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Realizable monotonicity for continuous-time Markov processes

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Abstract

We formalize and analyze the notions of stochastic monotonicity and realizable monotonicity for Markov Chains in continuous-time, taking values in a finite partially ordered set. Similarly to what happens in discrete-time, the two notions are not equivalent. However, we show that there are partially ordered sets for which stochastic monotonicity and realizable monotonicity coincide in continuous-time but not in discrete-time.

Keywords

Markov processes, coupling, partial ordering, stochastic monotonicity, realizable monotonicity, monotone random dynamical system representation

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1 Introduction

The use of Markov chains in simulation has raised a number of questions concerning qualitative and quantitative features of random processes, in particular in connection with mixing properties. Among the features that are useful in the analysis of effectiveness of Markov Chain Monte Carlo (MCMC) algorithms, a relevant role is played by monotonicity. Two notions of monotonicity have been proposed, for Markov chains taking values in a *partially ordered set* S (*poset* from now on). To avoid measurability issues, which are not relevant for our purposes, we shall always assume S to be finite. Moreover, all Markov chains are implicitly assumed to be time-homogeneous.

Definition 1.1. A Markov chain (η_t) , $t \in \mathbb{R}^+$ or $t \in \mathbb{Z}^+$, on the poset S , with transition probabilities $P_t(x, y) := P(\eta_t = y | \eta_0 = x)$, is said to be *stochastically monotone* (or, more briefly, *monotone*) if for each pair $w, z \in S$ with $w \leq z$ there exists a Markov chain $(X_t(w, z))$ on $S \times S$ such that

- i.) $X_0(w, z) = (w, z)$;
- ii.) each component $(X_t^i(w, z))$, $i = 1, 2$ is a Markov chain on S with transition probabilities $P_t(\cdot, \cdot)$;
- iii.) $X_t^1(w, z) \leq X_t^2(w, z)$ for all $t \geq 0$.

There are various equivalent formulations of stochastic monotonicity. For instance, defining the *transition operator* $T_t f(x) := \sum_{y \in S} f(y) P_t(x, y)$, then the chain is stochastically monotone if and only if for every $t \geq 0$ T_t maps increasing functions to increasing functions. See Theorem 5 in [KKO77] for this generalization to continuous-time dynamics of the well-known Strassen's result (Theorem 2.4 in [Lin92]). This characterization can be turned (see Section 2) into a simple algorithm for checking stochastic monotonicity of Markov chains in terms of the element of the transition matrix (in discrete-time) or in terms of the infinitesimal generator (in continuous-time).

References on the relations between this monotonicity concept and the existence and construction of a monotone coupling for some family of processes in continuous-time, such as diffusions or interacting particle systems, are [Che91, FYMP97, FF97, LS98] and references therein.

For various purposes, including simulation, a stronger notion of monotonicity has been introduced.

Definition 1.2. A Markov chain (η_t) , $t \in \mathbb{R}^+$ or $t \in \mathbb{Z}^+$, on the poset S , with transition probabilities $P_t(x, y) := P(\eta_t = y | \eta_0 = x)$, is said to be *realizably monotone* if there exists a Markov chain $(\xi_t(\cdot))$ on S^S such that

- i.) $\xi_0 = \text{Id}$;
- ii.) for every fixed $z \in S$, the process $(\xi_t(z))$ is a Markov chain with transition probabilities $P_t(\cdot, \cdot)$;
- iii.) if $w \leq z$, then for every $t \geq 0$ we have $\xi_t(w) \leq \xi_t(z)$.

In other words, realizable monotonicity means that we can simultaneously couple, in an order preserving way, all processes leaving any possible initial state. This property becomes relevant when one aims at sampling from the stationary measure of a Markov chain using the Propp and Wilson coupling from the past algorithm (see [PW96]) which we briefly summarize in Subsection 2.4. Notice that if realizable monotonicity holds, the simultaneous order preserving coupling ξ_t can be extended to all S^S . Indeed, for $f \in S^S$, we can define $\tilde{\xi}_t := \xi_t \circ f$. We have

$\tilde{\xi}_0 = f$ and property ii.) holds true for $\tilde{\xi}_t$, while property iii.) has to be replaced by $f(w) \leq f(z) \Rightarrow \tilde{\xi}_t(w) \leq \tilde{\xi}_t(z)$.

We recall that in [FM01] a more general definition of *stochastic monotonicity* and *realizable monotonicity* for a *system of probability measures* is considered:

Definition 1.3. Let \mathcal{A} and \mathcal{S} be two partially ordered sets. A system of probability measures on \mathcal{S} indexed in \mathcal{A} , $(\mathbb{P}_\alpha : \alpha \in \mathcal{A})$ is said to be stochastically monotone if \mathbb{P}_α is stochastically smaller than \mathbb{P}_β (denoted $\mathbb{P}_\alpha \preceq \mathbb{P}_\beta$) whenever $\alpha \leq_{\mathcal{A}} \beta$, i.e. $\int f d\mathbb{P}_\alpha \leq_{\mathbb{R}} \int f d\mathbb{P}_\beta$ for every increasing function $f : \mathcal{S} \rightarrow \mathbb{R}$ whenever $\alpha \leq_{\mathcal{A}} \beta$.

The system $(\mathbb{P}_\alpha : \alpha \in \mathcal{A})$ is said to be realizable monotone if there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a system of \mathcal{S} -valued random variables $(X_\alpha : \alpha \in \mathcal{A})$ with $\mathbb{P}(X_\alpha \in \cdot) = \mathbb{P}_\alpha(\cdot)$ such that $X_\alpha \leq_{\mathcal{S}} X_\beta$ whenever $\alpha \leq_{\mathcal{A}} \beta$.

If the transition probabilities, or the infinitesimal generator, are given, no simple rule for checking realizable monotonicity is known. Since, obviously, realizable monotonicity implies stochastic monotonicity, a natural question is to determine for which posets the converse is true. This problem has been completely solved in [FM01] for discrete-time Markov chain (see Theorem 1.5 here). More precisely, it has been solved as a particular case of the more general problem of *equivalence* between stochastic and realizable monotonicity for systems of probability measures \mathbb{P}_α w.r.t $(\mathcal{A}, \mathcal{S})$ where $\mathcal{A} = \mathcal{S} = S$, $\mathbb{P}_\alpha = P(\alpha, \cdot)$ with $P(\cdot, \cdot)$ denoting the Markov chain's transition probability on S . Notice that the realizable monotonicity of $(P(\alpha, \cdot), \alpha \in S)$ is equivalent to the one for discrete-time Markov chains given in Definition 1.2 through the construction

$$\begin{cases} \xi_0(z) = z, \\ \xi_t(z) = X_{\xi_{t-1}(z)}^{(t)} \quad \forall t \geq 1 \end{cases}$$

where $(X_\alpha^{(t)} : \alpha \in S)$ ($t \in \mathbb{Z}^+$) is an i.i.d. sequence of copies of $(X_\alpha : \alpha \in S)$, which realizes $(P(\alpha, \cdot), \alpha \in S)$.

In what follows, when such equivalence holds we shall say that *monotonicity equivalence holds*.

Let us now give the following definitions

Definition 1.4. The *Hasse diagram* of a poset is an oriented graph. Its vertices are the elements of the poset. There is an edge from x to y if $x \preceq y$ and $x \preceq z \preceq y$ implies $z = x$ or $z = y$; it is said that y *covers* x .

The *cover graph* of a poset S is the Hasse diagram regarded as an undirected graph.

By convention, y is drawn above x in the planar representation of the diagram in order to mean there is an edge from x to y . With this convention of reading the diagram from bottom to the top there is no need to direct any edges. See for example figures 1, 2, 3.

In the case of discrete-time Markov chains the following result holds. It is a consequence of Theorem 4.3 stated in [FM01], (see the previous comment).

Theorem 1.5. *Every stochastically monotone Markov chain in the poset S is also realizably monotone if and only if the cover graph of S is acyclic, i.e. there is no loop $x_0, x_1, \dots, x_n, x_{n+1} = x_0$ such that, for $i = 0, 1, \dots, n$,*

i.) $x_i \neq x_{i+1}$;

ii.) either x_i covers x_{i+1} or x_{i+1} covers x_i .

In the following we call acyclic a poset which has an acyclic cover graph. The nontrivial proof of the above statement consists of three steps.

1. For each *minimal* cyclic poset an example is found of a stochastically monotone Markov chain which is not realizably monotone.
2. Given a general cyclic poset, a stochastically monotone but not realizably monotone Markov chain is constructed by “lifting” one of the examples in step 1.
3. A proof by induction on the cardinality of the poset shows that, in an acyclic poset, stochastically monotone Markov chains are realizably monotone.

Note there is no contradiction with the fact that on some cyclic posets, such as product spaces, order preserving coupling may exist for *some* monotone Markovian dynamics. See for instance [Lou05].

Our aim in this paper is to deal with monotonicity equivalence in continuous-time for time-homogeneous *regular* Markov chains, i.e. Markov chains possessing an infinitesimal generator (or, equivalently, jumping a.s. finitely many times in any bounded time interval). It turns out that if in a poset S stochastic monotonicity implies realizable monotonicity in discrete-time, then the same holds true in continuous-time (see Corollary 2.8). The converse is not true, however; for the posets whose Hasse diagram is represented in Figure 1 (the diamond and the bowtie, following the terminology in [FM01]) equivalence between stochastic monotonicity and realizable monotonicity holds in continuous-time but not in discrete-time.

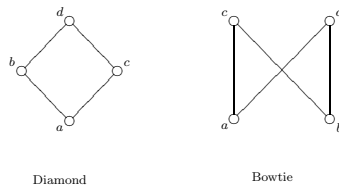


Figure 1: The four-points posets for which there is no equivalence between the two notions of monotonicity in discrete time.

In this paper we do not achieve the goal of characterizing all posets for which equivalence holds in continuous time. Via a computer-assisted (but exact) method we find a complete list of five- and six-point posets for which equivalence fails (see Propositions 3.1 and 3.8). Moreover we state in the Proposition 4.4 the following: in each poset containing one of the former as an induced sub-poset (see Definition 3.7), equivalence fails as well (this does not follow in a trivial way).

In Section 2 we give some preliminary notions, whose aim is to put the realizable monotonicity problem in continuous-time on a firm basis. In Section 3 we perform a systematic investigation of the monotonicity equivalence for five and six points posets, using the software *cdd+* (see

[cdd]). Extensions to larger posets are presented in Section 4. Some further considerations and conjectures are contained in Section 5.

2 Preliminaries

2.1 Characterization of realizable monotonicity

Let (S, \leq) be a finite poset, and $L = (L_{x,y})_{x,y \in S}$ be the infinitesimal generator of a regular continuous-time Markov chain on S . Assume the chain is realizably monotone and let $(\xi_t(\cdot))_{t \geq 0}$ be an order-preserving coupling. Since the original Markov chain is regular, also $(\xi_t(\cdot))_{t \geq 0}$ must be regular: if not, for at least one $z \in S$, $\xi_t(z)$ would be not regular, which is not possible by condition ii. in Definition 1.2. Thus $(\xi_t(\cdot))_{t \geq 0}$ admits an infinitesimal generator $\mathcal{L} = (\mathcal{L}_{f,g})_{f,g \in S^S}$. Let now be $\varphi : S \rightarrow \mathbb{R}$, $z \in S$, and define $F_{\varphi,z} : S^S \rightarrow \mathbb{R}$ by $F_{\varphi,z}(f) := \varphi(f(z))$. The fact that each component of the chain generated by \mathcal{L} is a Markov chain with generator L is equivalent to the following statement: for all choices of φ , z , and all $f \in S^S$,

$$\mathcal{L}F_{\varphi,z}(f) = L\varphi(f(z)). \quad (1)$$

By an elementary algebraic manipulation of (1), we can re-express (1) with the following statement: for every $z, x, y \in S$, $x \neq y$ and every $f \in S^S$ such that $f(z) = x$, we have

$$L_{x,y} = \sum_{g \in S^S: g(z)=y} \mathcal{L}_{f,g}. \quad (2)$$

Now let Id denote the identity on S , and define $\Lambda(f) := \mathcal{L}_{\text{Id},f}$. Note that since the Markov chain generated by \mathcal{L} is order preserving, necessarily $\mathcal{L}_{\text{Id},f} > 0 \Rightarrow f \in \mathcal{M}$, where \mathcal{M} is the set of increasing functions from S to S . Note that, by (2), for $x \neq y$,

$$L_{x,y} = \sum_{f \in \mathcal{M}: f(x)=y} \Lambda(f). \quad (3)$$

Identity (3) characterizes the generators of realizably monotone Markov chains, in the sense specified by the following proposition.

