that  $\|\pi_0(b_0)\| = 1$ . Let then  $\chi_0 : [0, \infty) \rightarrow [0, 1]$  be the continuous function defined by  $\chi_0(t) = t$  for  $t \leq 1$  and  $\chi_0(t) = 1$  for  $t \geq 1$ . Then we define  $a_0 = \chi_0(b_0)$ . Conditions (i–iv) are then satisfied.

To define  $a_n$  in terms  $a_{n-1}$ , we first define  $c_n$  and  $b_n = a_{n-1} + c_n$  as follows. By Lemma 4.1, there exists  $c_n \in J_n$ ,  $c_n \geq 0$ ,  $\|c_n\| \leq 2^{1-n}$ , such that  $\|\pi_0(b_n)\| \geq 1 + 2^{-n}$ . Let then  $\chi_n : [0, \infty) \rightarrow [0, 1]$  be the continuous function defined by  $\chi_n(t) = t$  for  $t \leq 1 - 2^{1-n}$ ,  $\chi_n$  linear on  $[1 - 2^{1-n}, 1]$  and on  $[1, 1 + 2^{-n}]$ ,  $\chi_n(1) = 1 - 2^{-n}$ , and  $\chi_n(t) = 1$  for  $t \geq 1 + 2^{-n}$ . Then we define  $a_n = \chi_n(b_n)$ .

*Claim.* The sequence  $a_n \in A$  just constructed satisfies conditions (i–iv).

Indeed, let us check these conditions one by one:

- (i) We have that  $a_{n-1}, c_n \geq 0$ , hence  $b_n := a_{n-1} + c_n \geq 0$ . Since  $0 \leq \chi_n \leq 1$ , we obtain that  $0 \leq a_n := \chi_n(b_n) \leq 1$ .
- (ii) Since  $\|\pi_0(b_n)\| \geq 1 + 2^{-n}$  and  $\chi_n(t) = 1$  for  $t \geq 1 + 2^{-n}$ , we have that  $\|\pi_0(a_n)\| = \|\pi_0(\chi_n(b_n))\| = \|\chi_n(\pi_0(b_n))\| = 1$ .
- (iii) Let  $\pi \in \hat{A}$  be such that  $\ker(\pi) \in \text{Prim}(J_k)^c := \text{Prim}(A) \setminus \text{Prim}(J_k)$ , for some  $k \leq n$ . Then  $\pi(c_n) = 0$ . Assume  $k < n$ . Then  $\|\pi(a_{n-1})\| \leq 1 - 2^{-k} \leq 1 - 2^{1-n}$ , by the induction hypothesis and hence

$$\pi(a_n) = \pi(\chi_n(b_n)) = \chi_n(\pi(b_n)) = \chi_n(\pi(a_{n-1})) = \pi(a_{n-1}),$$

since  $\chi_n(t) = t$  for  $t \leq 1 - 2^{1-n}$ . Consequently,  $\|\pi(a_n)\| = \|\pi(a_{n-1})\| \leq 1 - 2^{-k}$  for  $k < n$ . If  $k = n$ , then, similarly,  $\pi(a_n) = \chi_n(\pi(a_{n-1}))$ . Since  $\chi_n(t) \leq 1 - 2^{-n}$  for  $t \leq 1$  and  $0 \leq a_{n-1} \leq 1$ , we have that  $\|\pi(a_n)\| = \|\chi_n(\pi(a_{n-1}))\| \leq 1 - 2^{-n}$ .

- (iv) We have  $\|b_n\| \leq \|a_{n-1}\| + \|c_n\| \leq 1 + 2^{1-n}$ . Since  $|\chi(t) - t| \leq 2^{1-n}$  for all  $t$ , we have  $\|a_n - b_n\| \leq 2^{1-n}$ . Hence

$$\begin{aligned} \|a_n - a_{n-1}\| &= \|\chi_n(b_n) - a_{n-1}\| \leq \|\chi_n(b_n) - b_n\| + \|b_n - a_{n-1}\| \\ &\leq 2^{1-n} + \|c_n\| \leq 2^{2-n}. \end{aligned}$$

This completes the proof of our claim, and hence the sequence  $a_n$  constructed above satisfies Conditions (i–iv).

Let us now show how to use the fact that the sequence  $a_n \in A$  satisfies Conditions (i–iv) to construct  $a$  as in the statement of this corollary. First of all, Condition (iv) allows us to define  $a := \lim_{n \rightarrow \infty} a_n$ . Let us show that  $a \in A$  satisfied the desired conditions. Since Conditions (i–iii) are compatible with limits, we have

- (i)  $0 \leq a \leq 1$ ;
- (ii)  $\|\pi_0(a)\| = 1$ ;

(iii)  $\|\pi_0(a)\| \leq 1 - 2^{-k}$  on  $\text{Prim}(A) \setminus \text{Prim}(J_k)$  for  $k \geq 0$ .

