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# Symmetries of the $k$ -bounded partition lattice

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**Abstract.** We generalize the symmetry on Young's lattice, found by Suter, to a symmetry on the  $k$ -bounded partition lattice of Lapointe, Lascoux and Morse.

**Résumé.** Nous généralisons la symétrie sur le treillis de Young, découvert par Suter, à une symétrie sur le treillis des partages bornés par  $k$  et étudié par Lapointe, Lascoux and Morse.

**Keywords:** core partitions,  $k$ -Schur functions, cyclic symmetry

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## 1 Introduction

In [Su1], Suter found a dihedral symmetry which exists in Young's lattice, by taking all partitions whose bounding rectangle is contained within the staircase  $(k, k - 1, k - 2, \dots, 2, 1)$ . He recognized that these partitions would have the same symmetries as the affine Dynkin diagram of type  $A_k$ .

While studying  $k$ -Schur functions, we noticed that the rectangles which Suter uses are the same rectangles that appear in Morse and Lapointe's paper [LM3]. The rectangles in this picture correspond to special elements of the homology of the affine Grassmannian [L1, L2, L3]. For this reason, the lattice of  $k$ -bounded partitions related to the algebra of  $k$ -Schur functions is a natural place to view a generalization of the symmetry observed by Suter.

Recent results of Berg, Bergeron, Thomas and Zabrocki [BBTZ] developed some geometric properties of the affine hyperplane arrangement. We use this geometric picture to generalize the symmetry that Suter found to the  $k$ -bounded partition lattice of Lapointe, Lascoux and Morse [LLM]. We do this by recognizing that the  $k$ -bounded partitions which are contained in a concatenation of  $m$  rectangles with a  $k$  hook is isomorphic to an  $m$ -dilation in the geometric picture.

Recently, Nathan Williams [W] has identified an isomorphism between the geometric picture presented here and the set of words of length  $k + 1$  on  $\{0, 1, 2, \dots, m\}$  which sum to 0 (mod  $m + 1$ ) and a cyclic group action on these words.

### 1.1 From root systems in type $A_k$ to the affine Grassmannian

Let  $\alpha_1, \dots, \alpha_k$  denote the simple roots of type  $A_k$ , which form a basis for a vector space  $V$ .  $V$  has a symmetric bilinear form given by:

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{else.} \end{cases}$$

and we let  $\{\Lambda_i\}_{1 \leq i \leq k}$  denote the basis dual to  $\{\alpha_i\}_{1 \leq i \leq k}$  under this bilinear form. The  $\mathbb{Z}$  span of the  $\{\Lambda_i\}_{1 \leq i \leq k}$  will be called the *weights*.

For  $v \in V$ , we let  $H_v$  denote the hyperplane through the origin, perpendicular to  $v$ . We write  $H_i$  for  $H_{\alpha_i}$  and  $H_{v,p}$  for the points  $x$  satisfying  $\langle v, x \rangle = p$ .

Let  $s_i$  represent the reflection of a vector  $v$  through the hyperplane  $H_i$  so that the set of reflections  $s_1, \dots, s_k$  corresponding to the roots  $\alpha_1, \dots, \alpha_k$  generate a reflection group  $W_0$  which is isomorphic to the symmetric group  $S_{k+1}$ . The corresponding (finite) root system is  $\Phi_0$  is the closure of the set of vectors  $\{\alpha_i\}_{1 \leq i \leq k}$  under the action of  $W_0$ . The element  $\phi = \alpha_1 + \dots + \alpha_k$  is known as the *highest root* of the the root system.

The *affine arrangement* is the collection of all hyperplanes  $H_{\alpha,p}$  for  $\alpha \in \Phi_0$  and  $p \in \mathbb{Z}$ .

The *dominant chamber* is the (closed) collection of points in  $V$  which are bounded by the hyperplanes  $H_{\alpha_i,0}$ . We denote it by  $C$ . A weight is called *dominant* if it lies in the dominant chamber.

The *fundamental alcove* is bounded by the walls of the dominant chamber, together with the hyperplane  $H_{\phi,1}$ . We denote it by  $A_\emptyset$ .

The affine reflection group,  $W$ , has an additional generator  $s_0$ , which acts as reflection in  $H_{\phi,1}$ . The generators  $s_0, s_1, \dots, s_k$  satisfy the affine type A Coxeter relations:

$$\begin{aligned} s_i^2 &= 1 \text{ for } i \in \{0, 1, \dots, k\} \\ s_i s_j &= s_j s_i \text{ if } i - j \neq \pm 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \text{ for } i \in \{0, 1, \dots, k\} \end{aligned}$$

where  $i - j$  and  $i + 1$  are understood to be taken modulo  $k + 1$ .