Proposition 2.1. *A generator L is the generator of a realizably monotone Markov chain if and only if there exists $\Lambda : \mathcal{M} \rightarrow \mathbb{R}^+$ such that (3) holds.*

Proof. One direction has been proved above. For the converse, suppose (3) holds for some $\Lambda : \mathcal{M} \rightarrow \mathbb{R}^+$. For $f, g \in S^S$, define

$$\mathcal{L}_{f,g} := \sum_{h \in \mathcal{M}: g=h \circ f} \Lambda(h),$$

where the sum over the empty set is meant to be zero. It is easily checked that the Markov chain generated by \mathcal{L} is order preserving. Moreover, using (3), a simple computation shows that (2) holds, completing the proof. \square

2.2 The cones of stochastically monotone and realizably monotone generators

Let $S_2 := S \times S \setminus \{(x, x) : x \in S\}$. An infinitesimal generator is a matrix $L = (L_{x,y})_{x,y \in S}$ whose non-diagonal elements are nonnegative, while the terms in the diagonal are given by $L_{x,x} = -\sum_{y \neq x} L_{x,y}$. Thus L may be identified with an element of the cone $(\mathbb{R}^+)^{S_2}$. A subset $\Gamma \subseteq S$ is said to be an *up-set* (or *increasing set*) if

$$x \in \Gamma \text{ and } x \leq y \Rightarrow y \in \Gamma.$$

The following proposition (see e.g. [Mas87] for the proof) gives a characterization of the generators of stochastically monotone Markov chains.

Proposition 2.2. *An element $L \in (\mathbb{R}^+)^{S_2}$ is the generator of a stochastically monotone Markov chain if and only if for every up-set Γ the following conditions hold:*

$$\begin{aligned} \bullet \quad x \leq y \notin \Gamma &\Rightarrow \sum_{z \in \Gamma} L_{x,z} \leq \sum_{z \in \Gamma} L_{y,z}; \\ \bullet \quad x \geq y \in \Gamma &\Rightarrow \sum_{z \notin \Gamma} L_{x,z} \leq \sum_{z \notin \Gamma} L_{y,z}. \end{aligned}$$

Remark 2.3. In what follows we shall often call *monotone generator* (respectively *realizably monotone generator*) the infinitesimal generator of a stochastically monotone Markov Chain (resp. realizably monotone Markov Chain). Given a generator L on S , $x \in S$ and $\Gamma \subset S$, we shall use the symbol $L_{x,\Gamma}$ to denote $\sum_{z \in \Gamma} L_{x,z}$. Moreover, in order to check stochastic monotonicity of a generator we shall use the following condition, which is equivalent to the one given in Proposition 2.2:

- i) for every up-set Γ , $x \leq y \notin \Gamma \Rightarrow L_{x,\Gamma} \leq L_{y,\Gamma}$;
- ii) for every down-set Γ , $y \geq x \notin \Gamma \Rightarrow L_{x,\Gamma} \geq L_{y,\Gamma}$

where a down-set is a subset $\Gamma \subset S$ such that $x \in \Gamma$ and $y \leq x \Rightarrow y \in \Gamma$.

Let $V := \mathbb{R}^{S_2}$ be provided with the natural Euclidean scalar product $\langle \cdot, \cdot \rangle$. For given $x, y \in S$, Γ up-set, let $W^{\Gamma,x,y} \in \mathbb{R}^{S_2}$ be defined by

$$W_{v,z}^{\Gamma,x,y} = \begin{cases} 1 & \text{for } \begin{cases} x \leq y \notin \Gamma, v = y, z \in \Gamma \\ \text{or } x \geq y \in \Gamma, v = y, z \notin \Gamma; \end{cases} \\ -1 & \text{for } \begin{cases} x \leq y \notin \Gamma, v = x, z \in \Gamma \\ \text{or } x \geq y \in \Gamma, v = x, z \notin \Gamma; \end{cases} \\ 0 & \text{in all other cases.} \end{cases}$$

Proposition 2.2 can be restated as follows: $L \in (\mathbb{R}^+)^{S_2}$ generates a stochastically monotone Markov chain if and only if

$$\langle L, W^{\Gamma,x,y} \rangle \geq 0 \text{ for every } \Gamma, x, y. \quad (4)$$

In other words, denoting by \mathcal{G}_{mon} the set of monotone generators, the elements of \mathcal{G}_{mon} are characterized by the inequalities

$$\begin{aligned} \langle L, W^{\Gamma,x,y} \rangle &\geq 0 \\ L_{x,y} &\geq 0 \end{aligned}$$

for every Γ, x, y . In other words we are giving \mathcal{G}_{mon} through the *rays* of its *polar cone* (see [Zie95]), *i.e.* the family of vectors $\{W^{\Gamma,x,y}, \delta^{x,y} : (x, y) \in \mathcal{S}_2, \Gamma \text{ up-set in } \mathcal{S}\}$, where

$$\delta_{v,z}^{x,y} = \begin{cases} 1 & \text{if } (v, z) = (x, y) \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1 can also be restated to characterize the set of generators of realizably monotone Markov chains as a cone in V . For $f \in \mathcal{M}$, let $\mathbb{I}_f \in (\mathbb{R}^+)^{\mathcal{S}_2}$ be defined by

$$(\mathbb{I}_f)_{x,y} = \begin{cases} 1 & \text{if } f(x) = y \\ 0 & \text{otherwise.} \end{cases} \quad (5)$$

Then the set $\mathcal{G}_{r.mon}$ of generators of realizably monotone Markov chains is the cone given by linear combination with nonnegative coefficients of the vectors \mathbb{I}_f , *i.e.*

$$L = \sum_{f \in \mathcal{M}} \Lambda_f \mathbb{I}_f, \quad (6)$$

with $\Lambda_f \geq 0$. Note that in this case, since for each $f \in \mathcal{M}$, Γ up-set, $x, y \in \mathcal{S}$, we have $\langle \mathbb{I}_f, W^{\Gamma,x,y} \rangle \geq 0$, we recover the inclusion $\mathcal{G}_{r.mon} \subseteq \mathcal{G}_{mon}$. Our aim is to determine for which posets the converse inclusion holds true.

In the next sections, when we want to emphasize the dependence on \mathcal{S} , we shall use the notations $\mathcal{G}_{mon}(\mathcal{S})$ and $\mathcal{G}_{r.mon}(\mathcal{S})$.

2.3 Comparison with the discrete-time case

In this subsection we establish a comparison with the discrete-time case. The claim of Corollary 2.8 below relies on analogous representations in terms of cones for discrete-time transition matrices.

Consider a Markov chain with transition matrix P . We recall the following fact.

Proposition 2.4. *$P = (P_{x,y})_{x,y \in \mathcal{S}}$ is the transition matrix of a stochastically monotone Markov Chain if and only if, for every up-set Γ , the map $x \mapsto \sum_{y \in \Gamma} P_{x,y}$ is increasing.*

This Proposition 2.4 derives from the following general statement, which is an immediate consequence of the Definition 1.3 : a system of probability measures $(P_\alpha : \alpha \in \mathcal{A})$ on \mathcal{S} is stochastically monotone if and only if, for every up-set $\Gamma \subset \mathcal{S}$ the map $\alpha \mapsto P_\alpha(\Gamma)$ is increasing.

The discrete-time version of the argument in Subsection 2.1 shows that P is the transition matrix of a realizably monotone Markov chain if and only if there exists a probability Π on \mathcal{M} such that

$$P = \sum_{f \in \mathcal{M}} \Pi_f \mathbb{I}_f, \quad (7)$$

where, with a slight abuse of notation, \mathbb{I}_f given by (5) is now seen as a square matrix, with the diagonal terms too. A transition probability P may be seen as an element of the convex set $[0, 1]^{\mathcal{S}_2}$. Note in this representation the identity matrix I is the origin. Analogously to the continuous-time case, the set of stochastically monotone transition probabilities \mathcal{K}_{mon} (resp. realizable monotone $\mathcal{K}_{r.mon}$) is a convex set.

In what follows, we make use of the following simple fact: if L is a stochastically monotone generator, with any $\epsilon > 0$ such that $\epsilon \leq (\max_x \sum_{z \neq x} L_{x,z})^{-1}$, $P := I + \epsilon L$ is a stochastically

monotone transition probability. Conversely, for all $\epsilon > 0$, $L := \frac{1}{\epsilon}(P - I)$ is a stochastically monotone generator whenever P is a stochastically monotone transition probability.

Definition 2.5. The *weak monotonicity-equivalence* holds for a poset S if we can find a realizably monotone transition matrix $(1 - \lambda)I + \lambda P$ for some $\lambda \in]0, 1]$ whenever P is stochastically monotone.

Remark 2.6. By the correspondence between stochastically monotone generators and transitions probabilities, Definition 2.5 is equivalent to the following statement: for any stochastically monotone generator L on S there exists $\epsilon \in]0, 1]$ such that $I + \epsilon L$ is a realizably monotone transition probability on S .

Proposition 2.7. *Monotonicity equivalence holds for continuous-time Markov chain on S if and only if weak monotonicity equivalence holds for discrete-time Markov chains on S .*

Proof. In view of Remark 2.6, we have to show that the following two statements are equivalent for a generator L :

- i) L is a realizably monotone generator;
- ii) there exists $\epsilon > 0$ such that $I + \epsilon L$ is a realizably monotone transition probability.

We first show *ii) \Rightarrow i)*. Suppose that there exists $\epsilon > 0$ such that the transition probability $I + \epsilon L = \tilde{P}$ is realizably monotone, which means

$$\tilde{P} = \sum_{f \in \mathcal{M}} \Pi_f \mathbb{I}_f$$

for a suitable probability Π on \mathcal{M} . Thus the following identity holds in $(\mathbb{R}^+)^{S^2}$:

$$L = \frac{1}{\epsilon} \sum_{f \in \mathcal{M}} \Pi_f \mathbb{I}_f,$$

which implies that L is realizably monotone.

We now show *i) \Rightarrow ii)*. Let L be a realizably monotone generator. Then, for $\epsilon > 0$ sufficiently small, $\tilde{P} = I + \epsilon L$ is a stochastically monotone transition probability. Let us show that \tilde{P} is realizably monotone for $\epsilon > 0$ small enough. We have the following representation of L as an element of $(\mathbb{R}^+)^{S^2}$: $L = \sum_{f \in \mathcal{M}} \Lambda_f \mathbb{I}_f$. Note that $f = \text{Id}$ gives no contribution to the sum above, since \mathbb{I}_{Id} is the zero element of $(\mathbb{R}^+)^{S^2}$. We may therefore assume $\Lambda_{\text{Id}} = 0$. Now, consider the matrix in $\mathbb{R}^{S \times S}$ given by

$$H = \epsilon \sum_{f \in \mathcal{M}} \Lambda_f \mathbb{I}_f.$$

We have

$$H_{x,y} = \begin{cases} \epsilon \sum_{f \in \mathcal{M}} \Lambda_f \delta_{\{f(x),y\}} = \epsilon L_{x,y} & \text{if } x \neq y \\ \epsilon \sum_{f \in \mathcal{M}} \Lambda_f \delta_{\{f(x),x\}} = \epsilon L_{x,x} + \epsilon \sum_{f \in \mathcal{M}} \Lambda_f & \text{otherwise.} \end{cases}$$

where $\delta_{\{x,y\}}$ denotes the Kronecker delta.

