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Finite difference calculus for alternating permutations

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► **To cite this version:**

Dominique Foata, Guo-Niu Han. Finite difference calculus for alternating permutations. Journal of Difference Equations and Applications, 2013, 19, 10.1080/10236198.2013.794794 . hal-01283923

HAL Id: hal-01283923

<https://hal.science/hal-01283923>

Submitted on 7 Mar 2016

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are taken as initial values. To see this we first note that the equation $f_2(1) - 2f_2(2) + f_2(3) + 4f_1(1) = 0$ determines $f_2(2)$ uniquely. Assuming that the triangle $(f_{n'}(m))$ ($1 \leq m \leq 2n' - 1$, $n' \leq n$) has been determined, the system $\Delta^2 f_{n+1}(m) + 4f_n(m) = 0$ ($1 \leq m \leq 2n - 1$) consists of $(2n - 1)$ linear equations with $(2n - 1)$ unknowns, namely, $f_{n+1}(2), f_{n+1}(3), \dots, f_{n+1}(2n)$, the underlying matrix being trigonal of the form

$$F_{n+1} := \begin{pmatrix} -2 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -2 \end{pmatrix}.$$

As $\det F_{n+1} = -2n$ ($n \geq 1$), the system has a unique solution.

The purpose of this paper is to solve (1.2) in four cases, when the sets of initial values called [tan1], [tan2], [sec1], [sec2] are the following:

[tan1] $f_1(1) = 1$; $f_n(1) = 0$ and $f_n(2) = 2 \sum_k f_{n-1}(k)$ for $n \geq 2$;

[tan2] $f_1(1) = 1$; $f_n(1) = f_n(2n - 1) = 0$ for $n \geq 2$;

[sec1] $f_1(1) = 1$; $f_n(1) = \sum_k f_{n-1}(k)$ and $f_n(2) = 3 \sum_k f_{n-1}(k)$ for $n \geq 2$;

[sec2] $f_1(1) = 1$; $f_n(1) = f_n(2n - 1) = \sum_k f_{n-1}(k)$ for $n \geq 2$;

It will be proved (see Theorem 1.5) that both initial values [tan1] and [tan2] (resp. [sec1] and [sec2]) in fact lead to the same solution of the system and, furthermore, that the solutions found for the $f_n(k)$'s are non-negative *integral values*. To avoid any confusion the solutions of (1.2) will be denoted by $(g_n(k))$ (resp. $(h_n(k))$) when using [tan1] (resp. [sec1]). The first numerical values of those solutions are displayed in Fig. 1.1.

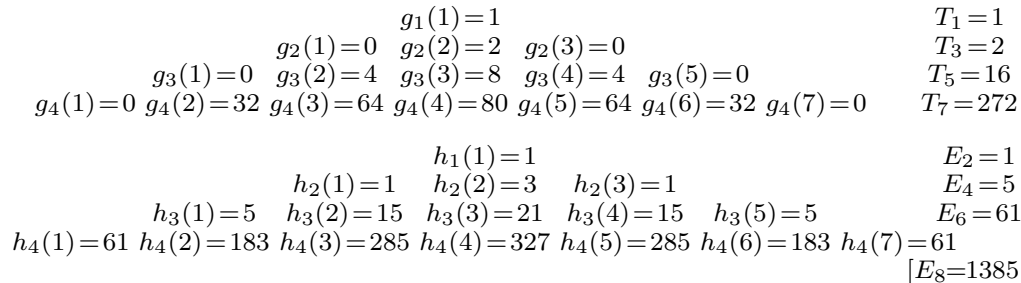


Fig. 1.1. The two triangles of the $g_n(k)$'s and $h_n(k)$'s.

To the right of each triangle have been calculated the row sums, which are equal, as stated in the next theorem, to the *tangent numbers* (resp. the *secant numbers*). Those classical numbers, denoted by T_{2n+1} and E_{2n} ,

appear in the Taylor expansions of $\tan u$ and $\sec u$:

$$(1.5) \quad \begin{aligned} \tan u &= \sum_{n \geq 0} \frac{u^{2n+1}}{(2n+1)!} T_{2n+1} \\ &= \frac{u}{1!} 1 + \frac{u^3}{3!} 2 + \frac{u^5}{5!} 16 + \frac{u^7}{7!} 272 + \frac{u^9}{9!} 7936 + \dots \end{aligned}$$

$$(1.6) \quad \begin{aligned} \sec u &= \frac{1}{\cos u} = \sum_{n \geq 0} \frac{u^{2n}}{(2n)!} E_{2n} \\ &= 1 + \frac{u^2}{2!} 1 + \frac{u^4}{4!} 5 + \frac{u^6}{6!} 61 + \frac{u^8}{8!} 1385 + \frac{u^{10}}{10!} 50521 + \dots \end{aligned}$$

(see, e.g., [Ni23, p. 177-178], [Co74, p. 258-259]).

Theorem 1.1. *Let $(g_n(k))$ (resp. $(h_n(k))$) be the unique solution of the finite difference equation system (1.2) when using the initial values [tan1] (resp. [sec1]). Then, the row sums of the solutions are equal to*

$$(1.7) \quad \sum_k g_n(k) = T_{2n-1} \quad (n \geq 1);$$

$$(1.8) \quad \sum_k h_n(k) = E_{2n} \quad (n \geq 1).$$

As further mentioned, Theorem 1.1 will appear as a consequence of Theorem 1.4. It will also be shown that the generating functions for the coefficients $g_n(k)$ and $h_n(k)$ can be evaluated in the following forms.

Theorem 1.2. *Let*

$$Z(x, y) := 1 + \sum_{n \geq 1} \sum_{1 \leq k \leq 2n+1} f_{n+1}(k) \frac{x^{2n+1-k} y^{k-1}}{(2n+1-k)! (k-1)!}$$

and $Z^{\tan}(x, y)$ (resp. $Z^{\sec}(x, y)$) when $f_n(k) := g_n(k)$ (resp. $f_n(k) := h_n(k)$). Then,

$$(1.9) \quad Z^{\tan}(x, y) = \sec(x+y) \cos(x-y);$$

$$(1.10) \quad Z^{\sec}(x, y) = \sec^2(x+y) \cos(x-y).$$

As $Z^{\tan}(y, x) = Z^{\tan}(x, y)$ and $Z^{\sec}(y, x) = Z^{\sec}(x, y)$, this implies the following Corollary.