Thus  $a$  has the properties stated in this corollary, which completes the proof.  $\square$

**Theorem 4.3.** *Let  $A$  be a  $C^*$ -algebra. Let us assume that every  $I \in \text{Prim}(A)$  has a countable base for its system of neighborhoods. Then every strictly norming family  $\mathcal{F}$  of representations of  $A$  is also exhausting.*

*Conversely, if every strictly norming family  $\mathcal{F}$  of representations of  $A$  is also exhausting, then every  $I \in \text{Prim}(A)$  has a countable base for its system of neighborhoods.*

*Proof.* Let us prove first the first part of the statement, so let us assume that every primitive ideal  $I \in \text{Prim}(A)$  has a countable base for its system of neighborhoods and let  $\mathcal{F}$  be a strictly norming family of representations of  $A$ . We need to show that  $\mathcal{F}$  is exhausting. We shall proceed by contradiction. Thus, let us assume that the family  $\mathcal{F}$  is not exhausting. Then there exists a primitive ideal  $I_0 = \ker(\pi_0) \in \text{Prim}(A) \setminus \cup_{\phi \in \mathcal{F}} \text{supp}(\phi)$ . Let

$$V_0 \supset \dots \supset V_n \supset V_{n+1} \dots \supset \{I_0\} = \cap_k V_k$$

be a basis for the system of neighborhoods of  $I_0$  in  $\text{Prim}(A)$ . We may assume without loss of generality that the neighbourhoods  $V_n$  consist of open sets. Therefore  $V_n = \text{Prim}(J_n)$  for some closed two-sided ideals  $J_n \subset A$ . Corollary 4.2 then yields  $a \in A$  such that  $\|a\| = \|\pi_0(a)\| = 1$ , but  $\|\pi(a)\| \leq 1 - 2^{-k}$  for any irreducible representation  $\pi$  such that  $\ker(\pi) \in \text{Prim}(A) \setminus V_k$ . Then, for every  $\phi \in \mathcal{F}$ , we have that  $\text{Prim}(A) \setminus \text{supp}(\phi)$  is a neighborhood of  $I_0$  in  $\text{Prim}(A)$ . Therefore there exists  $n$  such that  $V_n \subset \text{Prim}(A) \setminus \text{supp}(\phi)$  and hence  $\|\pi(a)\| \leq 1 - 2^{-n}$  for all  $\pi$  such that  $\ker(\pi) \in \text{supp}(\phi)$ . This gives  $\|\phi(a)\| \leq 1 - 2^{-n} < 1$ , thus contradicting the fact that  $\mathcal{F}$  is strictly norming. This proves the first half of the statement.

Let us prove the converse, that is, the second half of the statement, which is easier. Thus let us assume that every strictly norming family of representations of  $A$  is also exhausting and let us prove that every primitive ideal  $I_0 := \ker(\pi_0) \in \text{Prim}(A)$  has a countable basis for its system of neighbourhoods. Let us fix then  $I_0 := \ker(\pi_0) \in \text{Prim}(A)$  arbitrarily and show that it has a countable basis for its system of neighbourhoods. Also, we associate to each primitive ideal  $I \in \text{Prim}(A)$  an irreducible representation  $\phi_I$  with kernel  $I$ . By definition of an exhausting family, we have that the family of representations  $\mathcal{F} := \{\phi_I, I \in \text{Prim}(A), I \neq I_0\}$  is not exhausting. By our assumption, it is also not strictly norming. Hence by the definition of a strictly norming family, there exists  $a \in A$ , such that  $\|\pi(a)\| < \|a\|$  for all irreducible  $\pi$  with  $\ker(\pi) \neq I_0$ . Note that since the family  $\widehat{A}$  is strictly norming (see Example 3.2), we have that  $\|a\| = \max_{\pi \in \widehat{A}} \|\pi(a)\|$ , and hence  $\|a\| = \|\pi_0(a)\|$ . By rescaling, we can assume  $\|a\| = \|\pi_0(a)\| = 1$ . Then the sets

$$V_n := \{ \ker(\pi) \in \text{Prim}(A), \|\pi(a)\| > 1 - 2^{-n} \}$$

are open neighbourhoods of  $I_0 := \ker(\pi_0)$  in  $\text{Prim}(A)$  by Lemma 1.2. Let us show that they form a basis for the system of neighborhoods of  $I_0$ . Indeed, let  $G$  be an arbitrary open subset of  $\text{Prim}(A)$  containing  $I_0$ . Then there exists a two-sided ideal  $J \subset A$  such that  $G = \text{Prim}(J)$ . The set of irreducible representations of  $A/J$  identifies with  $\text{Prim}(J)^c := \text{Prim}(A) \setminus \text{Prim}(J)$ , and hence it does not contain  $\pi_0$ . Hence  $\|\pi(a)\| < 1$  for all  $\pi \in \text{Prim}(A/J)$ . Since  $\widehat{A/J}$  is a strictly norming family of representations of  $A/J$ , we obtain that  $\|a + J\|_{A/J} < 1$  (the norm is in  $A/J$ ). Let

$n$  be such that  $\|a + J\|_{A/J} \leq 1 - 2^{-n}$ . Then  $V_n \subset \text{Prim}(J) = G$ , which completes the proof of the second half of this theorem. The proof is now complete.  $\square$

It is easy to show that separable  $C^*$ -algebras satisfy the assumptions of the previous proposition.