Indeed, we have

$$\begin{aligned} \epsilon L_{x,x} &= - \sum_{y:y \neq x} \epsilon L_{x,y} = -\epsilon \sum_{y:y \neq x} \sum_{f \in \mathcal{M}} \Lambda_f \delta_{\{f(x),y\}} \\ &= -\epsilon \sum_{f \in \mathcal{M}} \sum_{y:y \neq x} \Lambda_f \delta_{\{f(x),y\}} = -\epsilon \sum_{f \in \mathcal{M}} \Lambda_f (1 - \delta_{\{f(x),x\}}) \\ &= \epsilon \sum_{f \in \mathcal{M}} \Lambda_f \delta_{\{f(x),x\}} - \epsilon \sum_{f \in \mathcal{M}} \Lambda_f. \end{aligned}$$

Therefore, we have

$$\tilde{P} = I + \epsilon L = I + \epsilon \sum_{f \in \mathcal{M}} \Lambda_f \mathbb{I}_f - (\epsilon \sum_{f \in \mathcal{M}} \Lambda_f) I = \sum_{f \in \mathcal{M}} \epsilon \Lambda_f \mathbb{I}_f + (1 - \sum_{f \in \mathcal{M}} \epsilon \Lambda_f) \mathbb{I}_{\text{Id}}.$$

If we choose ϵ sufficiently small, we can interpret the quantities

$$\Pi_f := \begin{cases} \epsilon \Lambda_f & \text{for } f \in \mathcal{M}, f \neq \text{Id} \\ 1 - \sum_{f \in \mathcal{M}} \epsilon \Lambda_f & \text{for } f = \text{Id} \end{cases}$$

as probabilities on \mathcal{M} and so \tilde{P} is realizable monotone. □

The following fact is an immediate consequence of Proposition 2.7.

Corollary 2.8. *Suppose that in the poset S stochastic monotonicity and realizable monotonicity are equivalent notions for discrete-time Markov chains. Then the equivalence holds for S -valued continuous-time Markov Chains as well.*

We summarize, in the following scheme, the main facts relating continuous and discrete-time. All matrices are thought as elements of \mathbb{R}^{S^2} , so the diagonal is not considered.

- $P \in \mathcal{K}_{mon} \Rightarrow \forall \epsilon > 0, L := \frac{1}{\epsilon} P \in \mathcal{G}_{mon};$
- $P \in \mathcal{K}_{r.mon} \Rightarrow \forall \epsilon > 0, L := \frac{1}{\epsilon} P \in \mathcal{G}_{r.mon};$
- $L \in \mathcal{G}_{mon} \Rightarrow \exists \epsilon_L, \forall \epsilon \leq \epsilon_L, P := \epsilon L \in \mathcal{K}_{mon};$
- $L \in \mathcal{G}_{r.mon} \Rightarrow \exists \epsilon_0, P := \epsilon_0 L \in \mathcal{K}_{r.mon};$
- the weak monotonicity equivalence means the segment $[I, P]$ intersects $\mathcal{K}_{r.mon}$ whenever $P \in \mathcal{K}_{mon}$.

2.4 The "coupling from the past" algorithm revisited

It is well known (see e.g. [Arn98, Bré99, FG, Min04]) that regular finite state Markov processes can be realized as *Random Dynamical Systems with independent increments* (shortly RDSI). To set up notations, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $(\theta_t)_{t \in \mathbb{R}}$ be a one-parameter group (*i.e.* $\theta_{t+s} = \theta_t \circ \theta_s$, $\theta_0 = \text{Id} = \text{identity map}$) of \mathbb{P} -preserving maps from Ω to Ω , such that the map $(t, \omega) \mapsto \theta_t \omega$ is jointly measurable in t and ω . We still denote by S a finite set, representing the state space.

Definition 2.9. A *Random dynamical system* is a measurable map $\varphi : \mathbb{R}^+ \times \Omega \rightarrow S^S$ such that

$$\varphi(0, \omega) \equiv \text{Id} \tag{8}$$

$$\varphi(t+s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega) \tag{9}$$

for every $s, t \geq 0$ and $\omega \in \Omega$.

Note that, for t fixed, $\varphi(t, \cdot)$ can be seen as a S^S -valued random variable.

Definition 2.10. A Random Dynamical System φ is said to have *independent increments* if for each $0 \leq t_0 < t_1 < \dots < t_n$ the random variables

$$\varphi(t_1 - t_0, \theta_{t_0} \cdot), \varphi(t_2 - t_1, \theta_{t_1} \cdot), \dots, \varphi(t_n - t_{n-1}, \theta_{t_{n-1}} \cdot)$$

are independent.

In what follows we consider the σ -fields

$$\begin{aligned} \mathcal{F}_t &:= \sigma\{\varphi(s, \cdot) : 0 \leq s \leq t\} \\ \mathcal{F}^- &:= \sigma\{\varphi(s, \theta_{-t} \cdot) : 0 \leq s \leq t, 0 \leq t \leq +\infty\}. \end{aligned}$$

The following proposition recalls Theorem 1.2.1 of [D97] (see section 2.1.3 in [Arn98] too).

Proposition 2.11. *For a RDSI, the random process $(\varphi(t, \cdot))_{t \geq 0}$ is a S^S -valued, time-homogeneous, \mathcal{F}_t -Markov process. Moreover, for any S -valued, \mathcal{F}^- -measurable random variable X (in particular any $X = x \in S$ constant), the process $(\varphi(t, \cdot)(X))_{t \geq 0}$ is a S -valued, time-homogeneous Markov process, whose transition probabilities do not depend on X .*

The processes $(\varphi(t, \cdot)(x))_{t \geq 0}$, with $x \in S$ are called the *one point motions* of φ . When the one point motions are Markov process with infinitesimal generator L , we say that φ *realizes* L . It is nothing else than a complete coupling: copies of the chain starting from every initial conditions are realized on the same probability space.

If we are given a generator L of a Markov chain, it is not difficult to realize it by a RDSI. Let \mathcal{H} be the set whose elements are the locally finite subsets of \mathbb{R} . An element $\eta \in \mathcal{H}$ can be identified with the σ -finite point measure $\sum_{t \in \eta} \delta_t$; the topology in \mathcal{H} is the one induced by vague convergence, and the associated Borel σ -field provides a measurable structure on \mathcal{H} . Set $\Omega' := \mathcal{H}^{S_2}$. For $\omega = (\omega_{xy})_{(x,y) \in S_2} \in \Omega'$, we define

$$\theta_t \omega := (\theta_t \omega_{xy})_{(x,y) \in S_2},$$

where $\tau \in \theta_t \omega_{xy} \iff \tau - t \in \omega_{xy}$. Consider now a probability \mathbb{P} on Ω' with the following properties:

- i.) for $(x, y) \in S_2$, ω_{xy} is, under \mathbb{P} , a Poisson process of intensity $L_{x,y}$;
- ii.) for $x \in S$ fixed, the point processes $(\omega_{xy})_{y \neq x}$ are independent under \mathbb{P} ;
- iii.) for every I, J disjoint intervals in \mathbb{R} , the two families of random variables

$$\{|\omega_{xy} \cap I| : (x, y) \in S_2\} \quad \text{and} \quad \{|\omega_{xy} \cap J| : (x, y) \in S_2\}$$

are independent under \mathbb{P} ;

- iv.) for every $t \in \mathbb{R}$, \mathbb{P} is θ_t -invariant.

It is easy to exhibit one example of a \mathbb{P} satisfying i.-iv.: if \mathbb{P}_{xy} is the law of a Poisson process of intensity $L_{x,y}$, then we can let \mathbb{P} be the product measure

$$\mathbb{P} := \otimes_{(x,y) \in S_2} \mathbb{P}_{xy}. \tag{10}$$

We now construct the map φ *pointwise* in ω . Define

$$\Omega = \{\omega \in \Omega' : \omega_{xy} \cap \omega_{xz} = \emptyset \text{ for every } (x, y), (x, z) \in S_2, y \neq z\}.$$

By condition ii. on \mathbb{P} , $\mathbb{P}(\Omega) = 1$, and clearly $\theta_t \Omega = \Omega$ for every $t \in \mathbb{R}$. For every $\omega \in \Omega$ the following construction is well posed:

- set $\varphi(0, \omega) = \text{Id}$ for every $\omega \in \Omega$;
- we run the time in the forward direction. Whenever we meet $t \in \bigcup_{(x,y) \in S_2} \omega_{xy}$ we use the following updating rule:

$$\text{if } \varphi(t^-, \omega)(x) = z \text{ and } t \in \omega_{zy} \text{ then } \varphi(t, \omega)(x) := y.$$

Proposition 2.12. (see Theorem 3.1 of [Min04]). *The map φ constructed above is a RDSI, and its one-point motions are Markov chains with generator L .*

Conditions i.- iii. leave a lot of freedom on the choice of \mathbb{P} . The choice corresponding to (10) is the simplest, but may be quite inefficient when used for simulations.

The following Theorem is just a version of the "coupling from the past" algorithm for perfect simulation ([PW96]).

Theorem 2.13. *Let L be the generator of an irreducible Markov chain on S , π be its stationary distribution, and let φ be a RDSI whose one-point motions are Markov chains with generator L . Define*

$$T(\omega) := \inf\{t > 0 : \varphi(t, \theta_{-t}\omega) = \text{constant}\}$$

where, by convention, $\inf \emptyset := +\infty$. Assume $T < +\infty$ \mathbb{P} -almost surely. Then for each $x \in S$ the random variable $\varphi(T, \theta_{-T}\cdot)(x)$ has distribution π .

Proof. Set $X(\omega) := \varphi(T, \theta_{-T}\omega)(x)$ that, by definition of T , is independent of $x \in S$. For $h > 0$

$$\varphi(T+h, \theta_{-T-h}\omega)(x) = \varphi(T, \theta_{-T}\omega)(\varphi(h, \theta_{-T-h}\omega)(x)) = X(\omega).$$

Thus, since we are assuming $T < +\infty$ a.s., we have

$$X(\omega) = \lim_{t \rightarrow +\infty} \varphi(t, \theta_{-t}\omega)(x).$$

In particular, this last formula shows that X is \mathcal{F}^- -measurable. Denote by ρ the distribution of X , i.e. $\rho(x) := P(X = x)$. We have:

$$\begin{aligned} \varphi(h, \omega)(X(\omega)) &= \lim_{t \rightarrow +\infty} \varphi(h, \omega)(\varphi(t, \theta_{-t}\omega)(x)) = \lim_{t \rightarrow +\infty} \varphi(t+h, \theta_{-t}\omega)(x) \\ &= \lim_{t \rightarrow +\infty} \varphi(t+h, \theta_{-t-h}(\theta_h\omega))(x) = X(\theta_h\omega). \end{aligned}$$

By Proposition 2.11, $\varphi(h, \cdot)(X(\cdot))$ has distribution ρe^{hL} . But, since θ_h is \mathbb{P} -preserving, $X(\theta_h\cdot)$ has distribution ρ . Thus $\rho = \rho e^{hL}$, i.e. ρ is stationary, and therefore $\rho = \pi$. □

The condition $\mathbb{P}(T < +\infty) = 1$ depends on the particular choice of the RDSI, and it is not granted by the irreducibility of L . For example, consider $S = \{0, 1\}$ and L given by $L_{0,1} = L_{1,0} = 1$. We can realize this chain by letting ω_{01} be a Poisson process of intensity 1, and $\omega_{10} = \omega_{01}$. Clearly $\varphi(t, \theta_{-t}\omega)(0) \neq \varphi(t, \theta_{-t}\omega)(1)$ for every $t > 0$, so $T \equiv +\infty$. On the other hand, and this holds in general, if we make the choice of \mathbb{P} given by (10), it is not hard to see that $(\varphi(t, \cdot))_{t \geq 0}$ is an irreducible Markov chain on S^S . By recurrence, any constant function is hit with probability 1 in finite time, so $T < +\infty$ a.s.

