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Non-Degeneracy of Kobayashi Volume Measures for Singular Directed Varieties

YA DENG

Abstract

In this note, we prove the *generic* Kobayashi volume measure hyperbolicity of singular directed varieties (X, V) , as soon as the canonical sheaf \mathcal{K}_V of V is big in the sense of Demailly.

1 Introduction

Let (X, V) be a *complex directed manifold*, i.e. X is a complex manifold equipped with a holomorphic subbundle $V \subset T_X$. Demailly's philosophy in introducing directed manifolds is that, there are certain functorial constructions which work better in the category of directed manifolds (ref. [Dem12]), even in the “absolute case”, i.e. the case $V = T_X$. Therefore, it is usually inevitable to allow singularities of V , and V can be seen as a coherent subsheaf of T_X such that T_X/V is torsion free. In this case V is a subbundle of T_X outside an analytic subset of codimension at least 2, which is denoted by $\text{Sing}(V)$. The Kobayashi volume measure can also be defined for (singular) directed manifolds.

Definition 1.1. Let (X, V) be a directed manifold with $\dim(X) = n$ and $\text{rank}(V) = r$. Then the Kobayashi volume measure of (X, V) is the pseudometric defined on any $\xi \in \wedge^r V_x$ for $x \notin \text{Sing}(V)$, by

$$\mathbf{e}_{X,V}^r(\xi) := \inf\{\lambda > 0; \exists f : \mathbb{B}_r \rightarrow X, f(0) = x, \lambda f_*(\tau_0) = \xi, f_*(T_{\mathbb{B}_r}) \subset V\},$$

where \mathbb{B}_r is the unit ball in \mathbb{C}^r and $\tau_0 = \frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}$ is the unit r -vector of \mathbb{C}^r at the origin. We say that (X, V) is *generic volume measure hyperbolic* if $\mathbf{e}_{X,V}^r$ is generically positive definite.

In [Dem12] the author also introduced the definition of *canonical sheaf* \mathcal{K}_V for any singular directed variety (X, V) , and he showed that the “bigness” of \mathcal{K}_V will imply that, any non-constant entire curve $f : \mathbb{C} \rightarrow (X, V)$ must satisfy certain global algebraic differential equations. In this note, we will study the Kobayashi volume measure of the singular directed variety (X, V) , and give another intrinsic explanation for the bigness of \mathcal{K}_V . Our main theorem is as follows:

Theorem 1.1. Let (X, V) be a compact complex directed variety (V is singular) with $\text{rank}(V) = r$ and $\dim(X) = n$. If V is of general type (see Definition 2.1 below), with the base locus $\text{Bs}(V) \subsetneq X$ (see also Definition 2.1), then Kobayashi volume measure of (X, V) is non-degenerate outside $\text{Bs}(V)$. In particular (X, V) is generic volume measure hyperbolic.

Remark 1.1. In the absolute case, Theorem 1.1 is proved in [Gri71] and [KO71]; for the smooth directed variety it is proved in [Dem12].

2 Proof of main theorem

Proof. Since the singular set $\text{Sing}(V)$ of V is an analytic set of codimension ≥ 2 , the top exterior power $\wedge^r V$ of V is a coherent sheaf of rank 1, and is included in its bidual $\wedge^r V^{**}$, which is an invertible sheaf (of course, a line bundle). We will give an explicit construction of the multiplicative cocycle which represent the line bundle $\wedge^r V^{**}$.

Since $V \subset T_X$ is a coherent sheaf, we can take an open covering $\{U_\alpha\}$ satisfying that on each U_α one can find sections $e_1^{(\alpha)}, \dots, e_{k_\alpha}^{(\alpha)} \in \Gamma(U_\alpha, T_X|_{U_\alpha})$ which generate the coherent sheaf V on U_α . Thus the sections $e_{i_1}^{(\alpha)} \wedge \dots \wedge e_{i_r}^{(\alpha)} \in \Gamma(U_\alpha, \wedge^r T_X|_{U_\alpha})$ with (i_1, \dots, i_r) varying among all r -tuples of $(1, \dots, k_\alpha)$ generate the coherent sheaf $\wedge^r V|_{U_\alpha}$, which is a subsheaf of $\wedge^r T_X|_{U_\alpha}$. Let $v_I^{(\alpha)} := e_{i_1}^{(\alpha)} \wedge \dots \wedge e_{i_r}^{(\alpha)}$, then by $\text{Cod}(\text{Sing}(V)) \geq 2$ we know that the common zero of all $v_I^{(\alpha)}$ is contained in $\text{Sing}(V)$, and thus all $v_I^{(\alpha)}$ are proportional outside $\text{Sing}(V)$. Therefore there exists a unique $v_\alpha \in \Gamma(U_\alpha, \wedge^r T_X|_{U_\alpha})$, and holomorphic functions $\{\lambda_I\}$ which do not have common factors, such that $v_I^{(\alpha)} = \lambda_I v_\alpha$ for all I . By this construction we can show that on $U_\alpha \cap U_\beta$, v_α and v_β coincide up to multiplication by a nowhere vanishing holomorphic function, i.e.