**Proposition 4.4.** *Let  $A$  be a separable  $C^*$ -algebra. Then every primitive ideal  $I \in \text{Prim}(A)$  has a countable base for its system of neighborhoods. Consequently, if  $\mathcal{F}$  is a strictly norming set of representations of  $A$ , then  $\mathcal{F}$  is exhausting.*

*Proof.* Let  $\{a_n\}$  be a dense subset of  $A$  and fix  $I_0 := \ker(\pi_0) \in \text{Prim}(A)$ . Define

$$V_n := \{ \ker(\pi) \in \text{Prim}(A), \|\pi(a_n)\| > \|\pi_0(a_n)\|/2 \}.$$

Then each  $V_n$  is open by Lemma 1.2. We claim that  $V_n$  is a basis of the system of neighborhoods of  $I_0 := \ker(\pi_0)$  in  $\text{Prim}(A)$ . Indeed, let  $G \subset \text{Prim}(A)$  be an open set containing  $I_0$ . Then  $G = \text{Prim}(J)$  for some two-sided ideal of  $A$  and  $\pi_0 \neq 0$  on  $J$ . Let  $a \in J$  such that  $\pi_0(a) \neq 0$ . By the density of the sequence  $a_n$  in  $A$ , we can find  $n$  such that  $\|a - a_n\| < \|\pi_0(a)\|/4$ . Then  $\|\pi'(a) - \pi'(a_n)\| < \|\pi_0(a)\|/4$  for any irreducible representation  $\pi'$ , and hence

$$(11) \quad \|\pi'(a)\| - \|\pi_0(a)\|/4 < \|\pi'(a_n)\| < \|\pi'(a)\| + \|\pi_0(a)\|/4, \quad \forall \pi' \in \hat{A}.$$

To show that  $V_n \subset G$ , it is enough to show that  $V_n \cap G^c = V_n \cap \text{Prim}(J)^c = \emptyset$ . Suppose the contrary and let  $\pi \in \hat{A}$  be such that  $\ker(\pi) \in V_n \cap \text{Prim}(J)^c$ . Then  $\|\pi(a_n)\| > \|\pi_0(a_n)\|/2$ , by the definition of  $V_n$ . Moreover,  $\pi(a) = 0$  since  $a \in J$  and  $\pi$  vanishes on  $J$ . Let us show that this is not possible. Indeed, using Equation (11) twice, for  $\pi' = \pi_0$  and for  $\pi' = \pi$ , we obtain

$$\frac{3}{8}\|\pi_0(a)\| < \frac{1}{2}\|\pi_0(a_n)\| < \|\pi(a_n)\| < \frac{1}{4}\|\pi_0(a)\|,$$

which is contradiction. Consequently  $V_n \subset G$  and hence  $\{V_n\}$  is a basis for the system of neighborhoods of  $\pi_0$  in  $\text{Prim}(A)$ , as claimed.  $\square$

The next two basic examples illustrate the differences between the notions of faithful and strictly norming families.

*Example 4.5.* Let in this example  $A$  be the  $C^*$ -algebra of continuous functions  $f$  on  $[0, 1]$  with values in  $M_2(\mathbb{C})$  such that  $f(1)$  is diagonal, which is a type I  $C^*$ -algebra, and thus we identify  $\hat{A}$  and  $\text{Prim}(A)$ . Then the maps  $\text{ev}_t: f \mapsto f(t) \in M_2(\mathbb{C})$ , for  $t < 1$ , together with the maps  $\text{ev}_1^i: f \mapsto f(1)_{ii}$  ( $i = 0, 1$ ) provide all the irreducible representations of  $A$  (up to equivalence). The family

$$\mathcal{F} = \{ \text{ev}_t, t < 1 \} \cup \{ \text{ev}_1^1 \}$$

is a faithful but not exhausting family. In fact the function  $t \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1-t \end{pmatrix}$  is not invertible in  $A$  but  $\pi(f)$  is invertible for all  $\pi \in \mathcal{F}$ . Of course, in this example, every  $\pi \in \hat{A} = \text{Prim}(A)$  has a countable base for its system of neighborhoods, so every strictly norming family of representations  $\mathcal{F}$  of  $A$  is also exhausting.

The next example is closely related to the examples we will be dealing with below.

*Example 4.6.* Let  $\mathcal{T}$  be the Toeplitz algebra, which is again a type I  $C^*$ -algebra, and thus we again identify  $\widehat{\mathcal{T}}$  and  $\text{Prim}(\mathcal{T})$ . The Toeplitz algebra  $\mathcal{T}$  is defined as the  $C^*$ -algebra generated by the operator defined by the unilateral shift  $S$ . (Recall that  $S$  acts on the Hilbert space  $L^2(\mathbb{N})$  by  $S: \epsilon_k \mapsto \epsilon_{k+1}$ .) As  $S^*S = 1$  and  $SS^* - 1$  is a rank 1 operator, one can prove that the following is an exact sequence

$$0 \rightarrow \mathcal{K} \rightarrow \mathcal{T} \rightarrow \mathcal{C}(S^1) \rightarrow 0,$$