Corollary 1.3. *The entries $g_n(k)$ and $h_n(k)$ have the symmetry property:*

$$(1.11) \quad g_n(k) = g_n(2n-k), \quad h_n(k) = h_n(2n-k) \quad (1 \leq k \leq 2n-1).$$

In view of (1.7) and (1.8), two finite sets \mathfrak{A}_{2n-1} and \mathfrak{A}_{2n} , of cardinalities T_{2n-1} and E_{2n} , are to be found, together with a statistic, call it “grn,” defined on those sets with the property that

$$(1.11) \quad \sum_{\sigma \in \mathfrak{A}_{2n-1}} x^{\text{grn } \sigma} = \sum_k g_n(k) x^k;$$

$$(1.12) \quad \sum_{\sigma \in \mathfrak{A}_{2n}} x^{\text{grn } \sigma} = \sum_k h_n(k) x^k.$$

We shall use Désiré André's old result [An1879, An1881], who introduced the notion of *alternating* permutation, as being a permutation $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ of $12\cdots n$ with the property that $\sigma(1) > \sigma(2)$, $\sigma(2) < \sigma(3)$, $\sigma(3) > \sigma(4)$, etc. in an alternating way. For each $n \geq 1$ let \mathfrak{A}_n denote the set of all alternating permutations of $12\cdots n$. He proved that $\#\mathfrak{A}_{2n-1} = T_{2n-1}$, $\#\mathfrak{A}_{2n} = E_{2n}$. The desired statistic "grn" is then the following.

Definition. Let $\sigma = \sigma(1)\sigma(2)\cdots\sigma(n)$ be an alternating permutation from \mathfrak{A}_n , so that $\sigma(i) = n$ for a certain i ($1 \leq i \leq n$). By convention, let $\sigma(0) = \sigma(n+1) := 0$. Define the **greater neighbor of n in σ** to be

$$(1.13) \quad \text{grn}(\sigma) := \max\{\sigma(i-1), \sigma(i+1)\}.$$

Also, let

$$(1.14) \quad \mathfrak{A}_{n,k} := \{\sigma \in \mathfrak{A}_n : \text{grn}(\sigma) = k\} \quad (0 \leq k \leq n-1).$$

Theorem 1.4. *Under the same assumptions as in Theorem 1.1 we have*

$$(1.15) \quad g_n(k) = \#\mathfrak{A}_{2n-1,k-1} \quad (n \geq 1, 1 \leq k \leq 2n-1);$$

$$(1.16) \quad h_n(k) = \#\mathfrak{A}_{2n,k} \quad (n \geq 1, 1 \leq k \leq 2n-1).$$

Example. There are $T_3 = 2$ alternating permutations of length 3, namely, 213 and 312, and $\text{grn}(213) = \text{grn}(312) = 1$, so that $g_2(1) = \#\mathfrak{A}_{3,0} = 0$, $g_2(2) = \#\mathfrak{A}_{3,1} = 2$, $g_2(3) = \#\mathfrak{A}_{3,2} = 0$; there are $E_4 = 5$ alternating permutations of length 4, namely, 4132, 4231, 3142, 3241, 2143, and $\text{grn}(4132) = 1$, $\text{grn}(4231) = \text{grn}(3142) = \text{grn}(3241) = 2$, $\text{grn}(2143) = 3$, so that $h_2(1) = 1$, $h_2(2) = 3$, $h_2(3) = 1$; in accordance with the numerical values in Fig. 1.1.

As $\#\mathfrak{A}_{2n-1} = T_{2n-1}$, $\#\mathfrak{A}_{2n} = E_{2n}$, following Désiré André's result, it is now clear that Theorem 1.1 is a consequence of Theorem 1.4. Thus, an analytical result is proved by combinatorial methods.

In Proposition 2.1 it will be proved that $\#\mathfrak{A}_{1,0} = 1$, $\#\mathfrak{A}_{2n-1,0} = \#\mathfrak{A}_{2n-1,2n-2} = 0$ ($n \geq 2$) and $\#\mathfrak{A}_{2n-1,1} = 2T_{2n-3}$ ($n \geq 2$); also, $\#\mathfrak{A}_{2,1} = 1$, $\#\mathfrak{A}_{2n,1} = \#\mathfrak{A}_{2n,2n-1} = E_{2n-2}$ ($n \geq 2$) and $\#\mathfrak{A}_{2n,2} = 3E_{2n-2}$ ($n \geq 2$). In view of Theorem 1.4 this implies that conditions [tan2] and [sec2] are fulfilled as soon as conditions [tan1] and [sec1] hold, and then the following theorem.

Theorem 1.5. *The entries $(g_n(k))$ and $(h_n(k))$ given by (1.15) and (1.16) are also solutions of the finite difference equation system (1.2) when the initial values [tan2] and [sec2] are used, respectively.*

There are other combinatorial models which are also counted by tangent and secant numbers, or in a one-to-one correspondence with alternating permutations, in particular, the *labeled, binary, increasing, topological trees*, also called “arbres binaires croissants complets” by Viennot [Vi88, chap. 3, p. 111]. The set of those trees having n labeled nodes is denoted by \mathfrak{T}_n . The statistic “pom” (**p**arent **o**f the **m**aximum leaf), introduced by Poupard [Po89] for her strictly ordered, binary trees can be extended to all of \mathfrak{T}_n . The usual bijection (see [Vi88]) $\gamma : \mathfrak{T}_n \rightarrow \mathfrak{A}_n$, called *projection*, has the property: $\text{pom}(t) = \text{grn}(\gamma(t))$, as proved in Theorem 5.1. We then have another combinatorial interpretation for the polynomials $\sum_k g_n(k)x^k$ and $\sum_k h_n(k)x^k$.

As such, the triangle $(g_n(k))$ ($n \geq 1, 1 \leq k \leq 2n - 1$) does not appear in Sloane’s On-Line Encyclopedia of Integer Sequences [Sl06], but the triangle $(g_n(k)/2^{n-1})$ does under reference A008301 and is called Poupard’s triangle, after her pioneering work on strictly ordered binary trees [Po89]. It is banal to verify that 2^{n-1} divides $g_n(k)$ when dealing with the combinatorial model \mathfrak{T}_n for n odd.