$$v_\alpha = g_{\alpha\beta} v_\beta$$

on $U_\alpha \cap U_\beta \neq \emptyset$, where $g_{\alpha\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$. This multiplicative cocycle $\{g_{\alpha\beta}\}$ corresponds to the line bundle $\wedge^r V^{**}$. Then fix a Kähler metric ω on X , it will induce a metric H_r on $\wedge^r T_X$ and thus also induce a singular hermitian metric h_s of $\wedge^r V^{***}$ whose local weight φ_α is equal to $\log |v_\alpha|_{H_r}^2$. It is easy to show that h_s has *analytic singularities*, and the set of singularities $\text{Sing}(h_s) = \text{Sing}(V)$. Indeed, we have $\text{Sing}(h_s) = \cup_\alpha \{p \in U_\alpha | v_\alpha(p) = 0\}$. Now we make the following definition.

Definition 2.1. With the notions above, V is called to be *of general type* if there exists a singular metric h on $\wedge^r V^{***}$ with analytic singularities satisfying the following two conditions:

- (1) The curvature current $\Theta_h \geq \epsilon\omega$, i.e., it is a Kähler current.
- (2) h is more singular than h_s , i.e., there exists a globally defined quasi-psh function χ which is bounded from above such that

$$e^\chi \cdot h = h_s.$$

Since h and h_s has both analytic singularities, χ also has analytic singularities, and thus e^χ is a continuous function. Moreover, $e^{\chi(p)} > 0$ if $p \notin \text{Sing}(h)$. We define *the base locus of V* to be

$$\text{Bs}(V) := \cap_h \text{Sing}(h),$$

where h varies among all singular metric on $\wedge^r V^{***}$ satisfying the Properties (1) and (2) above.

Now fix a point $p \notin \text{Bs}(V)$, then by Definition 2.1 we can find a singular metric h with analytic singularities satisfying the Property (1) and (2) above, and $p \notin \text{Sing}(h)$.

Let f be any holomorphic map from the unit ball $\mathbb{B}_r \subset \mathbb{C}^r$ to (X, V) such that $f(0) = p$, then locally we have

$$f_* \left(\frac{\partial}{\partial t_1} \wedge \dots \wedge \frac{\partial}{\partial t_r} \right) = a^{(\alpha)}(t) \cdot v_\alpha|_f,$$

where $a^{(\alpha)}(t)$ is meromorphic functions, with poles contained in $f^{-1}(\text{Sing}(V))$, and satisfies

$$|f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right)|_{\omega}^2 = |a^{(\alpha)}(t)|^2 \cdot |v_{\alpha}|_{H^r}^2 \leq C.$$

Since $f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right)$ can be seen as a (meromorphic!) section of $f^* \wedge^r V^{**}$, then we define

$$\delta(t) := |f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right)|_{h^{-1}}^2 = |a^{(\alpha)}(t)|^2 \cdot e^{\phi_{\alpha}},$$

where ϕ_{α} is the local weight of h . By Property (2) above, we have a globally defined quasi-psh function χ on X which is bounded from above such that

$$\delta(t) = e^{\chi} \cdot |f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right)|_{\omega}^2 \leq C_1. \quad (2.1)$$

Now we define a semi-positive metric $\tilde{\gamma}$ on \mathbb{B}_r by putting $\tilde{\gamma} := f^*\omega$, then we have

$$\frac{|f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right)|_{\omega}^2}{\det \tilde{\gamma}} \leq C_0(f(t)), \quad (2.2)$$

where $C_0(z)$ is a bounded function on X which does not depend on f , and we take C_2 to be its upper bound. One can find a conformal $\lambda(t)$ to define $\gamma = \lambda \tilde{\gamma}$ satisfying

$$\det \gamma = \delta(t)^{\frac{1}{2}}.$$

Combined (2.1) and (2.2) together we obtain

$$\lambda \leq C_2^{\frac{1}{2}} \cdot e^{\frac{\chi}{2r}}.$$

Since $\Theta_h \geq \epsilon\omega$, by (2.1) we have

$$\text{dd}^c \log \det \gamma \geq \frac{\epsilon}{2} f^*\omega = \frac{\epsilon}{2\lambda} \gamma \geq \frac{\epsilon}{2C_2^{\frac{1}{2}}} e^{-\frac{\chi}{2r}} \gamma.$$