where  $\mathcal{K}$  is the algebra of compact operator. Extend the unique irreducible representation  $\pi$  of  $\mathcal{K}$  to  $\mathcal{T}$  as in [12]. Also, the irreducible characters  $\chi_\theta$  of  $S^1$  pull-back to irreducible characters of  $\mathcal{T}$  vanishing on  $\mathcal{K}$ . Then the spectrum of  $\mathcal{T}$  is

$$\widehat{\mathcal{T}} = \{\pi\} \cup \{\chi_\theta, \theta \in S^1\},$$

with  $S^1$  embedded as a closed subset. A subset  $V \subset \text{Prim}(\mathcal{T})$  will be open if, and only if, it contains  $\pi$  and its intersection with  $S^1$  is open. We thus see that the single element set  $\{\pi\}$  defines an exhausting family. In other words  $\widehat{\mathcal{T}} = \overline{\{\pi\}} = \text{supp}(\pi)$ . Since every exhausting family is also strictly norming, by Proposition 3.4, the family  $\{\pi\}$  consisting of a single representation is also strictly norming. We can see also directly that the family  $\mathcal{F} = \{\pi\}$  (consisting of  $\pi$  alone) is strictly norming. Indeed, it suffices to notice that  $\|x\| = \|\pi(x)\|$  for all  $x$  since  $\pi$  is injective. In this example again every  $\pi' \in \widehat{\mathcal{T}} = \text{Prim}(\mathcal{T})$  has a countable base for its system of neighborhoods, so every strictly norming family of representations  $\mathcal{F}$  of  $\mathcal{T}$  is also exhausting.

Here are two more examples that show that the condition that  $A$  be separable is not necessary for the classes of exhausting families of representations and strictly norming families of representations to coincide.

*Example 4.7.* Let  $I$  be an infinite *uncountable* set. We endow it with the discrete topology. Then  $A_0 := \mathcal{C}_0(I)$  and  $A_1 := \mathcal{K}(\ell^2(I))$  (the algebra of compact operators on  $\ell^2(I)$ ) are not separable, however, if  $\mathcal{F}$  is a strictly norming family of representations of  $A_i$ ,  $i = 0, 1$ , then  $\mathcal{F}$  is also an exhausting family of representations of  $A_i$ .

## 5. UNBOUNDED OPERATORS

The results of the previous sections are relevant often in applications to unbounded operators, so we now extend Theorem 3.7 to (possibly) unbounded operators affiliated to  $C^*$ -algebras. We begin with an abstract setting.

**5.1. Abstract affiliated operators.** The notion of affiliated self-adjoint operator has been extensively and successfully studied, see [5, 9, 15, 34, 35] for example. In the sequel we will closely follow the definitions in [15], beginning with an abstract version of this notion. See [18, 28] for results on unbounded operators on Hilbert modules [7, 19, 24].

**Definition 5.1.** Let  $A$  be a  $C^*$ -algebra. An *observable*  $T$  affiliated to  $A$  is a morphism  $\theta_T : \mathcal{C}_0(\mathbb{R}) \rightarrow A$  of  $C^*$ -algebras. The observable  $T$  is said to be *strictly affiliated* to  $A$  if the space generated by elements of the form  $\theta_T(h)a$  ( $a \in A$ ,  $h \in \mathcal{C}_0(\mathbb{R})$ ), is dense in  $A$ .

As in the classical case, we now introduce the Cayley transform. To this end, let us notice that an observable affiliated to  $A$  extends to a morphism  $\theta_T^+ : \mathcal{C}_0(\mathbb{R})^+ \rightarrow A^+$  (the algebra obtained from  $A$  by adjunction of a unit). If moreover  $T$  is strictly

affiliated to  $A$ , then  $\theta_T$  extends to a morphism from  $C_b(\mathbb{R})$  to the multiplier algebra of  $A$ , but we shall not need this fact.

**Definition 5.2.** Let  $T$  be an observable affiliated to  $A$ . The *Cayley transform*  $u_T \in A^+$  of  $T$  is

$$(12) \quad u_T := \theta_T^+(h_0), \quad h_0(z) := (z + \iota)(z - \iota)^{-1}.$$

The Cayley transform allows us to reduce questions about the spectrum of an observable to questions about the spectrum of its Cayley transform. Let us first introduce, however, the *spectrum of an affiliated observable*. Let thus  $\theta_T : \mathcal{C}_0(\mathbb{R}) \rightarrow A$  be a self-adjoint operator affiliated to a  $C^*$ -algebra  $A$ . The kernel of  $\theta_T$  is then of the form  $\mathcal{C}_0(U)$ , for some open subset of  $\mathbb{R}$ . We define the spectrum  $\text{Spec}_A(T)$  as the complement of  $U$  in  $\mathbb{R}$ . Explicitly,

$$(13) \quad \text{Spec}_A(T) = \{\lambda \in \mathbb{R}, h(\lambda) = 0, \forall h \in \mathcal{C}_0(\mathbb{R}) \text{ such that } \theta_T(h) = 0\}.$$