In order to implement the Propp-Wilson algorithm, in principle one needs to run a Markov chain on S^S , which may be computationally unachievable. Some additional structure can make the algorithm much more effective.

Definition 2.14. Let S be a poset. A RDS on S is said to be *monotone* if for every $t \geq 0$ and $\omega \in Q$

$$\varphi(t, \omega) \in \mathcal{M}.$$

Suppose a Markov chain is realized by a monotone RDSI. Then

$$\varphi(t, \theta_{-t}\omega) \text{ is constant} \iff \varphi(t, \theta_{-t}\omega) \text{ is constant on } A,$$

where A is the set of points in S that are either maximal or minimal. Thus to implement the Propp-Wilson algorithm one needs to run Markov chains starting from every point of A , that may be much smaller than S . Moreover, the following result holds.

Theorem 2.15. *Let S be a connected poset (i.e. every two points in S are connected by a path of comparable points), and φ be a monotone RDSI whose one-point motions are irreducible Markov chains. Then $\mathbb{P}(T < +\infty) = 1$.*

Proof. Let x be minimal in S , and $y_1 > x$. By irreducibility, given $s > 0$

$$\mathbb{P}(\varphi(s, \cdot)(y_1) = x) > 0.$$

Since $\varphi(s, \omega) \in \mathcal{M}$ and \mathcal{M} is finite, there exists $f_1 \in \mathcal{M}$ with $f_1(y_1) = x$ and

$$\mathbb{P}(\varphi(s, \cdot) = f_1) > 0.$$

Note that, necessarily, $f_1(x) = x$, so f_1 is not bijective. Set $S_1 := f_1(S)$. Note that S_1 , with the order induced by S , is connected, being the image of a connected poset under an increasing function. Clearly $x \in S_1$ is still minimal in S_1 , and $|S_1| < |S|$. Unless $|S_1| = 1$, the same argument can be repeated. Take $y_2 \in S_1$, $y_2 > x$, and $f_2 \in \mathcal{M}$ such that $f_2(y_2) = x$ and $\mathbb{P}(\varphi(s, \omega) = f_2) > 0$. Again $f_2(x) = x$, so that $|S_2| := |f(S_1)| < |S_1|$. After a finite number of similar steps, we obtain a finite family $f_1, f_2, \dots, f_n \in \mathcal{M}$ such that $\mathbb{P}(\varphi(s, \omega) = f_i) > 0$ and

$$f_n \circ f_{n-1} \circ \dots \circ f_1 \equiv x \text{ is constant.} \tag{11}$$

Now, for $k = 1, 2, \dots, n$, consider the events

$$\{\varphi(s, \theta_{-ks}\cdot) = f_{n-k+1}\}.$$

Since θ_t is \mathbb{P} -preserving, all these events have nonzero probability and, by independence of the increments, they are all independent. Thus, by (11)

$$0 < \mathbb{P}\left(\bigcap_{k=1}^n \{\varphi(s, \theta_{-ks}\cdot) = f_{n-k+1}\}\right) \leq \mathbb{P}(\varphi(ns, \theta_{-ns}\cdot) = \text{const.}).$$

Now, let $t := ns$ and for $N \geq 1$ consider the events $\{\varphi(t, \theta_{-Nt}\cdot) = \text{const.}\}$. Since they are independent and with the same nonzero probability,

$$\mathbb{P}\left(\bigcup_N \{\varphi(t, \theta_{-Nt}\cdot) = \text{const.}\}\right) = 1.$$

Observing that $\varphi(t, \theta_{-Nt}\cdot) = \text{const.}$ implies $\varphi(Nt, \theta_{-Nt}\cdot) = \text{const.}$, we obtain

$$\mathbb{P}\left(\bigcup_N \{\varphi(Nt, \theta_{-Nt}\cdot) = \text{const.}\}\right) = 1.$$

from which $\mathbb{P}(T < +\infty) = 1$ follows. □

We conclude this section by remarking that a Markov chain with generator L can be realized by a monotone RDSI if and only if it is realizably monotone. Indeed, if such RDSI exists, then $(\varphi(t, \cdot))_{t \geq 0}$ is a Markov chain on S^S for which the conditions in Definition 1.2 are satisfied. Conversely, once we have the representation in (3), a RDSI with the desired properties is obtained as follows. For $f \in \mathcal{M}$, let \mathbb{P}_f be the law of a Poisson process on \mathbb{R} with intensity $\Lambda(f)$, and, on the appropriate product space whose elements are denoted by $\omega = (\omega_f)_{f \in \mathcal{M}}$, we define $\mathbb{P} := \otimes_{f \in \mathcal{M}} \mathbb{P}_f$. The map φ is constructed pointwise in ω via the updating rule: if $t \in \omega_f$ and $\varphi(t^-, \omega) = g$ then $\varphi(t, \omega) = f \circ g$.

3 Extremal generators of stochastically monotone Markov chains: the monotonicity equivalence for “small” posets

As seen in subsection 2.2, equivalence between realizable monotonicity and stochastic monotonicity of any Markov Chain on a poset S is equivalent to

$$\mathcal{G}_{r.mon} = \mathcal{G}_{mon}. \quad (12)$$

In this section we study monotonicity equivalence for posets with small cardinality.

First note that the cases $\sharp S = 2$, $\sharp S = 3$ are obvious: in these cases S is acyclic. According to Theorem 1.5, there is equivalence for discrete-time Markov chains and using the result of Proposition 2.8 the equivalence holds for continuous-time Markov chains as well.

In order to further investigate the equality (12) we developed computer programs. The cone \mathcal{G}_{mon} is defined as intersection of half spaces in (4) (so called *H-representation*). The cone $\mathcal{G}_{r.mon}$ is defined by its extremal rays in (6) (so called *V-representation*). The software *cdd+* (see [cdd]) is able to compute exactly one representation given the other one. This is a *C++* implementation for convex polyhedron of the Double Description Method (see for instance [FP96]). Finding the extremal rays of the cone \mathcal{G}_{mon} and the (minimal) set of inequalities defining the cone $\mathcal{G}_{r.mon}$, the inclusion $\mathcal{G}_{mon} \subseteq \mathcal{G}_{r.mon}$ can be easily checked.

We operated by first using the software *GAP* (see [GAP]) in order to

- i.) find the up-sets Γ related to the poset S , the vectors $W^{\Gamma, x, y} \in \mathbb{R}^{S^2}$ and then identify the H-representation of \mathcal{G}_{mon} ;
- ii.) compute all the increasing functions $f \in \mathcal{M}$, identify the vectors $\mathbb{I}_f \in (\mathbb{R}^+)^{S^2}$ and then find the V-representation of $\mathcal{G}_{r.mon}$.

We then use the software *cdd+* to produce the other representations of the cones, and the software *Scilab* to test if $\mathcal{G}_{mon} \subseteq \mathcal{G}_{r.mon}$.

The difficulty in applying this method to posets with high cardinality is mainly due to the combinatorial complexity of the step (ii) and to the computational time needed to *cdd+* to obtain the dual representation of the cone. Rather than to $\sharp S$, this time is related to the number of facets of the cones, which comes from the partial order structure. It should also be remarked that a systematic analysis, made by generating all posets with a given cardinality, is not doable for “moderate” cardinality. For instance, the number of different unlabeled posets structure – up to an order preserving isomorphism, not necessarily connected – for a given set of cardinality 16 is $\sim 4.48 \times 10^{15}$. It was stated in 2002, see [BM02]. For a set of cardinality 17, the number of unlabeled posets is till now unknown. For a set of cardinality 4, resp. 5, 6, 7, the number of posets is respectively 16, 63, 318, 2045. See [pos] for the list.

Nevertheless, we were able to completely study the cases when $\sharp S \leq 6$. For $\sharp S > 6$, the result of Proposition 4.4 in the next section gives the answer for some posets.

For $\sharp S = 4$ the two relevant poset-structure are the diamond and the bowtie. Their Hasse-Diagram are given by the Figure 1. For those two posets, the algorithm above ensures that $\mathcal{G}_{mon} = \mathcal{G}_{r.mon}$ holds. Note that this result is known to be *false* in discrete-time, see for instance examples 1.1 and 4.5 in [FM01].

Then, we studied all five-points posets which are not linearly totally ordered. For some of these posets (see Figure 2 below), we found extremal rays $L = (L_{x,y})_{(x,y) \in S_2}$ of \mathcal{G}_{mon} which are not in $\mathcal{G}_{r.mon}$. One example for each poset will be given below.

In what follows, a *symmetry* of a poset S is a bijective map from S to S which is either order preserving or order reversing.

Proposition 3.1. *The only posets S with $\sharp S \leq 5$ such that (12) does not hold are, up to symmetries, those whose Hasse-Diagrams are presented in Figure 2.*

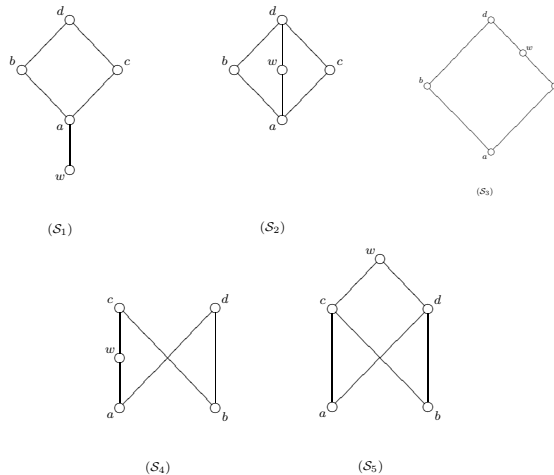


Figure 2: The five-points posets, for which there is no equivalence between the two notions of monotonicity in continuous-time.

As mentioned above, Proposition 3.1 has been obtained by exact computer-aided computations. For completeness, for each one of the posets in Figure 2, we give explicitly a generator in $\mathcal{G}_{mon} \setminus \mathcal{G}_{r.mon}$. We recall that, by Proposition 2.1, a generator L is realizably monotone if and only if, for some $\Lambda : \mathcal{M} \rightarrow \mathbb{R}^+$, we have $x, y \in S$, $x \neq y \Rightarrow L_{x,y} = \sum_{f \in \mathcal{M}: f(x)=y} \Lambda(f)$. In particular $L_{x,y} = 0 \Leftrightarrow \Lambda(f) = 0$ for all $f \in \mathcal{M}$ such that $f(x) = y$. Given a function Λ as above, we shall use the abbreviate notation $\mathcal{M}_{x \mapsto y}$ for the set $\{f \in \mathcal{M} : \Lambda(f) > 0 \text{ and } f(x) = y\}$.