In contrast to Christiane Poupard [Po89], who showed that the distribution of the strictly ordered binary trees satisfied the finite difference equation system $\Delta^2 f_{2n+1}(k) + 2 f_{2n-1}(k) = 0$, we have used the multiplicative factor “4” in equation (1.2) to make a unified study of the tangent *and* secant cases and deal with objects in one-to-one correspondence with alternating permutations. *Mutatis mutandis*, identities (1.15), as well as (1.11) concerning the tangent numbers, are due to her. She obtains the generating function for her trees in the form: $\sec((x+y)/\sqrt{2}) \cos((x-y)/\sqrt{2})$ instead of (1.9). However, the alternating permutation development in Sections 2 and 3, identity (1.10) and the combinatorial properties of the entries $h_n(k)$ are new.

The triangle of the $h_n(k)$ ’s appears in Sloane’s [Sl06] as sequence A125053. It was deposited there by Paul D. Hanna. The entries have been calculated by using a procedure equivalent to (1.2) and the initial condition [tan2]. No combinatorial interpretation is given and no generating function calculated.

Theorem 1.4 will be proved in Section 3, once evaluations of some cardinalities such as $\#\mathfrak{A}_{n,k}$ will be made, as done in Section 2. The proof of Theorem 1.2 is given in Section 4, together with further identities on the $g_n(k)$ and $h_n(k)$ ’s.

2. Some special values

The evaluations of $\#\mathfrak{A}_{2n-1,k-1}$ and $\#\mathfrak{A}_{2n,k}$ made in the next proposition for some values of n and k will facilitate the derivation of the proof of Theorem 1.2. They also have their own combinatorial interests.

Proposition 2.1. *The following relations hold:*

$$(2.1) \quad \#\mathfrak{A}_{1,0} = 1, \quad \#\mathfrak{A}_{2n-1,0} = \#\mathfrak{A}_{2n-1,2n-2} = 0 \quad (n \geq 2)$$

$$(2.2) \quad \#\mathfrak{A}_{2n-1,1} = \#\mathfrak{A}_{2n-1,2n-3} = 2T_{2n-3} \quad (n \geq 2);$$

$$(2.3) \quad \#\mathfrak{A}_{2n-1,2} = \#\mathfrak{A}_{2n-1,2n-4} = 4T_{2n-3} \quad (n \geq 3);$$

$$(2.4) \quad \#\mathfrak{A}_{2n-1,3} = \#\mathfrak{A}_{2n-1,2n-5} = 6T_{2n-3} - 8T_{2n-5} \quad (n \geq 3);$$

$$(2.5) \quad \sum_{k \geq 0} \#\mathfrak{A}_{2n-1,k} = T_{2n-1} \quad (n \geq 1).$$

$$(2.6) \quad \#\mathfrak{A}_{2,1} = 1;$$

$$(2.7) \quad \#\mathfrak{A}_{2n,1} = \#\mathfrak{A}_{2n,2n-1} = E_{2n-2} \quad (n \geq 2);$$

$$(2.8) \quad \#\mathfrak{A}_{2n,2} = \#\mathfrak{A}_{2n,2n-2} = 3E_{2n-2} \quad (n \geq 2);$$

$$(2.9) \quad \#\mathfrak{A}_{2n,3} = \#\mathfrak{A}_{2n,2n-3} = 5E_{2n-2} - 4E_{2n-4} \quad (n \geq 2);$$

$$(2.10) \quad \sum_{k \geq 1} \#\mathfrak{A}_{2n,k} = E_{2n} \quad (n \geq 1).$$

Proof. (2.1) The set \mathfrak{A}_1 is the singleton 1 and $\text{grn}(1) = 0$ by definition, so that $\#\mathfrak{A}_{1,0} = 1$. For $n \geq 2$ all alternating permutations from \mathfrak{A}_{2n-1} have a “grn” at least equal to 1. Hence, $\#\mathfrak{A}_{2n-1,0} = 0$. Finally, each alternating permutation of length $(2n-1)$ ($n \geq 2$) contains neither the factor $(2n-2)(2n-1)$, nor $(2n-1)(2n-2)$. Hence, $\#\mathfrak{A}_{2n-1,2n-2} = 0$.

(2.2) When $n \geq 2$, each alternating permutation from $\mathfrak{A}_{2n-1,1}$ starts with $(2n-1)1$, or ends with $1(2n-1)$. After removal of those two letters, there remains an alternating permutation on $\{2, 3, \dots, 2n-2\}$. Hence, $\#\mathfrak{A}_{2n-1,1} = 2T_{2n-3}$. Next, each permutation from $\mathfrak{A}_{2n-1,2n-3}$ must contain, either the three-letter factor $(2n-1)(2n-3)(2n-2)$, or $(2n-2)(2n-3)(2n-1)$. The removal of the factor $(2n-1)(2n-3)$ (resp. $(2n-3)(2n-1)$) yields an alternating permutation of the set $\{1, 2, \dots, (2n-4), (2n-2)\}$, of cardinality $(2n-3)$. This proves relation $\#\mathfrak{A}_{2n-1,2n-3} = 2T_{2n-3}$.

(2.3) Start with an alternating permutation on $\{1, 3, 4, \dots, 2n-2\}$, then having $(2n-3)$ elements. There are four possibilities to generate a permutation from $\mathfrak{A}_{2n-1,2}$: (1) insert $(2n-1)2$ to the left; (2) insert $2(2n-1)$ to the right; (3) insert $2(2n-1)$ just before 1; (4) insert $(2n-1)2$ just after 1. For the second identity in (2.3) proceed in the same way: in each alternating permutation on $\{1, 2, \dots, 2n-1\} \setminus \{2n-4, 2n-1\}$ the two letters $(2n-3)$, $(2n-2)$ are necessarily local maxima. There are four possibilities to obtain a permutation from $\mathfrak{A}_{2n-1,2n-4}$: insert $(2n-1)(2n-4)$ just before, either $(2n-3)$, or $(2n-2)$; also insert $(2n-4)(2n-1)$ just after, either $(2n-3)$, or $(2n-2)$.