We allow the case  $\text{Spec}_A(T) = \emptyset$ , which corresponds to the case  $T = \infty$  or  $u_T = 1$ . If  $\sigma : A \rightarrow B$  is a morphism of  $C^*$ -algebras, then  $\sigma \circ \theta_T : \mathcal{C}_0(\mathbb{R}) \rightarrow B$  is an observable  $\sigma(T)$  affiliated to the  $C^*$ -algebra  $B$  and

$$\text{Spec}_B(\sigma(T)) \subset \text{Spec}_A(T).$$

If  $\sigma$  is injective, then  $\text{Spec}_B(\sigma(T)) = \text{Spec}_A(T)$ , which shows that the spectrum is preserved by increasing the  $C^*$ -algebra  $A$ . Note that

$$(14) \quad \sigma(u_T) = u_{\sigma(T)}.$$

By classical results, if  $(u_T - 1)$  is injective, then we can define a true self-adjoint operator  $T := \iota(u_T + 1)(u_T - 1)^{-1} \in A$  such that  $\theta_T(h) = h(T)$ ,  $h \in \mathcal{C}_0(\mathbb{R})$  [11]. This is the case, for instance, if  $\text{Spec}(T)$  is a bounded subset of  $\mathbb{R}$ , in which case we shall say that  $T$  is *bounded*. In any case, bounded or unbounded, our definition of  $\text{Spec}(T)$  in terms of  $\theta_T$  coincides with the classical spectrum of  $T$  defined using the resolvent. Let  $h_0(z) := (z + \iota)(z - \iota)^{-1}$ , as before.

**Lemma 5.3.** *The spectrum  $\text{Spec}(T)$  of the an observable  $\theta_T : \mathcal{C}_0(\mathbb{R}) \rightarrow A$  affiliated to the  $C^*$ -algebra  $A$  and the spectrum  $\text{Spec}(u_T)$  of its Cayley transform are related by*

$$\text{Spec}(T) = h_0^{-1}(\text{Spec}(u_T)).$$

*Proof.* This follows from the fact that  $h_0$  is a homeomorphism of  $\mathbb{R}$  onto its image in  $S^1 := \{|z| = 1\}$  and from the properties of the functional calculus.  $\square$

Let us notice that the above lemma is valid also in the case when

$$T = \infty \Leftrightarrow \theta_T = 0 \Leftrightarrow \text{Spec}(T) = \emptyset \Leftrightarrow u_T = 1 \Leftrightarrow \sigma(u_T) = \{1\}.$$

One can make the relation in the above lemma more precise by saying that for bounded  $T$  we have  $h_0(\text{Spec}(T)) = \text{Spec}(u_T)$  whereas for unbounded  $T$  we have

$$(15) \quad \overline{h_0(\text{Spec}(T))} = h_0(\text{Spec}(T)) \cup \{1\} = \text{Spec}(u_T),$$

where  $h_0(z) := (z + \iota)(z - \iota)^{-1}$ , as before.

Here is our main result on (possibly unbounded) self-adjoint operators affiliated to  $C^*$ -algebras.

**Theorem 5.4.** *Let  $A$  be a unital  $C^*$ -algebra and  $T$  an observable affiliated to  $A$ . Let  $\mathcal{F}$  be a set of representations of  $A$ .*



(1) If  $\mathcal{F}$  is strictly norming, then

$$\text{Spec}(T) = \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(T)).$$

(2) If  $\mathcal{F}$  is faithful, then

$$\text{Spec}(T) = \overline{\cup}_{\phi \in \mathcal{F}} \text{Spec}(\phi(T)).$$

*Proof.* The proofs of (i) and (ii) are similar, starting with the relation  $\text{Spec}(T) = h_0^{-1}(\text{Spec}(u_T))$  of Lemma 5.3. We begin with (i), which is slightly easier. Since  $\mathcal{F}$  is strictly norming, we can then apply theorem 3.9 to  $u_T \in A^+$  and the family  $\sigma \in \mathcal{F}$ . We obtain

$$\begin{aligned} \text{Spec}(T) &= h_0^{-1}[\text{Spec}(u_T)] = h_0^{-1}[\cup_{\sigma \in \mathcal{F}} \text{Spec}(\sigma(u_T))] \\ &= h_0^{-1}[\cup_{\sigma \in \mathcal{F}} \text{Spec}(u_{\sigma(T)})] = \cup_{\sigma \in \mathcal{F}} h_0^{-1}[\text{Spec}(u_{\sigma(T)})] = \cup_{\sigma \in \mathcal{F}} \text{Spec}(\sigma(T)). \end{aligned}$$

If, on the other hand,  $\mathcal{F}$  is faithful, we apply proposition 2.6 after noting that  $h_0$  is a homeomorphism of  $\mathbb{R}$  onto its image in  $S^1 := \{|z| = 1\}$  and hence  $h_0^{-1}(\overline{S}) = \overline{h_0^{-1}(S)}$  for any  $S \subset S^1$ . The same argument then gives

$$\begin{aligned} \text{Spec}(T) &= h_0^{-1}[\text{Spec}(u_T)] = h_0^{-1}[\overline{\cup}_{\sigma \in \mathcal{F}} \text{Spec}(\sigma(u_T))] \\ &= h_0^{-1}[\overline{\cup}_{\sigma \in \mathcal{F}} \text{Spec}(u_{\sigma(T)})] = \overline{\cup}_{\sigma \in \mathcal{F}} h_0^{-1}[\text{Spec}(u_{\sigma(T)})] = \overline{\cup}_{\sigma \in \mathcal{F}} \text{Spec}(\sigma(T)). \end{aligned}$$

The proof is now complete.  $\square$

*Remark 5.5.* In view of the remarks preceding it, Theorem 5.4 remains valid for true self-adjoint operators  $T$ .