Example 3.2. For $S = S_1$ there is only one extremal ray L of \mathcal{G}_{mon} which is not in $\mathcal{G}_{r.mon}$. It is given by $L_{d,c} = L_{d,b} = L_{b,w} = L_{c,w} = L_{a,w} = 1$, and $L_{x,y} = 0$ for each other pair $x, y \in S_1$ with $x \neq y$. This generator is clearly monotone (the conditions of Proposition 2.2 are easily verified), but it is not realizably monotone: indeed, if Proposition 2.1 holds, we have $L_{d,b} = \sum_{f \in \mathcal{M}_{d \rightarrow b}} \Lambda(f) = 1$ and $L_{d,c} = \sum_{f \in \mathcal{M}_{d \rightarrow c}} \Lambda(f) = 1$. Note that $\mathcal{M}_{d \rightarrow b} \cap \mathcal{M}_{d \rightarrow c} = \emptyset$. Moreover, for each $f \in \mathcal{M}_{d \rightarrow b}$, since $c < d$ and $\Lambda(f) > 0$, by monotonicity of f and the fact that $L_{c,a} = L_{c,b} = 0$, we have necessarily $f(c) = w$ and then $f(a) = w$ to maintain the ordering, *i.e.*, $f \in \mathcal{M}_{a \rightarrow w}$.

Analogously, $\mathcal{M}_{d \rightarrow c} \subset \mathcal{M}_{a \rightarrow w}$, then $1 = L_{a,w} = \sum_{f \in \mathcal{M}_{a \rightarrow w}} \Lambda(f) \geq \sum_{f \in \mathcal{M}_{d \rightarrow b} \sqcup \mathcal{M}_{d \rightarrow c}} \Lambda(f) = 2$, so we obtain a contradiction.

Example 3.3. For $S = S_2$ a generator $L \in \mathcal{G}_{mon} \setminus \mathcal{G}_{r.mon}$ is given by $L_{a,c} = L_{w,c} = L_{d,c} = L_{b,d} = L_{b,a} = 1$ and $L_{x,y} = 0$ for each other pair $x, y \in S_2$ with $x \neq y$. According to Proposition 2.2 it is monotone. Assume it is realizably monotone. If f is an increasing function on S with $f(d) = f(a) = c$, since $a < b < d$ we have $f(b) = c$. But $L_b = 0$, then necessarily $\Lambda(f) = 0$. This means that $\mathcal{M}_{d \rightarrow c} \cap \mathcal{M}_{a \rightarrow c} = \emptyset$. Moreover, $(\mathcal{M}_{d \rightarrow c} \sqcup \mathcal{M}_{a \rightarrow c}) \subset \mathcal{M}_{w \rightarrow c}$ which gives the contradiction $1 = L_{w,c} \geq L_{d,c} + L_{a,c} = 2$.

Example 3.4. For $S = S_3$, consider the monotone generator given by $L_{a,w} = L_{b,w} = L_{c,d} = L_{w,d} = L_{d,w} = 1$ and $L_{x,y} = 0$ for each other pair $x, y \in S_3$ with $x \neq y$ and suppose it is realizably monotone. If $f \in \mathcal{M}$ and $f(d) = f(a) = w$, by monotonicity we have $f(c) = w$ and then, since $L_{c,w} = 0$, we have $\Lambda(f) = 0$. Then $\mathcal{M}_{d \rightarrow w} \cap \mathcal{M}_{a \rightarrow w} = \emptyset$ and by $(\mathcal{M}_{d \rightarrow w} \sqcup \mathcal{M}_{a \rightarrow w}) \subset \mathcal{M}_{b \rightarrow w}$, it follows that $L_{b,w} \geq 2$, which gives a contradiction.

Example 3.5. For $S = S_4$ we take the monotone generator given by $L_{a,b} = L_{w,b} = L_{d,b} = L_{b,d} = L_{c,d} = 1$ and $L_{x,y} = 0$ for each other pair $x, y \in S_4$ with $x \neq y$. It is clear that, if L was realizably monotone, we should have $\mathcal{M}_{d \rightarrow b} \cap \mathcal{M}_{b \rightarrow d} = \emptyset$ and the inclusions $\mathcal{M}_{d \rightarrow b} \subset \mathcal{M}_{a \rightarrow b} \subset \mathcal{M}_{w \rightarrow b}$ and $\mathcal{M}_{b \rightarrow d} \subset \mathcal{M}_{c \rightarrow d} \subset \mathcal{M}_{w \rightarrow b}$: but it is not possible, since in that case we should have $L_{w,b} \geq 2$.

Example 3.6. For $S = S_5$ consider the monotone generator given by $L_{c,a} = L_{d,a} = L_{b,a} = L_{w,c} = L_{w,d} = 1$ and $L_{x,y} = 0$ for each other pair $x, y \in S_5$ with $x \neq y$. If $L \in \mathcal{G}_{r.mon}$, we have $\mathcal{M}_{w \rightarrow d} \cap \mathcal{M}_{w \rightarrow c} = \emptyset$, $\mathcal{M}_{w \rightarrow d} \subset \mathcal{M}_{c \rightarrow a} \subset \mathcal{M}_{b \rightarrow a}$ and $\mathcal{M}_{w \rightarrow c} \subset \mathcal{M}_{d \rightarrow a} \subset \mathcal{M}_{b \rightarrow a}$, then we obtain the contradiction $1 = L_{b,a} \geq 2$.

For the sake of completeness, for the posets considered in examples 3.2,..., 3.6 we give the number of extremal rays generating the cone $\mathcal{G}_{r.mon}$, resp. \mathcal{G}_{mon} in \mathbb{R}^{20} , see Table 1.

Poset	$\mathcal{G}_{r.mon}$	\mathcal{G}_{mon}
S_1	40	41
S_2	41	47
S_3	40	42
S_4	46	50
S_5	49	53

Table 1: $\#S = 5$: minimal number of extremal rays generating the cone $\mathcal{G}_{r.mon}$, resp. \mathcal{G}_{mon} , in \mathbb{R}^{20}

We now recall the following definition:

Definition 3.7. A subset S' of S is said to be a *sub-poset* if for all $x, y \in S'$, $x \leq y$ in S' implies $x \leq y$ in S . S' is said to be an *induced sub-poset* if for all $x, y \in S'$, $x \leq y$ in S' if and only if $x \leq y$ in S .

For $\#S = 6$ we shall see in the next section that, if S has one of the 5-points posets above as an induced sub-poset, then there is no equivalence between stochastic monotonicity and realizable monotonicity. However, there are 6-points posets for which there is no equivalence and such that we have equivalence for each one of their sub-posets.

Proposition 3.8. *The only posets S with $\sharp S = 6$ such that (12) does not hold are, up to symmetries,*

- those having one of the posets in Proposition 3.1 as induced sub-poset;
- those whose Hasse-Diagrams are presented in Figure 3.

Following the terminology of [FM01], the first poset in Figure 3 is a *double diamond* and the second poset is a *3-crown*.

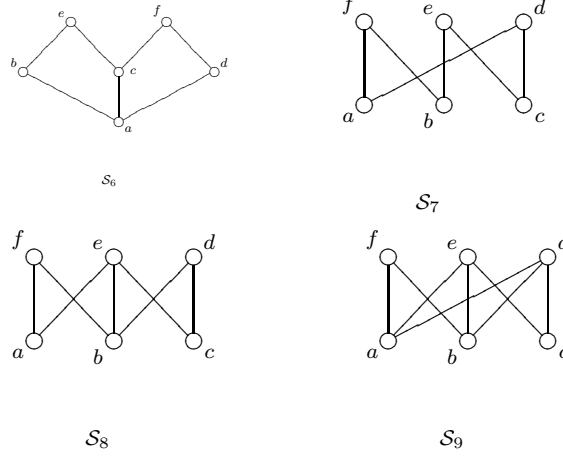


Figure 3: The double diamond S_6 and 3-crowns: S_7, S_8, S_9 .

In the next examples we shall see generators of $\mathcal{G}_{mon} \setminus \mathcal{G}_{r.mon}$ for posets S_6, S_7, S_8 and S_9 . For the posets considered in examples 3.9,... 3.12 we give the number of extremal rays generating the cone $\mathcal{G}_{r.mon}$, resp. \mathcal{G}_{mon} in \mathbb{R}^{30} in Table 2.

Poset	$\mathcal{G}_{r.mon}$	\mathcal{G}_{mon}
S_6	126	421
S_7	684	914
S_8	134	312
S_9	84	132

Table 2: $\sharp S = 6$: number of extremal rays generating the cone $\mathcal{G}_{r.mon}$, resp. \mathcal{G}_{mon} in \mathbb{R}^{30}

Example 3.9. Consider the monotone generator L on S_6 defined as follows: $L_{a,c} = L_{d,c} = L_{c,b} = L_{b,e} = L_{f,e} = 1$, and $L_{x,y} = 0$ for each other pair $x, y \in S_6$ with $x \neq y$. If L was realizably monotone we should have $\mathcal{M}_{a \rightarrow c} \cap \mathcal{M}_{c \rightarrow b} = \emptyset$, $\mathcal{M}_{a \rightarrow c} \subset \mathcal{M}_{d \rightarrow c}$ and $\mathcal{M}_{c \rightarrow b} \subset \mathcal{M}_{f \rightarrow e} \subset \mathcal{M}_{d \rightarrow c}$; but this would give the contradiction $1 = L_{d,c} \geq 2$.

Example 3.10. For $S = S_7$ we take $L \in \mathcal{G}_{mon}$ with $L_{a,c} = L_{b,c} = L_{f,c} = L_{d,c} = 1$, $L_{c,d} = L_{e,d} = 1$ and $L_{x,y} = 0$ for each other pair x, y with $x \neq y$. Suppose L is realizably monotone; then $\mathcal{M}_{d \rightarrow c} \cap \mathcal{M}_{c \rightarrow d} = \emptyset$, $\mathcal{M}_{d \rightarrow c} \subset \mathcal{M}_{a \rightarrow c}$ and $\mathcal{M}_{c \rightarrow d} \subset \mathcal{M}_{e \rightarrow d} \subset \mathcal{M}_{b \rightarrow c} \subset \mathcal{M}_{f \rightarrow c} \subset \mathcal{M}_{a \rightarrow c}$. But in this case we have $1 = L_{a,c} \geq L_{d,c} + L_{c,d} = 2$, then $L \notin \mathcal{G}_{r.mon}$.

Example 3.11. For $S = S_8$ we consider the same generator of Example 7; L is clearly monotone, but realizable monotonicity of L would imply $\mathcal{M}_{d \rightarrow c} \cap \mathcal{M}_{c \rightarrow d} = \emptyset$, $\mathcal{M}_{d \rightarrow c} \subset \mathcal{M}_{b \rightarrow c} \subset \mathcal{M}_{f \rightarrow c} \subset \mathcal{M}_{a \rightarrow c}$ and $\mathcal{M}_{c \rightarrow d} \subset \mathcal{M}_{e \rightarrow d} \subset \mathcal{M}_{a \rightarrow c}$, then $L_{ac} \geq 2$, which is not the case.