(2.4) Each permutation from $\mathfrak{A}_{2n-1,3}$ containing the factor 21 (resp. 12) starts with 21 (resp. ends with 12). Dropping the factor 21 (resp. 12) and subtracting 2 from the remaining letters yields an alternating

permutation from $\mathfrak{A}_{2n-3,1}$. There are then $2(2T_{2n-5})$ permutations from $\mathfrak{A}_{2n-1,3}$ containing, either 21 , or 12 .

If a permutation from $\mathfrak{A}_{2n-1,3}$ contains neither one of those two factors, it has one of the *six* properties: it starts with $(2n-1)3$, or contains one of the three-letter factor $1(2n-1)3$, $3(2n-1)1$, $2(2n-1)3$, $3(2n-1)2$, or still ends with $3(2n-1)$. After removal of the two-letter factor $(2n-1)3$ or $3(2n-1)$ there remains an alternating permutation on $\{1, 2, 4, \dots, (2n-2)\}$ *not starting with* 21 and *not ending with* 12 . There are then $6(T_{2n-3} - 2T_{2n-5})$ such permutations. Altogether, $\#\mathfrak{A}_{2n-1,3} = 4T_{2n-5} + 6(T_{2n-3} - 2T_{2n-5}) = 6T_{2n-3} - 8T_{2n-5}$.

The proof of the second identity in (2.4) follows a different pattern. If the letter $(2n-4)$ is a local minimum (i.e., less than its two adjacent letters) in a permutation σ from $\mathfrak{A}_{2n-1,2n-5}$, then σ necessarily contains one of the four five-letter factors $(2n-1)(2n-5)(2n-2)(2n-4)(2n-3)$, $(2n-1)(2n-5)(2n-3)(2n-4)(2n-2)$, $(2n-2)(2n-4)(2n-3)(2n-5)(2n-1)$, $(2n-3)(2n-4)(2n-2)(2n-5)(2n-1)$. Replacing this five-letter factor by $(2n-5)$ yields a permutation from \mathfrak{A}_{2n-5} . Thus, there are $4T_{2n-5}$ permutations from $\mathfrak{A}_{2n-1,2n-5}$ in which $(2n-4)$ is a local minimum.

In the other permutations from $\mathfrak{A}_{2n-1,2n-5}$ all the four letters $(2n-4)$, $(2n-3)$, $(2n-2)$, $(2n-1)$ are local maxima (i.e., greater than their adjacent letters). Let $\mathfrak{A}'_{2n-1,2n-5}$ be the set of those permutations. When the two-letter factor $(2n-1)(2n-5)$ or $(2n-5)(2n-1)$ is deleted from such a permutation, there remains a permutation on $\{1, 2, \dots, (2n-1)\} \setminus \{(2n-5), (2n-1)\}$ in which the third largest letter $(2n-4)$ is not a local minimum. Let \mathfrak{A}''_{2n-3} be the set of those permutations. But the alternating permutations on the latter set in which $(2n-4)$ is a local minimum necessarily contain the three-letter factor $(2n-3)(2n-4)(2n-2)$ or $(2n-2)(2n-4)(2n-3)$. There are then $2T_{2n-5}$ such permutations. Hence, $\#\mathfrak{A}''_{2n-3} = T_{2n-3} - 2T_{2n-5}$. To obtain a permutation from $\mathfrak{A}'_{2n-1,2n-5}$ it suffices to start from a permutation σ'' from \mathfrak{A}''_{2n-3} and insert $(2n-1)(2n-5)$ (resp. $(2n-5)(2n-1)$) just before (resp. just after) each one of the three letters $(2n-4)$, $(2n-3)$, $(2n-2)$ (which are all local maxima). There are then $6(T_{2n-3} - 2T_{2n-5})$ such permutations. Altogether, $\#\mathfrak{A}_{2n-1,2n-5} = 4T_{2n-5} + 6(T_{2n-3} - 2T_{2n-5}) = 6T_{2n-3} - 8T_{2n-5}$. No comment for (2.5) and (2.6).

(2.7) Simply note that the only alternating permutations from $\mathfrak{A}_{2n,1}$ and $\mathfrak{A}_{2n,2n-1}$ are, respectively, of the form: $(2n)1\sigma(3)\cdots\sigma(2n)$ and $\sigma(1)\sigma(2)\cdots(2n)(2n-1)$.

(2.8) Same proof as for (2.3): start with an alternating permutation on $\{1, 3, 4, \dots, (2n-1)\}$. There are exactly three possibilities to generate a permutation from $\mathfrak{A}_{2n,2}$: insert $(2n)2$ to the left, or just after the letter 1 ,

or still insert $2(2n)$ just before the letter 1. For the second identity start with a permutation on $\{1, 2, \dots, 2n\} \setminus \{2n-2, 2n\}$ and insert $(2n)(2n-2)$ either to the right, or just before $(2n-1)$, or still insert $(2n-2)(2n)$ just after $(2n-1)$.

(2.9) Each permutation from $\mathfrak{A}_{2n,3}$ containing the factor 21 is necessarily of the form $\sigma = 21\sigma(3)\cdots\sigma(2n)$, so that the alternating permutation $\sigma' := (\sigma(3)-2)\cdots(\sigma(2n)-2)$ belongs to $\mathfrak{A}_{2n-2,1}$. There are then E_{2n-4} such permutations. If a permutation from $\mathfrak{A}_{2n,3}$ does not contain 21, it has one of the *five* properties: it starts with $(2n)3$, or contains one of the three-letter factor $1(2n)3$, $3(2n)1$, $2(2n)3$, $3(2n)2$. After removal of the two-letter factor $(2n)3$ or $3(2n)$ there remains an alternating permutation on $\{1, 2, 4, \dots, (2n-1)\}$ *not starting with* 21. There are then $5(E_{2n-2} - E_{2n-4})$ such permutations. Altogether, $\#\mathfrak{A}_{2n,3} = E_{2n-4} + 5(E_{2n-2} - E_{2n-4}) = 5E_{2n-2} - 4E_{2n-4}$.