**5.2. The case of true operators.** We now look at concrete (true) operators.

**Definition 5.6.** Let  $A \subset \mathcal{L}(\mathcal{H})$  be a sub- $C^*$ -algebra of  $\mathcal{L}(\mathcal{H})$ . A (possibly unbounded) self-adjoint operator  $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$  is said to be *affiliated* to  $A$  if, for every continuous functions  $h$  on the spectrum of  $T$  vanishing at infinity, we have  $h(T) \in A$ .

*Remark 5.7.* We have that  $T$  is affiliated to  $A$  if, and only if,  $(T - \lambda)^{-1} \in A$  for one  $\lambda \notin \text{Spec}(T)$  (equivalently for all such  $\lambda$ ) [9]. We thus see that a self-adjoint operator  $T$  affiliated to  $A$  defines a morphism  $\theta_T : \mathcal{C}_0(\mathbb{R}) \rightarrow A$ ,  $\theta_T(h) := h(T)$  such that  $\text{Spec}(T) = \text{Spec}(\theta_T)$ . Thus  $T$  defines an observable affiliated to  $A$ .

When  $A \subset \mathcal{L}(\mathcal{H})$  is non degenerate, the correspondence between self-adjoint operators affiliated to  $A$  and observables affiliated to  $A$  given by  $T \mapsto \theta_T$  is actually bijective. This can be seen by using the unbounded functional calculus of normal operators.

Since in our paper we shall consider only the case when  $A \subset \mathcal{L}(\mathcal{H})$  is non degenerate, we shall not make a difference between operators and observables affiliated to  $A$ .

Recall that an unbounded operator  $T$  is invertible if, and only if, it is bijective and  $T^{-1}$  is bounded. This is also equivalent to  $0 \notin \text{Spec}(\theta_T)$ . We have the following analog of Proposition 2.3 and Theorem 3.7

**Theorem 5.8.** *Let  $A \subset \mathcal{L}(\mathcal{H})$  be a unital  $C^*$ -algebra and  $T$  a self-adjoint operator affiliated to  $A$ . Let  $\mathcal{F}$  be a set of representations of  $A$ .*

(1) *Let  $\mathcal{F}$  be strictly norming. Then  $T$  is invertible if, and only if  $\phi(T)$  is invertible for all  $\phi \in \mathcal{F}$ .*

(2) Let  $\mathcal{F}$  be faithful. Then  $T$  is invertible if, and only if  $\phi(T)$  is invertible for all  $\phi \in \mathcal{F}$  and the set  $\{\|\phi(T)^{-1}\|, \phi \in \mathcal{F}\}$  is bounded.

*Proof.* This follows from Theorem 5.4 as follows. First of all, we have that  $T$  is invertible if, and only if,  $0 \notin \text{Spec}(T)$ . Now, if  $\mathcal{F}$  is strictly norming, we have

$$0 \notin \text{Spec}(T) \Leftrightarrow 0 \notin \cup_{\phi \in \mathcal{F}} \text{Spec}(\phi(T)) \Leftrightarrow 0 \notin \text{Spec}(\phi(T)) \text{ for all } \phi \in \mathcal{F}.$$

This proves (i). To prove (ii), we proceed similarly, noticing also that the distance from 0 to  $\text{Spec}(T)$  is exactly  $\|T^{-1}\|$ .  $\square$

We have already remarked (Remark 5.5) that Theorem 5.4 extends to the framework of this subsection, that is, that of (possibly unbounded) self-adjoint operators on a Hilbert space.

## 6. PARAMETRIC PSEUDODIFFERENTIAL OPERATORS

Let  $M$  be a compact smooth Riemannian manifold and  $G$  be a Lie group (finite dimensional) with Lie algebra  $\mathfrak{g} := \text{Lie}(G)$ . We let  $G$  act by left translations on  $M \times G$ . We denote by  $\Psi^0(M \times G)^G$  the algebra of order 0,  $G$ -invariant pseudodifferential operators on  $M \times G$  and  $\overline{\Psi^0(M \times G)^G}$  be its norm closure acting on  $L^2(M \times G)$ . For any vector bundle  $E$ , we denote by  $S^*E$  the set of directions in its dual  $E^*$ . If  $E$  is endowed with a metric, then  $S^*E$  can be identified with the set of unit vectors in  $E^*$ . We shall be interested the the quotient

$$S^*(T(M \times G))/G = S^*(TM \times TG)/G = S^*(TM \times \mathfrak{g}).$$

We have that  $\overline{\Psi^0(M \times G)^G} \simeq C_r^*(G) \otimes \mathcal{K}$  and then obtain the exact sequence

$$(16) \quad 0 \rightarrow C_r^*(G) \otimes \mathcal{K} \rightarrow \overline{\Psi^0(M \times G)^G} \rightarrow \mathcal{C}(S^*(M \times \mathfrak{g})) \rightarrow 0,$$

[21, 22, 25, 34]. Note that the kernel of the symbol map will now have irreducible representations parametrised by  $\hat{G}_r$  the temperate unitary irreducible representations of  $G$ . Let  $T \in \Psi^m(M \times G)^G$  and denote by  $T^\sharp \in \Psi^m(M \times G)^G$  its formal adjoint (defined using the calculus of pseudodifferential operators). All operators considered below are closed with minimal domain (the closure of the operators defined on  $\mathcal{C}_c^\infty(M \times G)$ ). We denote by  $T^*$  the Hilbert space adjoint of a (possibly unbounded) densely defined operator.