There is an action of  $W$  on  $V$  defined by  $s_i$  reflecting across the hyperplane  $H_i$  for  $i \in \{1, 2, \dots, k\}$  and  $s_0$  reflecting across the hyperplane  $H_{\phi,1}$ .

We let  $A_w := w^{-1}A_\emptyset$ . The collection of  $A_w$  are called the *alcoves* of the affine arrangement. The hyperplanes  $H_{\alpha_i,n}$  will intersect with  $A_w$  either in the empty set, at a single weight, or in a facet of the alcove (the convex hull of  $k$  of the vertices of  $A_w$ ). An alcove  $A_w \subset C$  if and only if  $w$  is a minimal length coset representative of  $W/W_0$ . The set of minimal length coset representatives is denoted  $W^0$ . A permutation  $w \in W^0$  is called an *affine Grassmannian permutation*.

**Example 1.1** Let  $k = 4$ . Then  $s_4 s_1 s_0 = s_1 s_4 s_0$  is affine Grassmannian because all its reduced words end in  $s_0$ , but  $s_0 s_1 s_0 = s_1 s_0 s_1$  is not.

A partition  $\lambda$  is called a  $(k + 1)$ -core if  $\lambda$  has no removable  $(k + 1)$ -rim hook. Define the size of a  $(k + 1)$ -core,  $|\lambda|$ , to be the number of cells  $(i, j)$  with hook smaller than  $k + 1$  where the hook of a cell is  $\lambda_i + \lambda'_j - i - j + 1$ . Let  $\mathcal{C}^{(k+1)}$  denote the set of all  $(k + 1)$ -cores.

**Example 1.2** Let  $k = 3$  and  $\lambda = (4, 2, 2)$ . Then  $\lambda$  has no removable 4-rim hooks and the size of  $\lambda$  is 6.

$W$  has an action on  $\mathcal{C}^{(k+1)}$ . Let the *content* of a cell  $(i, j)$  in the Young diagram of  $\lambda$  be the integer  $j - i \pmod{k+1}$ . If  $\lambda$  is a  $(k+1)$ -core then  $s_i\lambda$  is  $\lambda$  union all addable cells of content  $i$ , if  $\lambda$  has such an addable cell,  $s_i\lambda$  is  $\lambda$  minus all removable boxes of content  $i$  from  $\lambda$  if  $\lambda$  has such a removable box (a  $(k+1)$ -core cannot have both a removable box and an addable position of the same content), and  $s_i\lambda = \lambda$  otherwise.

**Example 1.3** Let  $k = 3$  and  $\lambda = (4, 2, 2)$  as above. Then  $s_1\lambda = (4, 3, 2, 1)$  and  $s_3\lambda = (3, 2, 1)$ .

0	1	2	3
3	0		
2	3		

**Proposition 1.4** [Lascoux] There is a bijection between affine Grassmannian permutations of length  $r$  and the set of  $(k+1)$ -cores of size  $r$  by sending  $w \in W^0$  to the  $(k+1)$  core  $w\emptyset$  obtained by  $w$  acting on the empty core.

## 2 Background: Suter symmetry

For a fixed positive integer  $k$ , let  $R_1 = (1^k)$ ,  $R_2 = (2^{k-1})$ ,  $\dots$ ,  $R_k = (k)$  denote the rectangular partitions which have largest hook length equal to  $k$ . Let  $Y^k$  denote the (finite) sublattice of Young's lattice which contains everything smaller than  $R_1, R_2, \dots, R_k$ , i.e.  $Y^k = \{\lambda : \lambda \subset R_i \text{ for some } i\}$ .

Suter [Su1] noticed that  $Y^k$  had a dihedral symmetry, coming from the usual symmetry of partition transposition, as well as a  $k$ -fold rotational symmetry, as pictured in Figure 1.