By Property (2) in Definition 2.1 of h , there exists a constant $C_3 > 0$ such that

$$e^{-\frac{\chi}{2r}} \geq C_3.$$

Let $A := \frac{\epsilon C_3}{2C_2^{\frac{1}{2}}}$, and we know that it is a constant which does not depend on f . Then by Ahlfors-Schwarz Lemma (see Lemma 2.1 below) we have

$$\delta(0) \leq \left(\frac{r+1}{A}\right)^{2r}.$$

Since $p \notin \text{Sing}(h)$, then we have $e^{\chi(p)} > 0$, and thus

$$|f_*\left(\frac{\partial}{\partial t_1} \wedge \cdots \wedge \frac{\partial}{\partial t_r}\right)|_{\omega}^2(0) \leq e^{-\chi(p)} \delta(0) = e^{-\chi(p)} \cdot \left(\frac{r+1}{A}\right)^{2r}.$$

By the arbitrariness of f , and Definition 1.1, we conclude that (X, V) is generic volume measure hyperbolic and $\mathbf{e}_{X,V}^r$ is non-degenerate outside $\text{Bs}(V)$. \square

Lemma 2.1 (Ahlfors-Schwarz). Let $\gamma = \sqrt{-1} \sum \gamma_{jk}(t) dt_j \wedge dt_k$ be an almost everywhere positive hermitian form on the ball $B(0, R) \subset \mathbb{C}^r$ of radius R , such that

$$-\text{Ricci}(\gamma) := \sqrt{-1} \partial \bar{\partial} \log \det \gamma \geq A \gamma$$

in the sense of currents for some constant $A > 0$. Then

$$\det(\gamma)(t) \leq \left(\frac{r+1}{AR^2}\right)^r \frac{1}{\left(1 - \frac{|t|^2}{R^2}\right)^{r+1}}.$$

Remark 2.1. If V is regular, then V is of general type if and only if $\wedge^r V$ is a big line bundle. In this situation, the base locus $\text{Bs}(V) = \mathbf{B}_+(\wedge^r V)$, where $\mathbf{B}_+(\wedge^r V)$ is the *augmented base locus* for the big line bundle $\wedge^r V$ (ref. [Laz04]).

With the notions above, we define the coherent ideal sheaf $\mathcal{I}(V)$ to be germ of holomorphic functions which is locally bounded with respect to h_s , i.e., $\mathcal{I}(V)$ is the integral closure of the ideal generated by the coefficients of v_α in some local trivialization in $\wedge^r T_X$. Let $\wedge^r V^{***}$ be denoted by K_V , and $\mathcal{K}_V := K_V \otimes \mathcal{I}(V)$, then \mathcal{K}_V is the *canonical sheaf* of (X, V) defined in [Dem12]. It is easy to show that the zero scheme of $\mathcal{I}(V)$ is equal to $\text{Sing}(h_s) = \text{Sing}(V)$. \mathcal{K}_V is called to be a *big sheaf* iff for some log resolution $\mu: \tilde{X} \rightarrow X$ of $\mathcal{I}(V)$ with $\mu^* \mathcal{I}(V) = \mathcal{O}_{\tilde{X}}(-D)$, $\mu^* K_V - D$ is big in the usual sense. Now we have the following statement:

Proposition 2.1. V is of general type if and only if \mathcal{K}_V is big. Moreover, we have

$$\text{Bs}(V) \subset \mu(\mathbf{B}_+(\mu^* K_V - D)) \cup \text{Sing}(h_s) = \mu(\mathbf{B}_+(\mu^* K_V - D)) \cup \text{Sing}(V).$$

Proof. From Definition 2.1, the condition that \mathcal{K}_V is a big sheaf implies that K_V and $\mu^* K_V - D$ are both big line bundles. For $m \gg 0$, we have the following isomorphism

$$\mu^* : H^0(X, (mK_V - A) \otimes \mathcal{I}(V)^m) \xrightarrow{\sim} H^0(\tilde{X}, m\mu^* K_V - \mu^* A - mD). \quad (2.3)$$