The proof for the second identity in (2.9) is quite similar. Each permutation from $\mathfrak{A}_{2n,2n-3}$ containing the factor $(2n-1)(2n-2)$ necessarily ends with the four-letter factor $(2n)(2n-3)(2n-1)(2n-2)$. There are then E_{2n-4} such permutations. The other permutations from $\mathfrak{A}_{2n,2n-3}$ contain one of the four three-letter factors $(2n)(2n-3)(2n-2)$, $(2n-2)(2n-3)(2n)$, $(2n)(2n-3)(2n-1)$, $(2n-1)(2n-3)(2n)$, or ends with $(2n)(2n-3)$. After removal of the two-letter factor $(2n)(2n-3)$ or $(2n-3)(2n)$ there remains an alternating permutation on $\{1, 2, \dots, (2n-4), (2n-2), (2n-1)\}$, not ending with the two-letter factor $(2n-1)(2n-2)$. There are $E_{2n-2} - E_{2n-4}$ such permutations. Altogether, $\#\mathfrak{A}_{2n,2n-3} = E_{2n-4} + 5(E_{2n-2} - E_{2n-4})$.

No comment for (2.10). \square

3. Proof of Theorem 1.4

Let $a_n(k) := \#\mathfrak{A}_{2n-1,k-1}$ and $b_n(k) := \#\mathfrak{A}_{2n,k}$. From Proposition 2.1 it follows that the initial conditions [tan1] and [tan2] hold when $f_n(k) = a_n(k)$, and [sec1] and also [sec2] when $f_n(k) = b_n(k)$. It remains to prove that in each case (1.2) holds.

By means of identities (2.2)–(2.4) and (2.7)–(2.9) we easily verify that (1.2) holds for both $a_n(k)$ and $b_n(k)$ when $n = 2, 3$ and $1 \leq k \leq 2n-3$. It also holds for $a_n(k)$ when $n \geq 4$ and $k = 1, 2, 2n-4, 2n-3$, and for $b_n(k)$ when $n \geq 4$ and $k = 1, 2n-3$.

What is left to prove is: $\Delta^2 a_n(k) + 4a_{n-1}(k) = 0$, that is, $\Delta^2 \mathfrak{A}_{2n-1,k-1} + 4\mathfrak{A}_{2n-3,k-1} = 0$ for $n \geq 4$ and $3 \leq k \leq 2n-5$ —by identifying each finite set with its cardinality—and also $\Delta^2 b_n(k) + 4b_{n-1}(k) = 0$, that is, $\Delta^2 \mathfrak{A}_{2n,k} + 4\mathfrak{A}_{2n-2,k} = 0$ for $n \geq 4$ and $2 \leq k \leq 2n-4$; altogether,

$$(2.11) \quad \Delta^2 \mathfrak{A}_{n,k} + 4\mathfrak{A}_{n-2,k} = 0 \quad \text{for } n \geq 7 \text{ and } 2 \leq k \leq n-4 \text{ (} n \text{ even)} \\ 2 \leq k \leq n-5 \text{ (} n \text{ odd)}.$$

FINITE DIFFERENCE CALCULUS

Let $v = y_1 \cdots y_m$ be a nonempty word with *distinct* letters from the set $\{0, 1, 2, \dots, n\}$ and $\tilde{v} = y_m \cdots y_1$ be its mirror-image. If $m = 1$ and $y_1 = 0$, let $[v] = [0]$ be the empty set. If $m \geq 2$ and $y_1 = 0$ (resp. $y_m = 0$), let $[v]$ be the set of all alternating permutations from \mathfrak{A}_n , if any, whose left factors are equal to $y_2 \cdots y_m$, or whose right factors are equal to $y_m \cdots y_2$. When $y_1 \geq 1$, let $[v]$ be the set of all alternating permutations from \mathfrak{A}_n , if any, containing, either the factor v , or the factor \tilde{v} . Finally, let $[\tilde{v}] := [v]$.

Using those notations we get

$$\begin{aligned} \mathfrak{A}_{n,k} &= \sum_{0 \leq y \leq k-1} [ynk] \\ &= \sum_{0 \leq y \leq k-1} [ynk(k+1)] + \sum_{\substack{0 \leq y \leq k-1 \\ k+2 \leq z \leq n-1}} [ynkz] + \sum_{1 \leq y \leq k-1} [ynk0]; \\ \mathfrak{A}_{n,k+1} &= \sum_{0 \leq y \leq k} [yn(k+1)] \\ &= [kn(k+1)] + \sum_{\substack{0 \leq y \leq k-1 \\ k+2 \leq z \leq n-1}} [yn(k+1)z] + \sum_{1 \leq y \leq k-1} [yn(k+1)0]. \end{aligned}$$

The transposition $(k, k+1)$ maps the set $[ynkz]$ onto the set $[yn(k+1)z]$ for $z \in \{k+2, \dots, n-1\} \cup \{0\}$, so that we may write

$$\begin{aligned} \Delta \mathfrak{A}_{n,k} &= \mathfrak{A}_{n,k+1} - \mathfrak{A}_{n,k} = [kn(k+1)] - \sum_{0 \leq y \leq k-1} [ynk(k+1)] \\ &= [kn(k+1)] - \sum_{\substack{0 \leq y_1, y_2 \leq k-1 \\ y_1 \neq y_2}} [y_1 nk(k+1)y_2]; \end{aligned}$$

$$\begin{aligned} \Delta \mathfrak{A}_{n,k+1} &= \mathfrak{A}_{n,k+2} - \mathfrak{A}_{n,k+1} \\ &= [(k+1)n(k+2)] - \sum_{0 \leq y \leq k} [yn(k+1)(k+2)] \\ &= [(k+1)n(k+2)] - [kn(k+1)(k+2)] \\ &\quad - \sum_{0 \leq y \leq k-1} [yn(k+1)(k+2)k] - \sum_{\substack{0 \leq y_1, y_2 \leq k-1 \\ y_1 \neq y_2}} [y_1 n(k+1)(k+2)y_2]. \end{aligned}$$