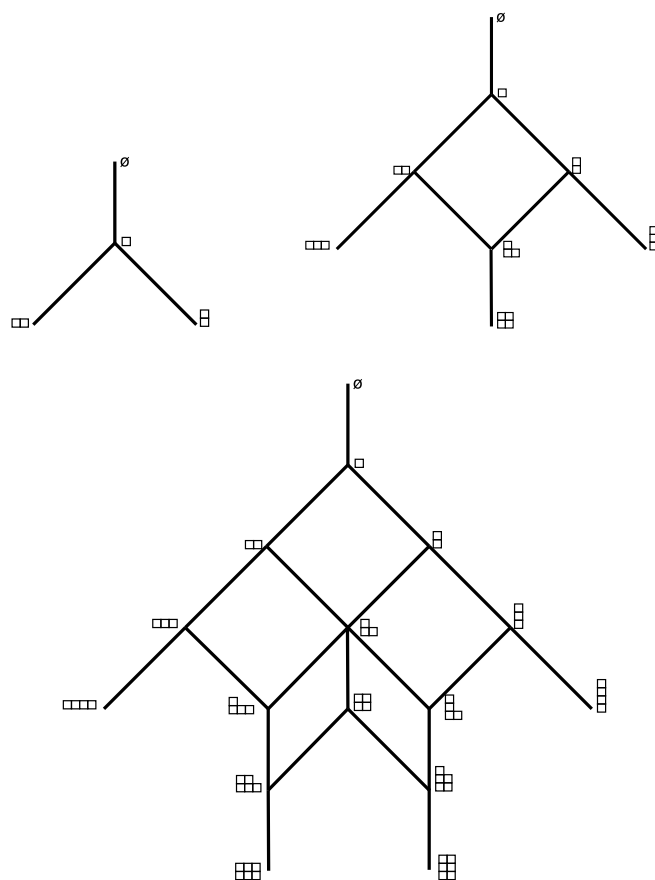
Suter defined a cyclic action on  $Y^k$  of order  $k+1$ , described on a Young diagram of a partition. We will not present this here; our generalization comes from a different description of this cyclic action which we now introduce.

### 2.1 Suter symmetry on alcoves

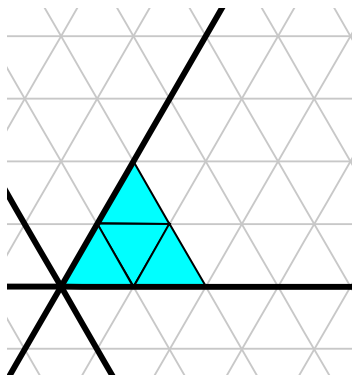
Since every partition in  $Y^k$  is a  $(k+1)$ -core, we can associate each partition  $\lambda \in Y^k$  with some affine Grassmannian permutation, or equivalently, to an alcove  $A_w$  in the dominant chamber. It was noticed by Suter in [Su2] that all partitions whose hook is smaller than or equal to  $k$  are in bijection with the alcoves in the fundamental chamber bounded by  $H_{\phi, 2}$ . The elements of  $Y^k$ , viewed as alcoves, now form a 2 fold dilation of the fundamental alcove. The fundamental alcove has a  $k+1$  cyclic symmetry (cycling the vertices of the dilated alcove) and so the elements of  $Y^k$  also have this symmetry. We will generalize this version of Suter symmetry in Section 4.

## 3 Combinatorics of $k$ -bounded partitions

Lapointe and Morse [LM2] introduced a bijection between  $(k+1)$ -cores and  $k$ -bounded partitions (a partition is  $k$ -bounded if all of its parts are less than or equal to  $k$ ). The bijection sends a  $(k+1)$ -core  $\mu$  to the  $k$ -bounded partition  $\lambda$  whose  $i^{\text{th}}$  part is equal to the number of cells  $(i, j)$  in  $\mu$  with hook less than  $k+1$ . For a  $(k+1)$ -core  $\mu$ , we let  $\mathfrak{p}(\mu)$  denote the corresponding  $k$ -bounded partition, and we will let  $\mathfrak{c}$  denote the inverse map (so  $\mathfrak{c}(\mathfrak{p}(\mu)) = \mu$ ).



**Fig. 1:** Three examples of the  $k + 1$  dihedral symmetry of  $Y^k$  for  $k \in \{2, 3, 4\}$ .



**Fig. 2:** A dilation of the fundamental alcove of  $\tilde{A}_2$  by multiplying the edge lengths by 2. The highlighted cells are in bijection with the partitions  $\{\emptyset, (1), (2), (1, 1)\}$ .

Lapointe, Lascoux and Morse [LLM] introduced a  $k$ -version of Young's lattice. It is a sublattice of Young's lattice whose vertices are labeled by  $k$ -bounded partitions. It is the lattice generated by the covering relation  $\lambda < \mu$  if  $|\lambda| + 1 = |\mu|$  and  $s_i c(\lambda) = c(\mu)$  for some  $i = 0, 1, \dots, k$ .