**Lemma 6.1.** *Let  $T \in \Psi^m(M \times G)^G$  be elliptic. Then  $T^* = T^\sharp$ . Thus, if also  $T = T^\sharp$ , then  $T$  is self-adjoint and  $(T + \iota)^{-1} \in C_r^*(G)$ , and hence it is affiliated to  $C_r^*(G)$ .*

*Proof.* This is a consequence of the fact that  $\Psi^\infty(M \times G)^G$  is closed under multiplication and formal adjoints. See [21, 22, 34] for details.  $\square$

In other words, any elliptic, formally self-adjoint  $T \in \Psi^m(M \times G)^G$ ,  $m > 0$ , is actually self-adjoint.

Let us assume  $G = \mathbb{R}^n$ , regarded as an abelian Lie group. Then our exact sequence (16) becomes

$$(17) \quad 0 \rightarrow \mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{K} \rightarrow \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}} \rightarrow \mathcal{C}(S^*(TM \times \mathbb{R}^n)) \rightarrow 0.$$

This shows that  $A := \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$  is a type I  $C^*$ -algebra, and hence we can identify  $\hat{A}$  and  $\text{Prim}(A)$ . Then we use that, to each  $\lambda \in \mathbb{R}^n$ , there corresponds an irreducible representation  $\phi_\lambda$  of  $\mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{K}$ . Recalling that every irreducible

(bounded,  $*$ ) representation of an ideal  $I$  in a  $C^*$ -algebra  $A$  extends uniquely to a representation of  $A$ , we obtain that  $\phi_\lambda$  extends uniquely to an irreducible representation of  $\overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$  denoted with the same letter. It is customary to denote by  $\hat{T}(\lambda) := \phi_\lambda(T)$  for  $T$  a pseudodifferential operator in  $\Psi^m(M \times \mathbb{R}^n)^{\mathbb{R}^n}$ ,  $m \geq 0$ . To define  $\hat{T}(\lambda)$  for  $m > 0$ , we can either use the Fourier transform or, notice that  $\Delta$  is affiliated to the closure of  $\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}$ . This allows us to define  $\hat{\Delta}(\lambda)$ . In general, we write  $T = (1 - \Delta)^k S$ , with  $S \in \Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}$  and define  $\hat{T}(\lambda)q = \widehat{(1 - \Delta)}(\lambda)^k \hat{S}(\lambda)$ . (We consider the ‘‘analyst’s’’ Laplacian, so  $\Delta \leq 0$ .)

**Lemma 6.2.** *Let  $A := \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$ . Then the primitive ideal spectrum of  $A$ ,  $\text{Prim}(A)$ , is in a canonical bijection with the disjoint union  $\mathbb{R}^n \cup S^*(TM \times \mathbb{R}^n)$ , where the copy of  $\mathbb{R}^n$  corresponds to the open subset  $\{\phi_\lambda, \lambda \in \mathbb{R}^n\}$  and the copy of  $S^*(TM \times \mathbb{R}^n)$  corresponds to the closed subset  $\{e_p, p \in S^*(TM \times \mathbb{R}^n)\}$ . The induced topologies on  $\mathbb{R}^n$  and  $S^*(TM \times \mathbb{R}^n)$  are the standard ones. Let  $S^*M := S^*(TM) \subset S^*(TM \times \mathbb{R}^n)$  correspond to  $T^*M \subset T^*M \times \mathbb{R}^n$ . Then the closure of  $\{\phi_\lambda\}$  in  $\text{Prim}(A)$  is  $\{\phi_\lambda\} \cup S^*M$ .*

*Proof.* By standard properties of  $C^*$ -algebras (the definition of the Jacobson topology), the ideal  $\mathcal{C}_0(\mathbb{R}^n) \otimes \mathcal{K} \subset A$  defines an open subset of  $\text{Prim}(A)$  with complement  $\text{Prim}(A/I)$  with the induced topologies. This proves the first part of the statement.

In order to determine the closure of  $\{\phi_\lambda\}$  in  $\text{Prim}(A)$ , let us notice that the principal symbol of  $\hat{T}(\lambda)$  can be calculated in local coordinate carts on  $M$  (more precisely, on sets of the form  $U \times \mathbb{R}^n$ , with  $U$  a coordinate chart in  $M$ ). This gives that the principal symbol of  $\hat{T}(\lambda)$  is given by the restriction of the principal symbol of  $T$  to  $S^*M$ .