Example 3.12. For $S = S_9$ let L be defined by $L_{a,c} = L_{b,c} = L_{e,c} = 1$, $L_{b,e} = L_{f,e} = L_{d,e} = 1$, $L_{a,d} = L_{f,d} = L_{e,d} = 1$ and $L_{x,y} = 0$ for each other pair x, y with $x \neq y$. L is a monotone generator. Suppose $L \in \mathcal{G}_{r.mon}$. Then, inequalities $a < e$ and $a < f$ imply respectively $\mathcal{M}_{a \rightarrow d} \cap \mathcal{M}_{e \rightarrow c} = \emptyset$ and $\mathcal{M}_{a \rightarrow d} \subset \mathcal{M}_{f \rightarrow d}$. Note that we have also $\mathcal{M}_{e \rightarrow c} \subset \mathcal{M}_{f \rightarrow d}$: indeed $\mathcal{M}_{e \rightarrow c} \subset (\mathcal{M}_{f \rightarrow e} \cup \mathcal{M}_{f \rightarrow d})$ and, since $\mathcal{M}_{b \rightarrow e} \subset \mathcal{M}_{f \rightarrow e}$ and $L_{b,e} = L_{f,e} = 1$ we have necessarily $\mathcal{M}_{b \rightarrow e} = \mathcal{M}_{f \rightarrow e}$ and so $\mathcal{M}_{e \rightarrow c} \cap \mathcal{M}_{b \rightarrow e} = \mathcal{M}_{e \rightarrow c} \cap \mathcal{M}_{f \rightarrow e} = \emptyset$. Therefore we obtain the contradiction $L_{f,d} \geq 2$.

Remark 3.13. We recall that, in discrete-time, equivalence does not hold if the graph corresponding to the Hasse diagram of the poset has a subgraph which is a cycle (in the graph-theoretic sense). So, a look at the figures above could suggest that in continuous-time, a sufficient condition for the failure of (12) is the presence of *two* cycles in the Hasse diagram. The poset in Figure 4 (the complete 3-crown) gives a counterexample: it has more than two cycles, but for this set we have *equivalence* between the two concepts of monotonicity. In fact, more generally, if we have a poset $S = \{a_1, \dots, a_n, b_1, \dots, b_m\}$ with $a_i < b_j$ for $i = 1, \dots, n, j = 1, \dots, m$, we can show that every monotone generator on S is realizable monotone. We use an argument analogous to the one used in section 5 of [FM01]. Let L be a monotone generator on S and consider the poset $\tilde{S} = \{a_1, \dots, a_n, c, b_1, \dots, b_m\}$ with $a_i < c < b_j$ for $i = 1, \dots, n, j = 1, \dots, m$. This poset admits S as induced sub-poset and it is acyclic. Now, we take a generator \tilde{L} on \tilde{S} defined as follows: $\tilde{L}_{x,y} = L_{x,y}$ if $x, y \in S$, $\tilde{L}_{x,c} = 0$ for each $x \in S$, $\tilde{L}_{c,a_i} = \sum_{h=1}^m L_{b_h,a_i}$ for $i = 1, \dots, n$, $\tilde{L}_{c,b_j} = \sum_{k=1}^n L_{a_k,b_j}$ for $j = 1, \dots, m$. It is not hard to check that this generator is monotone, and its restriction to S is given by L . Since \tilde{S} is acyclic, then \tilde{L} is realizable monotone. But a monotone realization of \tilde{L} gives a monotone realization of L too. Therefore, L is realizable monotone.

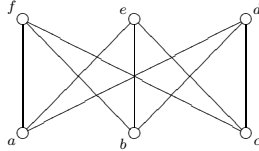


Figure 4: Complete 3-crown

4 Extensions to larger posets

In this section we prove some sufficient conditions on a poset S under which $\mathcal{G}_{mon}(S) \neq \mathcal{G}_{r.mon}(S)$. The general argument we use is analogous to the one used in the discrete-time case (see [FM01], Theorem 4.2): we take a generator L in $\mathcal{G}_{mon}(S) \setminus \mathcal{G}_{r.mon}(S)$ for a "small" poset S and we define a "monotone extension of L to a larger poset S' ", i.e., a generator $L' \in \mathcal{G}_{mon}(S')$, where S' is a poset having S as induced sub-poset, such that $L'_{xy} = L_{xy}$ for all $x, y \in S$. If this construction is possible, L' is not realizable monotone too. This is a consequence of the following Lemma

Lemma 4.1. *Let S be an induced sub-poset of a given poset S' and let L be a monotone generator on S which has a monotone extension L' to S' . Then $L' \in \mathcal{G}_{r.mon}(S') \Rightarrow L \in \mathcal{G}_{r.mon}(S)$.*

Proof. We denote by \mathcal{M}' and \mathcal{M} the sets of increasing functions on S' and S respectively. Assume $L' \in \mathcal{G}_{r.mon}(S')$. Then, by Proposition 2.1 there exists $\bar{\Lambda} : \mathcal{M}' \rightarrow \mathbb{R}^+$ such that (3) holds for L' . Let us define $\Lambda : \mathcal{M} \rightarrow \mathbb{R}^+$ by

$$\Lambda(f) = \sum_{f' \in \mathcal{M}', f'|_S = f} \bar{\Lambda}(f')$$

for each $f \in \mathcal{M}$. Now, since $L'_{x,y} = L_{x,y}$ for all $x, y \in S$, in particular we have $L'_{x,x} = L_{x,x}$ for all $x \in S$ and then necessarily $L'_{x,z} = 0$ for all $x \in S, z \in S' \setminus S$. Then, by condition (3), for every $f' \in \mathcal{M}'$ with $\bar{\Lambda}(f') > 0$ we have $f'(S) \subset S$; moreover, S is an *induced* sub-poset of S' , then we have also $f'|_S \in \mathcal{M}$. Therefore, for $x, y \in S, x \neq y$

$$\begin{aligned} \sum_{f \in \mathcal{M}: f(x)=y} \Lambda(f) &= \sum_{f \in \mathcal{M}: f(x)=y} \left(\sum_{f' \in \mathcal{M}', f'|_S = f} \bar{\Lambda}(f') \right) \\ &= \sum_{\substack{f' \in \mathcal{M}', f'(x)=y \\ f'|_S \in \mathcal{M}}} \bar{\Lambda}(f') \\ &= \sum_{f' \in \mathcal{M}'_{x \rightarrow y}} \bar{\Lambda}(f') = L'_{x,y} = L_{x,y}, \end{aligned}$$

then the proof is complete using for L Proposition 2.1. \square

As the Example below shows, Lemma 4.1 is false if S is a (not necessarily induced) sub-poset of S' , *i.e.* a subset of S' such that, for all $x, y \in S, x \leq y$ in S implies $x \leq y$ in S' .

Example 4.2. Let S be the poset S_8 of Figure 3: it is a (not induced) sub-poset of the complete crown of Figure 3, which we denote by S' . Now let us consider the generator L on S defined by $L_{f,e} = L_{b,e} = L_{d,e} = 1, L_{e,d} = L_{a,c} = L_{c,a} = 1$ and $L_{x,y} = 0$ for each other pair $x, y \in S$ with $x \neq y$. It is easy to check that L is monotone as a generator both on S and on S' . Moreover, $L \notin \mathcal{G}_{r.mon}(S)$. Indeed, if L was realizably monotone and \mathcal{M} denotes the set of increasing functions on S we should have $\mathcal{M}_{e \rightarrow d} \subset \mathcal{M}_{a \rightarrow c} \subset \mathcal{M}_{f \rightarrow e}, \mathcal{M}_{b \rightarrow e} \subset \mathcal{M}_{f \rightarrow e}$ and $\mathcal{M}_{b \rightarrow e} \cap \mathcal{M}_{e \rightarrow d} = \emptyset$ which implies the contradiction $L_{f,e} \geq 2$. On the other hand, as a generator on the complete crown, L is a monotone extension of itself and, by Remark 3.13, $L \in \mathcal{G}_{r.mon}(S')$.

It must be stressed that the method of monotone extension of generators to larger posets does not always work. First of all, note that, if $L \in \mathcal{G}_{mon}(S) \setminus \mathcal{G}_{r.mon}(S)$ and there is an *acyclic* poset S' which has S as an induced sub-poset, it is impossible to construct a monotone extension of L to S' : indeed, by Theorem 1.5 and Proposition 2.8 such an extension would be a generator of $\mathcal{G}_{r.mon}(S')$ and so, by Lemma 4.1 we should have also $L \in \mathcal{G}_{r.mon}(S)$.

As an example, consider the poset S_9 of Figure 3 and the generator L of Example 3.12. Consider the poset S'_9 (see Figure 5) obtained by adding to S_9 the points w, w_1, w_2 in such a way that $a < w_1 < w < w_2 < d, b < w_1 < f$ and $c < w_2 < e$: we obtain a 9-points poset which is *acyclic*. Therefore, it is impossible to obtain a monotone extension of L to S'_9 .

Moreover, the example below shows that, even when S is not "enlargeable" to an acyclic poset (*i.e.*, every poset having S as an induced sub-poset is *non-acyclic*), there are generators which cannot be extended to every "larger" poset.

Example 4.3. Let L be the generator on poset S_8 (Figure 3) defined in Example 4.2. Note that (see Proposition 5.11 of [FM01]) S_8 is not enlargeable to any acyclic poset. However, consider

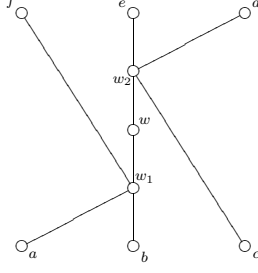


Figure 5: An extension from poset S_9 : acyclic poset S'_9

a poset S'_8 obtained from S_8 by adding a point z with $z > f$, $z > e$ and $z \not> d$. Suppose L' is a "monotone extension" of L . Since $\{d\}$ is a up-set, $L_{e,d} = 1$ and $z > e$, it must be $L'_{z,d} \geq 1$. Moreover $\Gamma := \{d, b, c\}$ is a down-set; since $L'_{z,\Gamma} \geq 1$ and $z > f$, it should be $L_{f,\Gamma} \geq 1$, which is false, being $L_{f,\Gamma} = 0$. Thus there can be no such extension.

However, as we shall see in subsection 4.2, the method of monotone extension works for S_8 , in the sense that for any poset S' having S_8 as induced sub-poset, *there exists* a generator L in S_8 which is monotone but not realizably monotone, that can be extended to a monotone generator in S' . But this generator L has to be chosen appropriately. For the reasons explained above, the extensions given in Sections 4.1 and 4.2 have to be made "case by case".

4.1 From 5-points posets to larger posets

In this subsection we show that monotonicity equivalence does not hold for any poset S' having one of the 5-points posets of Figure 2 as an induced sub-poset. Note that, in this case, S' is *non-acyclic*: this is an immediate consequence of Proposition 2.7 of [FM01].

Proposition 4.4. *If a poset S' admits as induced sub-poset a poset S , whose Hasse-Diagram is one of those in Figure 2 (up to symmetries), then monotonicity equivalence fails in S' as well.*

Proof. For each $i = 1, \dots, 5$, let S'_i be a poset which has S_i as an induced sub-poset.

According to Lemma 4.1, it is enough to show that, if we choose a monotone generator L on S_i which is not realizably monotone, then we can define a stochastically monotone generator L' on S'_i such that $L'_{x,y} = L_{x,y}$ for all $x, y \in S_i$.

In each case considered below we shall pose $\bar{S} = S'_i \setminus S_i$ and define L' in such a way that the only new transitions allowed are the ones from elements of \bar{S} to elements of S_i . In other words, for each $i = 1, \dots, 5$, we shall pose $L'_{xy} = L_{xy}$ for all $x, y \in S_i$ and $L'_{xy} = 0$ for all $x \in S'$ and $y \in \bar{S}$. Note that, if Γ' is an up-set in S'_i , then $\Gamma = \Gamma' \cap S_i$ is an up-set in S_i and by the construction of L' it follows that, for each $x \notin \Gamma'$, $L'_{x,\Gamma'} = L'_{x,\Gamma}$. The same property holds for down-sets.

Then (see Remark 2.3), in order to verify that L' is monotone, it will be sufficient to check that, for all $x, y \in S'_i$, with $x < y$, if Γ is an up-set in S_i (resp. a down-set) and $x, y \notin \Gamma$, we have $L'_{x,\Gamma} \leq L'_{y,\Gamma}$ (resp. $L'_{x,\Gamma} \geq L'_{y,\Gamma}$).