For $2 \leq k \leq n-4$ the permutation $\begin{pmatrix} k & k+1 & k+2 \\ k+1 & k+2 & k \end{pmatrix}$ maps $[y_1 nk(k+1)y_2]$ onto $[y_1 n(k+1)(k+2)y_2]$ in a bijective manner. Hence,

$$\begin{aligned} \Delta^2 \mathfrak{A}_{n,k} &= \Delta \mathfrak{A}_{n,k+1} - \Delta \mathfrak{A}_{n,k} \\ &= [(k+1)n(k+2)] - [kn(k+1)(k+2)] \\ &\quad - \sum_{0 \leq y \leq k-1} [yn(k+1)(k+2)k] - [kn(k+1)]. \end{aligned}$$

But

$$[kn(k+1)] = [(k+2)kn(k+1)] + [kn(k+1)(k+2)] \\ + \sum_{\substack{z_1, z_2 \in \{k+3, \dots, n-1\} \cup \{0\} \\ z_1 \neq z_2}} [z_1 kn(k+1)z_2].$$

Again, the permutation $\begin{pmatrix} k & k+1 & k+2 \\ k+1 & k+2 & k \end{pmatrix}$ maps the last sum onto the set $[(k+1)n(k+2)]$. Altogether, as $[(k+2)kn(k+1)] = [kn(k+1)(k+2)]$, we have

$$\Delta^2 \mathfrak{A}_{n,k} = -3 [kn(k+1)(k+2)] - \sum_{0 \leq y \leq k-1} [yn(k+1)(k+2)k].$$

When removing the factor $(k+1)(k+2)$ and replacing each integer $z \geq k+2$ by $(z-2)$, in each alternating permutation, both sets $[kn(k+1)(k+2)]$ and $\sum_{0 \leq y \leq k-1} [yn(k+1)(k+2)k]$ are transformed into $\mathfrak{A}_{n-2,k}$; so that $\Delta^2 \mathfrak{A}_{n,k} = -4 \mathfrak{A}_{n-2,k}$.

4. The bivariate generating functions

Let $f = (f_n(k))$ ($n \geq 1, 1 \leq k \leq 2n-1$) be the family of rational numbers, as displayed in (1.1), that satisfies the finite-difference equation system (1.2) under the initial conditions `[tan2]` of `[sec2]`. We know that the system has then a unique solution. With the triangle f associate the infinite matrix

$$(4.1) \quad \Gamma = (\gamma_{ij})_{(i \geq 0, j \geq 0)} := \begin{pmatrix} f_1(1) & 0 & f_2(3) & 0 & f_3(5) & 0 & f_4(7) & \cdots \\ 0 & f_2(2) & 0 & f_3(4) & 0 & f_4(6) & \cdots & \\ f_2(1) & 0 & f_3(3) & 0 & f_4(5) & \cdots & & \\ 0 & f_3(2) & 0 & f_4(4) & \cdots & & & \\ f_3(1) & 0 & f_4(3) & \cdots & & & & \\ 0 & f_4(2) & \cdots & & & & & \\ f_4(1) & \cdots & & & & & & \end{pmatrix}.$$

In other words, define $\gamma_{ij} := 0$ when $i+j$ is odd, and $\gamma_{ij} := f_n(k)$ with $k := j+1$, $2n = 2+i+j$ when $i+j$ is even. For $i+j$ even the mapping $(i, j) \mapsto (n, k)$ is one-to-one, the reverse mapping being for $n \geq 1, 1 \leq k \leq 2n-1$ given by $i = 2n-1-k, j = k-1$.

In terms of the entries γ_{ij} relation (1.2) may be written in the form

$$(4.2) \quad \gamma_{i,j} = 2\gamma_{i-1,j-1} + \frac{1}{2}(\gamma_{i-1,j+1} + \gamma_{i+1,j-1}) \quad (i \geq 1, j \geq 1);$$

$$(4.3) \quad \gamma_{ij} = 0, \quad \text{if } i+j \text{ odd.}$$

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Furthermore, the full matrix $\Gamma = (\gamma_{i,j})$ ($i \geq 0, j \geq 0$) is completely determined as soon as its *first row* $(\gamma_{0,j})$ ($j \geq 0$) and *first column* $(\gamma_{i,0})$ ($i \geq 0$) are known. Let $f \mapsto \Gamma$ denote the above correspondence between those triangles and matrices.

Let $Z(x, y) := \sum_{i \geq 0, j \geq 0} \gamma_{i,j} \frac{x^i}{i!} \frac{y^j}{j!}$. It is easily verified that $Z(x, y)$ satisfies the partial differential equation

$$(4.4) \quad \frac{\partial^2 Z(x, y)}{\partial x \partial y} = 2 Z(x, y) + \frac{1}{2} \frac{\partial^2 Z(x, y)}{\partial x^2} + \frac{1}{2} \frac{\partial^2 Z(x, y)}{\partial y^2},$$

if and only if the coefficients $\gamma_{i,j}$ satisfy relation (4.2). Hence, $Z(x, y)$ is fully determined by (4.4) and by the generating functions $Z(x, 0) = \sum_{i \geq 0} \gamma_{i,0} x^i / i!$ and $Z(0, y) = \sum_{j \geq 0} \gamma_{0,j} y^j / j!$ for the first column and first row of the matrix Γ .

But for any given formal power series in one variable $f(x) = 1 + \sum_{n \geq 1} f_{2n} \frac{x^{2n}}{(2n)!}$ it can be also verified that the bivariate formal power series

$$(4.5) \quad Z(x, y) = \sum_{i \geq 0, j \geq 0} \gamma_{i,j} \frac{x^i}{i!} \frac{y^j}{j!} = f(x+y) \sec(x+y) \cos(x-y)$$

satisfies (4.4) and that the generating functions for its first column and first row are given by $f(x)$ and $f(y)$, respectively. This proves the following proposition.