Indeed, let  $U = \mathbb{R}^k$ . A translation invariant pseudodifferential  $P$  operator on  $U \times \mathbb{R}^n = \mathbb{R}^{k+n}$  is of the form  $P = a(x, y, D_x, D_y)$  with  $a$  independent of  $y$ :  $a(x, y, \xi, \eta) = \tilde{a}(x, \xi, \eta)$ . With this notation, we have  $\hat{P}(\lambda) = \tilde{a}(x, D_x, \lambda)$ . The principal symbol of  $\hat{P}(\lambda)$  is then the principal symbols of the (global) symbol  $\mathbb{R}^k \ni (x, \xi) \rightarrow \tilde{a}(x, \xi, \lambda)$ , and is seen to be independent of the (finite) value of  $\lambda \in \mathbb{R}^n$  and is the restriction from  $S^*(TU \times \mathbb{R}^n)$  to  $S^*(TU \times \{0\})$  of the principal symbol of  $\tilde{a}$ .

Comming back to the general case, the same reasoning gives that the image of  $\phi_\lambda$  is  $\overline{\Psi^0(M)}$ . The primitive ideal spectrum of this algebra is canonically homeomorphic to the closure of  $\{\phi_\lambda\}$ , and this is enough to complete the proof.  $\square$

By the exact sequence (16), in addition to the irreducible representations  $\phi_\lambda$ ,  $\lambda \in \mathbb{R}^n$  (or, more precisely, their kernels),  $\text{Prim}(A)$  contains also (the kernels of) the irreducible representations  $e_p(T) = \sigma_0(T)(p)$ ,  $p \in S^*(TM \times \mathbb{R}^n)$ .

**Proposition 6.3.** *Let  $\mathcal{F} := \{\phi_\lambda, \lambda \in \mathbb{R}^n\} \cup \{e_p, p \in S^*(TM \times \mathbb{R}^n) \setminus S^*M\}$ .*

- (i) *The family  $\mathcal{F}$  is a strictly norming family of representations of  $\overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}$ .*
- (ii) *Let  $P \in \Psi^m(M \times \mathbb{R}^n)^{\mathbb{R}^n}$ , then  $P : H^s(M \times \mathbb{R}^n) \rightarrow H^{s-m}(M \times \mathbb{R}^n)$  is invertible if, and only if  $\hat{P}(\lambda) : H^s(M) \rightarrow H^{s-m}(M)$  is invertible for all  $\lambda \in \mathbb{R}^n$  and the principal symbol of  $P$  is non-zero on all rays not intersecting  $S^*M$ .*
- (iii) *If  $T \in \Psi^m(M \times \mathbb{R}^n)^{\mathbb{R}^n}$ ,  $m > 0$ , is formally self-adjoint and elliptic, then  $\text{Spec}(e_p(T)) = \emptyset$ , and hence*

$$\text{Spec}(T) = \cup_{\lambda \in \mathbb{R}^n} \text{Spec}(\hat{T}(\lambda)).$$

*Proof.* (i) follows from Lemma 6.2. To prove (ii), let us denote by  $\Delta_M \leq 0$  the (non-positive) Laplace operator on  $M$ . Then the Laplace operator  $\Delta$  on  $M \times \mathbb{R}^n$  is  $\Delta = \Delta_{\mathbb{R}^n} + \Delta_M$ . Note that  $(1 - \Delta)^{m/2} : H^s(M \times \mathbb{R}^n) \rightarrow H^{s-m}(M \times \mathbb{R}^n)$  and  $(c - \Delta_M)^{m/2} : H^s(M) \rightarrow H^{s-m}(M)$ ,  $c > 0$ , are isomorphisms. By [22], we have that

$$P_1 := (1 - \Delta)^{(s-m)/2} P (1 - \Delta)^{-s/2} \in A := \overline{\Psi^0(M \times \mathbb{R}^n)^{\mathbb{R}^n}}.$$

It is then enough to prove that  $P_1$  is invertible on  $L^2(M \times \mathbb{R}^n)$ . Moreover from part (i) we have just proved and Theorem 3.7 we know that  $P_1$  is invertible on  $L^2(M \times \mathbb{R}^n)$  if, and only if,  $\hat{P}_1(\lambda) := \phi_\lambda(P_1)$  is invertible on  $L^2(M)$  for all  $\lambda \in \mathbb{R}^n$  and the principal symbol of  $P_1$  is non-zero on all rays not intersecting  $S^*M$ . But, using also  $\widehat{1 - \Delta}(\lambda) = (1 + |\lambda|^2 - \Delta_M)$ , we have

$$\hat{P}_1(\lambda) = (1 + |\lambda|^2 - \Delta_M)^{(s-m)/2} \hat{P}(\lambda) (1 + |\lambda|^2 - \Delta_M)^{-s/2},$$

which is invertible by assumption.

To prove (iii), we recall that  $T$  is affiliated to  $A$ , by Lemma 6.1. The result then follows from Theorem 5.4(1) (See also Remark 5.5).  $\square$

Operators of the kind considered in this subsection were used also in [1, 8, 10, 23, 26, 32, 33]. They turn out to be useful also for general topological index theorems [13, 27]. A more class of operators than the ones considered in this subsection were introduced in [3, 4]. The above result has turned out to be useful for the study of layer potentials [29].

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