Case I. Let us consider S_1 and the generator L given in Example 3.2.

We have to define only transition rates from elements of \bar{S} to elements of S_1 . Consider the partition of S'_1 given by the sets $A = \{z \in S'_1 : z \leq b\} \cup \{z \in S'_1 : z \leq c\}$ and $B = S'_1 \setminus A$. Then, if $z \in \bar{S} \cap A$ we pose $L'_{zw} = 1$ and $L'_{zy} = 0$ for each other $y \in S_1$; if $z \in \bar{S} \cap B$, we pose $L'_{zb} = L'_{zc} = 1$ and $L'_{zy} = 0$ for each other $y \in S_1$.

L' is a monotone generator. Indeed, let us take $x, y \in S'_1$ with $x < y$. There are only three possibilities: $x, y \in B$, $x, y \in A$ or $x \in A$ and $y \in B$. If $x, y \in B$, then $L'_{xz} = L'_{yz}$ for each $z \neq x, y$ and there is nothing to verify. Now we show that if Γ is an up-set in S_1 such that $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$ then $L'_{x,\Gamma} \leq L'_{y,\Gamma}$. Suppose that $x, y \in A$; since, for $x \in A$ we have $L'_{x,\Gamma} \neq 0$ if and only if $\Gamma = S_1$, then necessarily $x, y \in \bar{S} \cap A$ and $L'_{x,S_1} = L'_{y,S_1} = 1$. For the same reason, if $x \in A$ and $y \in B$, then $\Gamma = S_1$ and $1 = L'_{x,S_1} < L'_{y,S_1} = 2$.

Analogously, consider a down-set Γ in S_1 with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$. If $x, y \in A$, since each down-set in S_1 contains w , we have $x, y \neq w$ and so $L'_{x,\Gamma} = L'_{y,\Gamma}$. If $x \in A$ and $y \in B$, then $\Gamma = \{b, a, w\}$ or $\{c, a, w\}$. Indeed, $y \in B$ and $L'_{y,\Gamma} \neq 0$, implies that $\Gamma \cap \{b, c\} \neq \emptyset$. But x is smaller or equal than at least one element of the set $\{b, c\}$ and $x \notin \Gamma$, so we cannot have $\{b, c\} \subset \Gamma$. Therefore $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$.

Case II. Consider the partition of S'_2 given by the sets $D = \{z \in S'_2 : z \leq b\} \cup \{z \in S'_2 : z \leq w\} \cup \{z \in S'_2 : z \leq c\}$ and $D^c = S'_2 \setminus D$. We pose, for $z \in \bar{S} \cap D$, $L'_{z,a} = L'_{z,c} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_2$. If $z \in \bar{S} \cap D^c$, we pose $L'_{z,c} = L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_2$. Let us take $x, y \in S'_2$ with $x < y$ and let Γ be an up-set in S_2 such that $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$. Suppose first that $x, y \in D$. Since $L'_{x,\Gamma} \neq 0$ we must have $c \in \Gamma$ which implies $\{c, d\} \subset \Gamma$. Now, if $x = a$ and $y \notin \{b, w\}$, we have $\Gamma \neq S_2$ and $L_{x,\Gamma} = 1 = L_{y,\Gamma}$; if $x \neq a$ and $y \in \{b, w\}$ we have $\{a, b, w\} \notin \Gamma$ and so $L_{x,\Gamma} = 1 = L_{y,\Gamma}$. Now, suppose $x, y \in D^c$. Since $L'_{x,\Gamma} \neq 0$ we must have $d \in \Gamma$ and $x, y \neq d$: this means that $x, y \in \bar{S} \cap D^c$ and so $L_{x,\Gamma} = L_{y,\Gamma}$. Finally, suppose that x and y are not in the same subset of the given partition. This means that $x \in D$ and $y \in D^c$. As we saw above $L'_{x,\Gamma} \neq 0 \Rightarrow \{c, d\} \subset \Gamma$, then necessarily $y \neq d$ and $L_{y,\Gamma} = 2 \geq L_{x,\Gamma}$. In order to check monotonicity of L' , we should consider also down-sets of S_2 , but in this case the argument for down-sets is symmetric to the one given above for up-sets.

Case III. For the poset S_3 we consider the generator L of Example 3.4 and take the partition of S'_3 given by $A = \{z \in S'_3 : z \leq b\} \cup \{z \in S'_3 : z < c\}$ and $B = S'_3 \setminus A$. Then, if $z \in \bar{S} \cap A$ we pose $L'_{z,w} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_3$; if $z \in \bar{S} \cap B$, we pose $L'_{z,w} = L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_3$.

In order to verify monotonicity of L' , take $x, y \in S'_3$ with $x < y$. Note that $y \in A \Rightarrow x \in A$, therefore we can have $x, y \in A$, $x, y \in B$ or $x \in A, y \in B$. If $x, y \in A$, then $L'_{x,z} = L'_{y,z}$ for each $z \neq x, y$ and there is nothing to check. Suppose that $x \in A, y \in B$. Let us take an up-set Γ in S_3 with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$; then, $\{w, d\} \subset \Gamma$ and $L'_{x,\Gamma} = 1 \leq L'_{y,\Gamma}$. If we take a down-set Γ in S_3 with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, since $x \in A \Rightarrow x < d$, we have $d \notin \Gamma$, therefore $L'_{y,\Gamma} = 1 = L'_{x,\Gamma}$. Now, suppose that $x, y \in B$. Let us take an up-set Γ with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$. Then, $x, y \neq d$ and, if $w \notin \Gamma$, $L'_{x,\Gamma} = L'_{y,\Gamma}$. If $\{w, d\} \subset \Gamma$ we have $x \neq w, d$ and (since $y = c \Rightarrow x \in A$) $y \neq w, d, c$, therefore $L'_{y,\Gamma} = 2 \geq L'_{x,\Gamma}$. If Γ is a down-set with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, then $\Gamma = S_3$ or $\{w, a, c\} \subset \Gamma$. If $\Gamma = S_3$, then $L'_{x,\Gamma} = L'_{y,\Gamma} = 2$. If $\{w, a, c\} \subset \Gamma$ we have $x, y \neq w, a, c$ and so $L'_{x,\Gamma} = L'_{y,\Gamma} = 1$.

Case IV. For S_4 take the generator L given in Example 3.5. Then we consider the partition of S'_4 given by $A = \{z \in S'_4 : b \leq z \leq d\}$, $B = \{z \in S'_4 : z \geq b\} \setminus A$ and $C = S'_4 \setminus (A \cup B)$. Then, if $z \in \bar{S} \cap A$ we pose $L'_{z,b} = L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_4$; if $z \in \bar{S} \cap B$, we pose $L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_4$; if $z \in C$ we pose $L'_{z,b} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_4$.

Now we take $x, y \in S'_4$ with $x < y$. Suppose that $x \in A$; then $y > x \Rightarrow y \in A \cup B$. If Γ is an up-set in S_4 with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$, then $x, y \neq d$ and $x \notin \Gamma \Rightarrow b \notin \Gamma$, therefore $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$. If Γ is a down-set in S_4 with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, note that, $d \in \Gamma \Rightarrow x \in \Gamma$,

therefore $\Gamma \cap \{b, d\} = \{b\}$ and so $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$. Suppose now that $x \in B$; since $y > x$ we have also $y \in B$ and there is nothing to verify. Finally, for $x \in C$, if Γ is an up-set with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$ then $b \in \Gamma$, and so $\{b, d\} \subset \Gamma$, which implies that $y \not\leq b$, *i.e.* $y \in C$ and so $L'_{x,\Gamma} = L'_{y,\Gamma}$. If Γ is a down-set with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, then $b \in \Gamma$, $y \in A \cup C$. Moreover, if $y \in A$ we have $d \notin \Gamma$, therefore $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$.

Case V. Consider the poset S_5 and the generator L of Example 3.6. We take the partition of S'_5 given by $A = \{z \in S'_5 : z \leq c\} \cup \{z \in S'_5 : z \leq d\}$, $B = S'_5 \setminus A$. Then, if $z \in \bar{S} \cap A$ we pose $L'_{z,a} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_5$; if $z \in \bar{S} \cap B$, we pose $L'_{z,c} = L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_5$.

Now we consider $x, y \in S'_5$ with $x < y$. If both x, y belong to A or B , then $L'_{x,z} = L'_{y,z}$ for each $z \neq x, y$, therefore there is nothing to verify. If x, y are not in the same set of the partition, since $y \in A \Rightarrow x \in A$, we can have only $x \in A$ and $y \in B$. Suppose that Γ is an up-set in S_5 with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$. Then $a \in \Gamma$ and so $c, d \in \Gamma$, which implies that $L'_{y,\Gamma} = 2 \geq L'_{x,\Gamma}$. If Γ is a down-set in S_5 with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, we have $\{c, d\} \cap \Gamma \neq \emptyset$. Moreover, since $x \leq c$ or $x \leq d$ and $x \notin \Gamma$, we have $\Gamma = \{c, a, b\}$ or $\Gamma = \{d, a, b\}$ and in both cases we have $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$. \square

4.2 From 6-points posets to larger posets

As we saw in the preceding subsection, for a poset of cardinality 6 having one of the 5-points posets of Figure 2 as an induced sub-poset, there is not equivalence between stochastic monotonicity and realizable monotonicity. Therefore, the only six-points posets for which we have to construct monotone extensions are posets S_6, S_7, S_8 of Figure 3.

Proposition 4.5. *If a poset S' admits as induced sub-poset a poset S , whose Hasse-Diagram is one of the posets S_6, S_7, S_8 of Figure 3 then monotonicity equivalence fails in S as well.*

Proof. Case I. As we did in the preceding section, we call S' a poset which has S_6 (the double diamond of Figure 2) as an induced sub-poset, $\bar{S} = S' \setminus S_6$, and we take the monotone generator L on S_6 defined in Example 3.9. This generator is not realizable monotone.

Now, we want to define L' on S' as a monotone extension of L . We consider the partition of S' given by the sets $A = \{z \in S' : z > b\} \cup \{z \in S' : z > c\} \cup \{z \in S' : z > d\}$ and $B = S' \setminus A$. Then, if $z \in \bar{S} \cap A$ we pose $L'_{z,e} = 1$ and $L'_{z,y} = 0$ for each other $y \in S'$; if $z \in \bar{S} \cap B$, we pose $L'_{z,a} = L'_{z,c} = 1$ and $L'_{z,y} = 0$ for each other $y \in S'$.

L' is a monotone generator. Indeed, let us suppose that $x, y \in S'$ with $x < y$ and at least one of them does not belong to S_6 . We have only three possibilities: $x, y \in A$, $x, y \in B$ or $x \in B$ and $y \in A$. If $x, y \in A$, and $x, y \neq e$, there is nothing to verify, since x and y make the same transitions with the same rate. On the other hand, if $x = e$ or $y = e$, then, for every up-set (down-set) Γ with $x, y \notin \Gamma$ we have $e \notin \Gamma$ and so $L'_{x,\Gamma} = L'_{y,\Gamma} = 0$.

If $x, y \in B$ we have to consider only the cases in which x and y make different transitions, *i.e.* when $y \in \{a, b, c, d\}$ and $x \neq a$ or when $y \notin \{a, b, c, d\}$ and $x = a$.