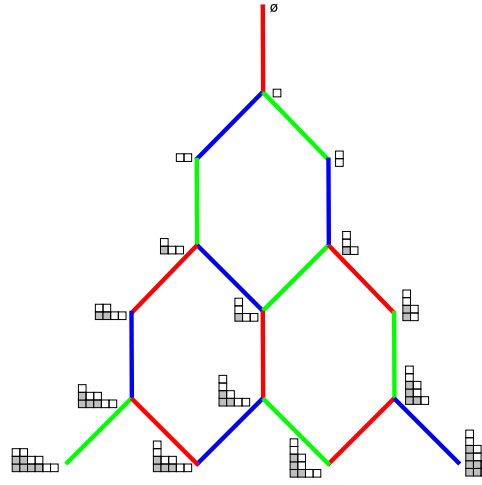
The rectangles  $R_1, \dots, R_k$  described above play an important role in the study of  $k$ -Schur functions.  $k$ -Schur functions, first introduced by Lapointe, Lascoux and Morse [LLM], were motivated in the study of Macdonald polynomials, but have since appeared in other contexts (see, in particular, [L2, L3, LS, LM3]). Each  $k$ -Schur function  $s_\lambda^{(k)}$  is indexed by a  $k$ -bounded partition  $\lambda$  (or equivalently a  $(k + 1)$ -core, or an affine Grassmannian permutation).

An important open problem in the study of  $k$ -Schur functions is to understand their multiplication rule. One special case is very explicitly understood, due to the following theorem of Lapointe and Morse. For two partitions  $\lambda$  and  $\mu$ , let  $\lambda \cup \mu$  denote the partition obtained by combining the parts of  $\lambda$  and  $\mu$  and placing them into non-increasing order.

**Theorem 3.1 (Lapointe, Morse [LM3])**  $s_\lambda^{(k)} s_R^{(k)} = s_{\lambda \cup R}^{(k)}$  for a rectangle  $R = R_1, \dots, R_k$ .

## 4 Generalized Suter symmetry

We now fix an integer  $m > 1$ . With Theorem 3.1 in mind, we will study all partitions contained in a product of  $m$  rectangles. Let  $Y_m^k$  denote the subposet of the  $k$ -Young's lattice which contains all partitions contained in a stack of  $m - 1$  of the  $k$ -rectangles (so  $\lambda \in Y_m^k$  if  $\lambda \subset R_{i_1} \cup R_{i_2} \cup \dots \cup R_{i_{m-1}}$  for some  $i_1, \dots, i_{m-1}$ ). By this definition,  $Y_2^k = Y^k$  from the beginning of Section 2. As exhibited in Figure 4, the set  $Y_m^k$  also has a  $k + 1$  cyclic symmetry. We will prove this by appealing to the geometric description of Suter symmetry. The collection of alcoves in the dominant chamber which are bounded by the affine hyperplane  $H_{\phi, m}$  again inherits the cyclic  $k + 1$  symmetry of the fundamental alcove, thus proving that a cyclic  $k + 1$  symmetry exists on the alcoves. It remains to be shown that the alcoves in the dominant chamber bounded by the hyperplane  $H_{\phi, m}$  correspond to the partitions which are contained in a product of  $m - 1$  rectangles. Once we have shown this, our main theorem, that  $Y_m^k$  has a cyclic  $k + 1$  action, will follow.



**Fig. 3:** The poset  $Y_4^2$  labeled by cores which exhibits a dihedral 3-fold symmetry. A reflection in this symmetry is realized through conjugation of the 3-cores. The red indicates the cells added to the core are content  $0 \pmod 3$ , blue at the cells are content  $1 \pmod 3$ , green the cells are content  $2 \pmod 3$

### 5 The affine Nil-Coxeter algebra and rectangle $k$ -Schur functions

The affine nilCoxeter algebra  $\mathbb{A}$  is the algebra generated by  $u_i$  for  $i \in \{0, 1, \dots, k\}$ , subject to the relations (see for instance [L1]):

$$\begin{aligned}
 u_i^2 &= 0 \text{ for } i \in \{0, 1, \dots, k\} \\
 u_i u_j &= u_j u_i \text{ if } i - j \neq \pm 1 \\
 u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} \text{ for } i \in \{0, 1, \dots, k\}
 \end{aligned}$$

where  $i - j$  and  $i + 1$  are understood to be taken modulo  $k + 1$ .