Fix an very ample divisor A , then for $m \gg 0$, the base locus (in the usual sense) $\mathbf{B}(m\mu^* K_V - mD - \mu^* A)$ is stably contained in $\mathbf{B}_+(\mu^* K_V - D)$ (ref. [Laz04]). Thus we can take a $m \gg 0$ to choose a basis $s_1, \dots, s_k \in H^0(\tilde{X}, m\mu^* K_V - mD - \mu^* A)$, whose common zero is contained in $\mathbf{B}_+(\mu^* K_V - D)$. Then by (2.3) there exists $\{e_i\}_{1 \leq i \leq k} \subset H^0(X, (mK_V - A) \otimes \mathcal{I}(V)^m)$ such that

$$\mu^*(e_i) = s_i.$$

Then we can define a singular metric h_m on $mK_V - A$ by putting $|\xi|_{h_m}^2 := \frac{|\xi|^2}{\sum_{i=1}^k |e_i|^2}$ for $\xi \in (mK_V - A)_x$. We take a smooth metric h_A on A such that the curvature $\Theta_A \geq \epsilon \omega$ is a smooth Kähler form. Then $h := (h_m h_A)^{\frac{1}{m}}$ defines a singular metric on K_V with analytic singularities, such that its curvature current $\Theta_h \geq \frac{1}{m} \Theta_A$. From the construction we know that h is more singular than h_s , and $\text{Sing}(h) \subset \mu(\mathbf{B}_+(\mu^* K_V - D)) \cup \text{Sing}(h_s)$. \square

Remark 2.2. From Proposition 2.1 we can take Definition 2.1 as another equivalent definition of the bigness of \mathcal{K}_V , and it is more analytic. From Theorem 1.1 we can replace the condition that V is of general type by the bigness of \mathcal{K}_V , and it means that the definition of canonical sheaf of singular directed varieties is very natural.

A direct consequence of Theorem 1.1 is the following corollary, which was suggested in [GPR13]:

Corollary 2.1. Let (X, V) be directed varieties with $\text{rank}(V) = r$, and f be a holomorphic map from \mathbb{C}^r to (X, V) with generic maximal rank. Then if \mathcal{K}_V is big, the image of f is contained in $\text{Bs}(V) \subsetneq X$.

The famous conjecture by Green-Griffiths stated that in the absolute case the converse of Theorem 1.1 should be true. It is natural to ask whether we have similar results for any directed varieties. A result by Marco Brunella (ref. [Bru10]) gives a weak converse of Theorem 1.1 for 1-directed variety:

Theorem 2.1. Let X be a compact Kähler manifold equipped with a singular holomorphic foliation \mathcal{F} by curves. Suppose that \mathcal{F} contains at least one leaf which is hyperbolic, then the canonical bundle $K_{\mathcal{F}}$ is pseudoeffective.

Indeed, Brunella proved more than the results stated in the theorem above. By putting on $K_{\mathcal{F}}$ precisely the Poincaré metric on the hyperbolic leaves, he construct a singular hermitian metric h (maybe not with analytic singularities) on $K_{\mathcal{F}}$, such that the set of points where h is unbounded locally are polar set $\text{Sing}(\mathcal{F}) \cup \text{Parab}(\mathcal{F})$, where $\text{Parab}(\mathcal{F})$ are the union of parabolic leaves, and the curvature Θ_h of the metric h is a positive current. Therefore, it seems that Brunella's theorem can be strengthened, i.e., not only $K_{\mathcal{F}}$ is pseudo-effective, but also the canonical sheaf $\mathcal{K}_{\mathcal{F}}$ is pseudoeffective. However, as is shown in the following example, even if all the leaves of \mathcal{F} are hyperbolic, the canonical sheaf can not be pseudoeffective.

Example 2.1. A foliation by curves of degree d on the complex projective space $\mathbb{C}P^n$ is generated by a global section

$$s \in H^0(\mathbb{C}P^n, T_{\mathbb{C}P^n} \otimes \mathcal{O}(d-1)).$$

From the results by Lins Neto and Soares [LS96] and Brunella [Bru06], we know that a *generic* one-dimensional foliation \mathcal{F} of degree d satisfies the following property:

- (a) the set of the singularities $\text{Sing}(\mathcal{F})$ of \mathcal{F} is discrete;
- (b) each singularity $p \in \text{Sing}(\mathcal{F})$ is non-degenerate, i.e. the Milnor number of \mathcal{F} at p is 1;
- (c) no $d+1$ points in $\text{Sing}(\mathcal{F})$ lie on a projective line;
- (d) all the leaves of \mathcal{F} are hyperbolic.