Proposition 4.1. *Let $f(x) = 1 + \sum_{n \geq 1} f_{2n} \frac{x^{2n}}{(2n)!}$ be given and $\Gamma = (\gamma_{ij})$ ($i \geq 0, j \geq 0$) be an infinite matrix, whose entries satisfy relations (4.2) and (4.3), on the one hand, and such that $\gamma_{0,0} = 1, \gamma_{2n+1,0} = \gamma_{0,2n+1} = 0$ for $n \geq 0$ and $\gamma_{2n,0} = \gamma_{0,2n} = f_{2n}$ for $n \geq 1$, on the other hand. Then, identity (4.5) holds.*

Using the correspondence $\gamma_{ij} \leftrightarrow f_n(k)$ above mentioned, the series $Z(x, y)$ can be rewritten

$$(4.6) \quad Z(x, y) = 1 + \sum_{n \geq 1} \sum_{1 \leq k \leq 2n+1} f_{n+1}(k) \frac{x^{2n+1-k}}{(2n+1-k)!} \frac{y^{k-1}}{(k-1)!},$$

which is then equal to $f(x+y) \sec(x+y) \cos(x-y)$ under the assumptions of the previous proposition.

Now, consider the two triangles described in Fig. 1.1 and let $\Gamma^{\tan} = (\gamma_{ij}^{\tan})$ and $\Gamma^{\sec} = (\gamma_{ij}^{\sec})$ be the two Γ -matrices attached to them:

$$\Gamma^{\tan} = (\gamma_{ij}^{\tan}) = \begin{pmatrix} g_1(1) & 0 & g_2(3) & 0 & g_3(5) & 0 & g_4(7) & \cdots \\ 0 & g_2(2) & 0 & g_3(4) & 0 & g_4(6) & \cdots & \\ g_2(1) & 0 & g_3(3) & 0 & g_4(5) & \cdots & & \\ 0 & g_3(2) & 0 & g_4(4) & \cdots & & & \\ g_3(1) & 0 & g_4(3) & \cdots & & & & \\ 0 & g_4(2) & \cdots & & & & & \\ g_4(1) & \cdots & & & & & & \end{pmatrix};$$

$$\Gamma^{\sec} = (\gamma_{ij}^{\sec}) = \begin{pmatrix} h_1(1) & 0 & h_2(3) & 0 & h_3(5) & 0 & h_4(7) & \cdots \\ 0 & h_2(2) & 0 & h_3(4) & 0 & h_4(6) & \cdots & \\ h_2(1) & 0 & h_3(3) & 0 & h_4(5) & \cdots & & \\ 0 & h_3(2) & 0 & h_4(4) & \cdots & & & \\ h_3(1) & 0 & h_4(3) & \cdots & & & & \\ 0 & h_4(2) & \cdots & & & & & \\ h_4(1) & \cdots & & & & & & \end{pmatrix}.$$

The exponential generating function for the first row and first column of Γ^{\tan} is equal to $f(x) = 1$. On the other hand, as $h_n(1) = h_n(2n+1) = \#\mathfrak{A}_{2n,1} = E_{2n-2}$ by [sec2], (1.8) and (1.16), the exponential generating function for the first row and first column of Γ^{\sec} is equal to $h_1(1) + h_2(1)x^2/2! + h_3(1)x^4/4! + \cdots = E_0 + E_2x/2! + E_4x^4/4 + \cdots = \sec(x)$. Theorem 1.2 is then a consequence of the previous Proposition.

When (x, y) is equal to (x, x) , then to $(x, -x)$ in (1.9) and (1.10), we obtain: $Z^{\tan}(x, x) = \sec(2x)$; $Z^{\tan}(x, -x) = \cos(2x)$; $Z^{\sec}(x, x) = \sec^2(2x) = 1 + \sum_{n \geq 1} 4^n T_{2n+1} x^{2n} / (2n)!$ and $Z^{\sec}(x, -x) = \cos(2x)$. Looking for the coefficients of $x^{2n} / (2n)!$ on both sides in the first (resp. last) two formulas yields four further identities

$$(4.7) \quad \sum_{1 \leq k \leq 2n+1} \binom{2n}{k-1} g_{n+1}(k) = 4^n E_{2n} \quad (n \geq 1);$$

$$(4.8) \quad \sum_{1 \leq k \leq 2n+1} (-1)^k \binom{2n}{k-1} g_{n+1}(k) = (-1)^n 4^n \quad (n \geq 1);$$

$$(4.9) \quad \sum_{1 \leq k \leq 2n+1} \binom{2n}{k-1} h_{n+1}(k) = 4^n T_{2n+1} \quad (n \geq 1)$$

$$(4.10) \quad \sum_{1 \leq k \leq 2n+1} (-1)^k \binom{2n}{k-1} h_{n+1}(k) = (-1)^n 4^n \quad (n \geq 1).$$

The last two ones are mentioned in Sloane's Encyclopedia [SI07] (sequence A125053) without proofs.

5. Alternating permutations and binary trees

In this Section the traditional vocabulary on trees, such as node, leaf, child, root, ... is used. In particular, when a node is not a leaf, it is said to be an *internal node*.

Definition. An *n*-labeled, binary, increasing, topological tree is defined by the following axioms:

- (1) it is a *labeled* tree with *n* nodes, labeled $1, 2, \dots, n$; the node labeled 1 is called the *root*;
- (2) each node has no child (then called a *leaf*), or one child, or two children;
- (3) the label of each node is smaller than the label of its children, if any;
- (4) the tree is planar and each child of a node is, either on the left (it is then called the *left child*), or on the right (the *right child*);
- (5) when *n* is odd, each node is, either a leaf, or a node with two children; when *n* is even, each node is, either a leaf, or a node with two children, except the rightmost node (uniquely defined) which has one *left* child, but no right child. It will be referred to as being the *one-son child*.

Each such binary tree *t* may be drawn on a Euclidean plane: the root has coordinates $(0, 0)$, the left son of the root $(-1, 1)$ and the right son $(1, 1)$, the grandsons $(-3/2, 2), (-1/2, 2), (1/2, 2), (3/2, 2)$, respectively, the great-grandsons $(-7/4, 3), (-5/4, 3), \dots, (7/4, 3)$, etc. With this convention all the nodes have different abscissas. Let *t* have *n* nodes and make the orthogonal projections of those nodes on a horizontal axis. Writing the labels of the projected *n* nodes yields a permutation $\sigma = \sigma(1)\sigma(2) \cdots \sigma(n)$ of $1\ 2 \cdots n$. We say that σ is the *projection* of *t* and *t* the *spreading out* of σ . Moreover, σ is *alternating*. For instance, the two trees *t*₁ and *t*₂ in Fig. 1.2 are labeled, binary, increasing, topological trees, with 7 and 8 nodes, respectively. Their projections $\sigma_1 = 6\ 1\ 5\ 4\ 7\ 2\ 3$ and $\sigma_2 = 6\ 1\ 5\ 4\ 8\ 2\ 7\ 3$ are alternating.