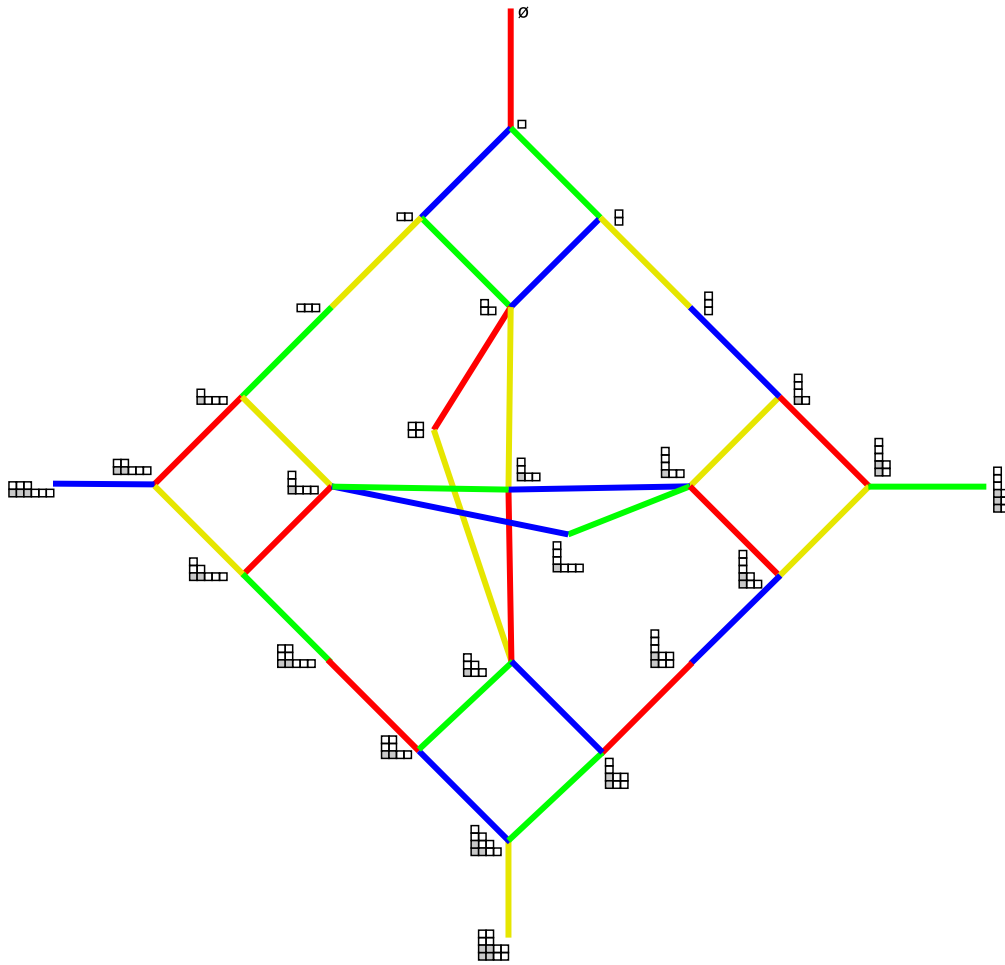
If  $s_{i_1} \dots s_{i_m}$  is a reduced word for an element  $w \in W$ , we let  $\mathbf{u}(w) = u_{i_1} \dots u_{i_m}$ , then  $U := \{\mathbf{u}(w) : w \in W\}$  is a basis of  $\mathbb{A}$ .

The affine nilCoxeter algebra has an action on the free abelian group with basis the  $(k + 1)$ -cores. Let  $\nu \in \mathcal{C}^{(k+1)}$  and then define  $u_i \nu$  to be the  $(k + 1)$ -core formed by adding all addable boxes of content  $i$  if  $\nu$  has at least one such addable box, and  $u_i \nu$  is 0 otherwise.

Within the affine nilCoxeter algebra, Lam [L1] found elements  $\mathbf{h}_i$  for  $1 \leq i \leq k$  which generate a subalgebra isomorphic to the subring of symmetric functions generated by the complete homogenous symmetric functions  $h_1, \dots, h_k$ .