In the first case, if we take an up-set Γ with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$, then, since $y \geq a$ we have $a \notin \Gamma$ and $\{c, e\} \subset \Gamma$. This implies $L'_{x,\Gamma} = L'_{y,\Gamma} = 1$; if we take a down-set Γ with $x, y \notin \Gamma$, then $a \in \Gamma$ and $L'_{x,\Gamma} \geq 1 \geq L'_{y,\Gamma}$. In the second case we have $x = a$, then, for any up-set Γ with $y, a \notin \Gamma$ we have $L'_{a,\Gamma} \neq 0 \Rightarrow c \in \Gamma \Rightarrow L'_{a,\Gamma} = L'_{y,\Gamma} = 1$; on the other hand, for each down-set Γ in S' with $L'_{y,\Gamma} \neq 0$ we have $x = a \in \Gamma$ and there is nothing to verify.

Now, suppose $x \in B$ and $y \in A$. If Γ is an up-set with $L'_{x,\Gamma} \neq 0$ and $x, y \notin \Gamma$, since $y > a$, we have $a \notin \Gamma$, so $L'_{x,\Gamma} \leq 1$ and $L'_{x,\Gamma} = 1$ if and only if $\{c, e\} \subset \Gamma$ which implies $L'_{y,\Gamma} = 1$; if Γ is a down-set with $L'_{y,\Gamma} \neq 0$, we have necessarily $e \in \Gamma$, which implies $a, c \in \Gamma$, then $2 = L'_{x,\Gamma} \geq L'_{y,\Gamma}$.

Cases II, III. Now, consider the monotone generator L on S_7 given in Example 3.10. Note that L has the same property also as a generator on the poset S_8 . If in the proof which follows we consider S_8 instead of S_7 , we obtain the same result.

Let S'_7 a poset which has S_7 as induced sub-poset and $\bar{S} = S'_7 \setminus S_7$. We take the partition of S'_7 given by $A = \{z \in S'_7 : c \leq z \leq d\}$, $B = \{z \in S'_7 : z \geq c\} \setminus A$, $C = S'_7 \setminus (A \cup B)$ and we define a monotone extension L' of L as follows: if $z \in \bar{S} \cap A$ we pose $L'_{z,c} = L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_7$; if $z \in \bar{S} \cap B$, we pose $L'_{z,d} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_7$; if $z \in C$ we pose $L'_{z,c} = 1$ and $L'_{z,y} = 0$ for each other $y \in S_7$.

Now we take $x, y \in S'_7$ with $x < y$. Suppose that $x \in A$; then $y > x \Rightarrow y \in A \cup B$. If Γ is an up-set in S_7 with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$, then $x, y \neq d$ and $x \notin \Gamma \Rightarrow c \notin \Gamma$, therefore $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$. If Γ is a down-set in S_7 with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, note that, $d \in \Gamma \Rightarrow x \in \Gamma$, therefore $\Gamma \cap \{c, d\} = \{c\}$ and so $L'_{x,\Gamma} = 1 = L'_{y,\Gamma}$. Suppose that $x \in B$; since $y > x$ we have also $y \in B$ and there is nothing to verify. Finally, for $x \in C$, if Γ is an up-set with $x, y \notin \Gamma$ and $L'_{x,\Gamma} \neq 0$ then $c \in \Gamma$, and so $\{c, d\} \subset \Gamma$, which implies that $y \not\geq c$, *i.e.* $y \in C$ and so $L'_{x,\Gamma} = L'_{y,\Gamma}$. If Γ is a down-set with $x, y \notin \Gamma$ and $L'_{y,\Gamma} \neq 0$, then $c \in \Gamma$ and $L'_{x,\Gamma} = 1$. On the other hand, if $y \in A$, then necessarily $d \notin \Gamma$, therefore in any case we have $L'_{x,\Gamma} = 1 \geq L'_{y,\Gamma}$. \square

Remark 4.6. The procedure used in Case II can be applied also to show that monotonicity equivalence fails for every poset which has a k -crown (see Figure 6) with $k \geq 3$ as induced sub-poset.

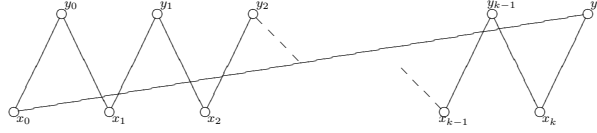


Figure 6: k -crown

Let S be a k -crown and $\{x_0, \dots, x_k\}$, resp. $\{y_0, \dots, y_k\}$, be the sets of its minimal, resp. maximal, elements (with $x_k < y_k$, $x_k < y_{k-1}$, $x_{k-1} < y_{k-1}$, $x_{k-1} < y_{k-2} \dots x_0 < y_0$, $x_0 < y_k$). The generator defined by $L_{x_k, y_k} = L_{y_{k-1}, y_k} = 1$, $L_{x_i, x_k} = 1$ for $i = 0, \dots, k-1$, $L_{y_k, x_k} = 1$, $L_{y_i, x_k} = 1$ for $i = 0, \dots, k-2$ and $L_{x, y} = 0$ for each other pair $x, y \in S$ with $x \neq y$, is monotone. Suppose L is realizably monotone. Then $\mathcal{M}_{x_k \mapsto y_k} \subset \mathcal{M}_{y_{k-1} \mapsto y_k} \subset \mathcal{M}_{x_{k-1} \mapsto x_k}$ and $\mathcal{M}_{y_k \mapsto x_k} \subset \mathcal{M}_{x_0 \mapsto x_k} \subset \mathcal{M}_{y_0 \mapsto x_k} \subset \dots \subset \mathcal{M}_{x_{k-1} \mapsto x_k}$. Therefore, since $\mathcal{M}_{x_k \mapsto y_k} \cap \mathcal{M}_{y_k \mapsto x_k} = \emptyset$ we obtain $L_{x_{k-1}, x_k} \geq 2$, which gives a contradiction.

Now, as we did for the 3-crown, we consider a poset S' which has a k -crown as induced sub-poset, we pose $\bar{S} = S' \setminus S$ and, in order to construct a monotone extension of the generator L given above, we take the partition of S' given by $A = \{z \in S' : x_k \leq z \leq y_k\}$, $B = \{z \in S' : z \geq x_k\} \setminus A$, $C = S' \setminus (A \cup B)$. For $z \in \bar{S} \cap A$ we pose $L'_{z, x_k} = L'_{z, y_k} = 1$ and $L'_{z, w} = 0$ for each other $w \in S$; if $z \in \bar{S} \cap B$, we pose $L'_{z, y_k} = 1$ and $L'_{z, w} = 0$ for each other $w \in S$; if $z \in C$ we pose $L'_{z, x_k} = 1$ and $L'_{z, w} = 0$ for each other $w \in S$. The same arguments used above for the 3-crown show that L' is monotone.

5 Conclusions

In this paper we have obtained partial results concerning the relationship between stochastic monotonicity and realizable monotonicity for continuous-time Markov chains on partially ordered sets. We have provided sufficient conditions on the poset for the monotonicity equivalence to hold or to fail, and given complete classifications for posets of cardinality ≤ 6 . Unlike what

Fill and Machida have obtained in the discrete-time case, we have not been able to find a characterization of posets for which monotonicity equivalence holds, in terms of their Hasse diagram. We remark, as the example in Figure 5 shows, that there are posets with an acyclic extension for which monotonicity equivalence fails. Therefore, in general, non-equivalence is not preserved by extending the poset.

For posets with no acyclic extensions, we believe the following fact holds true.

Conjecture. Let S be a connected poset having no acyclic extension. Then monotonicity equivalence holds if and only if the following conditions hold:

- i) the Hasse diagram of S has a unique cycle, which is a diamond;
- ii) S has no Y -shaped subposet (see Figure 7) having at most one point in common with the cycle in point i) and there is no induced subposet of the types from Figure 8 (up to symmetries).

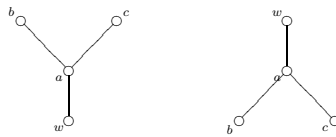


Figure 7: Y shapes

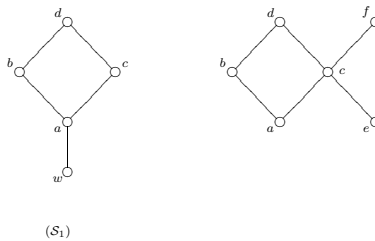


Figure 8: Forbidden posets

The necessity of conditions i) and ii) should actually be not too hard to prove, although many different cases have to be considered. We have tried harder to prove sufficiency of i), ii) by induction on the cardinality of the poset, but, unfortunately, we have not succeeded.

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References

- [Arn98] L. Arnold, *Random dynamical systems*, Springer, Berlin Heidelberg New York, 1998.
- [Bré99] P. Brémaud, *Markov chains, gibbs fields, monte carlo simulation, and queues*, Springer-Verlag, New York, 1999.
- [BM02] G. Brinkmann and B. D. McKay, *Posets on up to 16 points*, Order **19** (2002), 147–179.
- [cdd] *cdd/cddplus, 1995–2008*, Software, available at http://www.ifor.math.ethz.ch/~fukuda/cdd_home/.
- [Che91] Mu Fa Chen, *On coupling of jump processes*, Chin. Ann. of Math. **12B** (1991), no. 4, 385–399.
- [D97] D. Dubischar. *Representation of Markov Processes by Random Dynamical Systems*. Report 393, Institut für Dynamische Systeme, Universität Bremen, 1997.
- [FYMP97] W. Feng Yu and X. Mei Ping, *On order-preservation of couplings for multidimensional diffusion processes*, Chin. J. of Appl. Proba. and Stat. **13** (1997), no. 2, 142–148.
- [FG] P. A. Ferrari and A. Galves, *Construction of stochastic processes, coupling and regeneration*, Sociedad Venezolana de Matematicas, 2000
- [FM01] J. A. Fill and M. Machida, *Stochastic monotonicity and realizable monotonicity*, Ann. Probab. **29** (2001), no. 2, 938–978.
- [FF97] F. Forbes and O. François, *Stochastic comparison for markov processes on a product of partially ordered sets*, Stat. and Probab. Letters **33** (1997), 309–320.
- [FP96] K. Fukuda and A. Prodon, *Double description method revisited*, Combinatorics and computer science (Brest, 1995), Lecture Notes in Comput. Sci., vol. 1120, Springer, Berlin, 1996, pp. 91–111.
- [GAP] *Gap software - groups, algorithms, programming - a system for computational discrete algebra, 1986–2006.*, Software, available at <http://www.gap-system.org/>.
- [KKO77] T. Kamae, U. Krengel, and G.L. O'Brien, *Stochastic inequalities on partially ordered spaces*, Ann. Probab. **5** (1977), no. 6, 899–912.
- [Lin92] T. Lindvall, *Lectures on the coupling method*, John Wiley & Sons Inc., New-York , 1992, ISBN 0-471-54025-0
- [LS98] F. J. López and G. Sanz, *Stochastic comparisons and couplings for interacting particle systems*, Statist. Probab. Lett. **40** (1998), 93–102.
- [Lou05] P.-Y. Louis, *Increasing coupling of probabilistic cellular automata*, Statist. Probab. Lett. **74** (2005), 1–13.
- [Mas87] W. A. Massey, *Stochastic orderings for Markov processes on partially ordered spaces*, Math. Oper. Res. **12** (1987), no. 2, 350–367.
- [Min04] I. G. Minelli, *Pathwise convergence to equilibrium of the supercritical contact process*, Ph.D. thesis, Università di Padova, 2004.

- [pos] *The online encyclopedia of integer sequences*, Website, available at <http://www.research.att.com/~njas/sequences/?q=posets&language=english&go=Search>.
- [PW96] J.G. Propp and D.B. Wilson, *Exact sampling with coupled Markov chains and applications to statistical mechanics*, *Random Structures and Algorithms* **9** (1996), no. 1&2, 223–252.
- [Zie95] G. Ziegler, *Lectures on polytopes*, *Graduate Texts in Mathematics*, vol. 152, Springer-Verlag, New York, 1995.