Hence the Baum-Bott formula states that

$$\begin{aligned} \#\text{Sing}(\mathcal{F}) &= c_n(T_{\mathbb{C}P^n} + \mathcal{O}(d-1)) \\ &= \sum_{i=0}^n c_{n-i}(\mathbb{C}P^n) c_1(\mathcal{O}(d-1))^i \\ &= \sum_{i=0}^n \binom{n+1}{i+1} (d-1)^i \\ &= \frac{d^{n+1} - 1}{d-1}, \end{aligned}$$

and thus the canonical sheaf $\mathcal{K}_{\mathcal{F}} = \mathcal{O}(d-1) \otimes \mathcal{I}_{\text{Sing}(\mathcal{F})}$, where $\mathcal{I}_{\text{Sing}(\mathcal{F})}$ is the maximal ideal of $\text{Sing}(\mathcal{F})$. By property (c) above it is easy to prove that for $d \gg 0$ $\mathcal{K}_{\mathcal{F}}$ is not pseudo-effective.

Remark 2.3. In [McQ08] the author introduces the definition *canonical singularities* for foliations, in dimension 2 this definition is equivalent to *reduced singularities* in the sense of Seidenberg. The generic foliation by curves of degree d in $\mathbb{C}P^n$ is another example of canonical singularities. In this situation, one can not expect to improve the “bigness” of canonical sheaf $\mathcal{K}_{\mathcal{F}}$ by blowing-up. Indeed, this birational model is “stable” in the sense that, $\pi_*\mathcal{K}_{\tilde{\mathcal{F}}} = \mathcal{K}_{\mathcal{F}}$ for any birational model $\pi : (\tilde{X}, \tilde{\mathcal{F}}) \rightarrow (X, \mathcal{F})$. However, on the complex surface endowed with a foliation \mathcal{F} with reduced singularities, if f is an entire curve tangent to the foliation, and $T[f]$ is the Ahlfors current associated with f , then in [McQ98] it is shown that the positivity of $T[f] \cdot T_{\mathcal{F}}$ can be improved by an infinite sequence of blowing-ups, due to the fact that certain singularities of \mathcal{F} appearing in the blowing-ups are “small”, i.e. the lifted entire curve will not pass to these singularities. Since $T[f] \cdot T_{\mathcal{F}}$ is related to value distribution, and thus these small singularities are negligible. In [Den16] this “Diophantine approximation” idea has been generalized to higher dimensions.

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References

- [Bru06] M. Brunella, Inexistence of invariant measures for generic rational differential equations in the complex domain. Bol. Soc. Mat. Mexicana (3), 2006, 12(1): 43-49.
- [Bru10] M. Brunella, Uniformisation of foliations by curves. Holomorphic dynamical systems. Springer Berlin Heidelberg, 2010: 105-163.
- [Dem12] J.P. Demailly, Kobayashi pseudo-metrics, entire curves and hyperbolicity of algebraic varieties, Lecture notes in Summer School in Mathematics 2012.
- [Dem14] J.P. Demailly, Towards the Green-Griffiths-Lang conjecture, arXiv: 1412.2986v2 [math.AG]
- [Den16] Y. Deng, Degeneracy of entire curves into higher dimensional complex manifolds, in preparation.
- [GPR13] C. Gasbarri, G. Pacienza, E. Rousseau, Higher dimensional tautological inequalities and applications. Mathematische Annalen, 2013, 356(2): 703-735.
- [Gri71] P.A. Griffiths, Holomorphic mapping into canonical algebraic varieties. Annals of Mathematics, 1971: 439-458.
- [KO71] S. Kobayashi, T. Ochiai, Mappings into compact complex manifolds with negative first Chern class. Journal of the Mathematical Society of Japan, 1971, 23(1): 137-148.

- [Laz04] R. Lazarsfeld, Positivity in algebraic geometry II. Springer-Verlag. 2004.
- [LS96] A. Lins Neto, M.G. Soares, Algebraic solutions of one-dimensional foliations. *Journal of Differential Geometry*, 1996, 43(3): 652-673.
- [McQ98] M. McQuillan, Diophantine approximations and foliations. *Publications Mathématiques de l'Institut des Hautes études Scientifiques*, 1998, 87(1): 121-174.
- [McQ08] M. McQuillan, Canonical models of foliations. *Pure and applied mathematics quarterly*, 2008, 4(3).

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