For each $n \geq 1$ let \mathfrak{T}_n be the set of all *n*-labeled, binary, increasing, topological trees. Then, $t \mapsto \sigma$ is a bijection of \mathfrak{T}_n onto \mathfrak{A}_n , so that we also have: $\#\mathfrak{A}_{2n+1} = T_{2n+1}$, $\#\mathfrak{A}_{2n} = E_{2n}$. Each tree *t* from \mathfrak{T}_n is said to be *tangent* (resp. *secant*), if *n* is odd (resp. even).

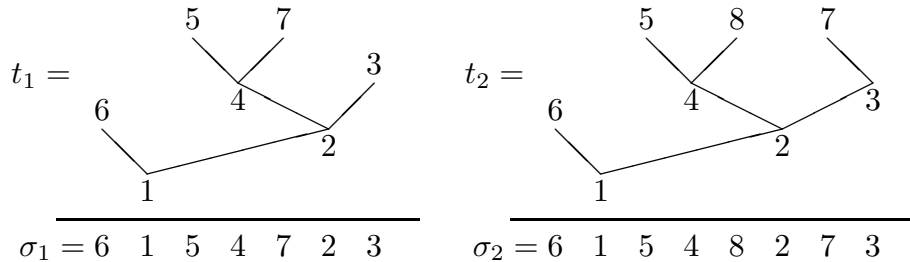


Fig. 1.2. Tangent, secant trees and alternating permutations.

The two statistics “emc” (“end of minimal chain”) and “pom” (“parent of maximum leaf”) we now define on each set \mathfrak{T}_n have been introduced by Christiane Poupard [Po89] in her study of the strictly ordered binary trees and provide two other combinatorial interpretations for the entries $g_n(k)$ and $h_n(k)$. Their definitions are also valid for all binary increasing trees, in particular, for secant and tangent trees. Let $n \geq 2$ and t be a binary increasing tree, with n nodes labeled $1, 2, \dots, n$. Let a be the label of an internal node. If the node has two children labeled b and c , define $\min a := \min\{b, c\}$; if it has one child b , let $\min a := b$. The *minimal chain* of t is defined to be the sequence $a_1 \rightarrow a_2 \rightarrow a_3 \rightarrow \dots \rightarrow a_{j-1} \rightarrow a_j$, with the following properties:

- (i) $a_1 = 1$ is the label of the root;
- (ii) for $i = 1, 2, \dots, j-1$ the i -th term a_i is the label of an internal node and $a_{i+1} = \min a_i$;
- (iii) a_j is the label of a leaf.

Define the *end of the minimal chain* in t to be: $\text{eoc}(t) := a_j$. As t is increasing, there is a unique leaf with label n . If that leaf is incident to a node labeled k , define the (*parent of the maximum leaf*) in t to be: $\text{pom}(t) := k$. By convention, $\text{eoc}(t) = 1$ and $\text{pom}(t) = 0$ for the unique $t \in \mathfrak{T}_1$.

The minimal chain of the tree t_1 (resp. t_2) displayed in Fig. 1.2 is $1 \rightarrow 2 \rightarrow 3$ (resp. $1 \rightarrow 2 \rightarrow 3 \rightarrow 7$). Then $\text{eoc}(t_1) = 3$, $\text{eoc}(t_2) = 7$. Also, $\text{pom}(t_1) = 4$ and $\text{pom}(t_2) = 4$.

Theorem 5.1. *Let $\sigma = \sigma(1)\sigma(2)\dots\sigma(n)$ be the projection of the n -labeled, binary, increasing, topological tree t . Then $\text{pom}(t) = \text{grn}(\sigma)$. In other words, the parent of the maximum leaf in t is the greater neighbor of n in σ .*

Proof. Let $\sigma(i) = n$ with $2 \leq i \leq n-1$. The parent of the node labeled n in t is, either the node labeled $\sigma(i-1)$, or the node labeled $\sigma(i+1)$. Let $\sigma(j)$ be the label of the node of the common ancestor of the previous two nodes in t . Then, $\sigma(j) \leq \min\{\sigma(i-1), \sigma(i+1)\}$, $j \neq i$ and $i-1 \leq j \leq i+1$. Hence, either $\sigma(j) = \sigma(i-1) < \sigma(i+1)$, or $\sigma(j) = \sigma(i+1) < \sigma(i-1)$. In the first (resp. the second) case the parent of n is $\sigma(i+1)$ (resp. $\sigma(i-1)$) and $\text{grn}(\sigma) = \max\{\sigma(i-1), \sigma(i+1)\}$. \square

In [Ha12] an explicit bijection of \mathfrak{T}_n onto itself is constructed that maps the statistic “eoc-1” onto the statistic “pom.” For each of the polynomials $\sum_k g_n(k)x^k$, $\sum_k h_n(k)x^k$, we then have *three* combinatorial interpretations: one on alternating permutations by the statistic “grn,” and two on labeled, binary, increasing, topological trees by “pom” and “eoc.”

Recently, there has been a revival of studies on arithmetical and combinatorial properties of both tangent and secant numbers. Ordering the alternating permutations according to their leftmost elements has led to the *Entringer recurrence*, having interesting properties ([En66, FZ71, FZ71a, Po97, GHZ10]). The geometry of those permutations has been fully exploited ([KPP94, St10]), in particular by looking at their quadrant marked mesh patterns in [KR12], or for defining and studying natural q -analogs of the tangent and secant numbers ([AG78, AF80, Fo81], [St99, p. 148-149]). Further q -analogs were also introduced, based no longer on alternating permutations, but on the so-called *doubloon* model (see [FH10, FH10a, FH11]). The classical continued fraction expansions of secant and tangent have made possible the discovery of other q -analogs (see [Pr08, Pr00, Fu00, HRZ01, Jos10, SZ10]).

Acknowledgement. The authors should like to thank the referee for his careful reading and his knowledgeable remarks.

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