**Definition 5.1** An element  $u = u_{i_1} u_{i_2} \dots u_{i_m} \in U$  is said to be cyclically decreasing if each of  $i_1, \dots, i_m$  are distinct, and whenever  $j = i_s$  and  $j + 1 = i_t$  then  $t < s$  (here  $j + 1$  is taken modulo  $k + 1$ ). To a strict subset  $D \subset \{0, 1, \dots, k\}$ , we let  $u_D$  denote the unique element of  $U$  which is cyclically decreasing and is a product of the generators  $u_m$  for  $m \in D$ .

**Example 5.2** Let  $k = 7$  and let  $D = \{0, 1, 4, 7\}$ . Then  $u_D = u_1 u_0 u_7 u_4$ .



**Fig. 4:** The poset  $Y_3^3$  exhibits a cyclic 4 symmetry. The vertices are labelled by 4-cores, and corresponding 3-bounded partitions are obtained by deleting shaded boxes and left justifying the partition. The edge colors correspond to the integer modulo 4 of the content of the cells being added; red is 0, blue is 1, yellow is 2 and green is 3.



Lam then defines elements  $\mathbf{h}_i := \sum_{|D|=i} u_D \in \mathbb{A}$  for  $i \in \{0, 1, \dots, k\}$ .

**Theorem 5.3 (Lam [L1] Corollary 14)** *The  $\mathbf{h}_i$  for  $i \in \{1, 2, \dots, k\}$  generate a subalgebra isomorphic to the ring generated by the complete homogeneous symmetric functions  $h_i$  for  $i \in \{1, 2, \dots, k\}$ . The isomorphism identifies  $\mathbf{h}_i$  and  $h_i$ .*

One can then define the  $k$ -Schur functions.

**Definition 5.4** *Let  $\lambda$  be a  $k$ -bounded partition. Then we define  $s_\lambda^{(k)}$  to be the unique elements of the subring generated by the  $\mathbf{h}_i$  which satisfy the following rule, known as the  $k$ -Pieri rule:*

$$\mathbf{h}_i s_\lambda^{(k)} = \sum_{\mu} s_\mu^{(k)}; \quad s_\emptyset^{(k)} = 1.$$

where  $\mu = \mathbf{u}(y)\lambda$  and  $y$  is a cyclically decreasing element of length  $i$ .

**Remark 5.5** *In general, expanding  $s_\lambda^{(k)} = \sum_w c_w \mathbf{u}(w)$  is an open problem, and has been shown to be equivalent to understanding the structure coefficients of  $k$ -Schur functions (called the  $k$ -Littlewood Richardson coefficients).*

## 5.1 Expression of rectangle $k$ -Schur functions as pseudo-translations

In [BBTZ], the authors introduced the notion of a pseudo-translation in order to describe the expansion of  $k$ -Schur functions corresponding to  $R_1, \dots, R_k$  in  $\mathbb{A}$ . Pseudo-translations have since been realized by Lam and Shimozono as being translations of the extended affine Weyl group (see [LS2]).

**Definition 5.6** *Let  $\eta$  be a weight. We say  $y \in W$  is a pseudo-translation of  $A_w$  in direction  $\eta$  if  $A_{yw} = A_w + \eta$ .*

For a weight  $\gamma$  we let  $z_\gamma$  denote the pseudo-translation of the fundamental alcove  $A_\emptyset$  in direction  $\gamma$ .

**Theorem 5.7 (Berg, Bergeron, Thomas, Zabrocki [BBTZ])** *Inside  $\mathbb{A}$ ,*

$$s_{R_i}^{(k)} = \sum_{\gamma \in W_0 \Lambda_i} \mathbf{u}(z_\gamma).$$

## 5.2 Alcoves with a facet on the hyperplane $H_{\phi, m}$

Let  $R$  be the  $k$ -bounded partition  $R_{i_1} \cup \dots \cup R_{i_{m-1}}$  with  $i_j \in \{1, 2, \dots, k\}$ . Then  $s_R^{(k)} = s_{R_{i_{m-1}}}^{(k)} \cdots s_{R_{i_1}}^{(k)}$  by Theorem 3.1. By Theorem 5.7, the  $k$ -bounded partition  $R$  corresponds to the alcove  $A_\emptyset + (\Lambda_{i_1} + \dots + \Lambda_{i_{m-1}})$ .

**Lemma 5.8** *There are  $\binom{m-1+k-1}{k-1}$  distinct  $k$ -bounded partitions of the form  $R = R_{i_1} \cup \dots \cup R_{i_{m-1}}$ .*

**Proof:** The partition  $R$  will be the union some number (possibly 0) of each of the different rectangles  $R_1, R_2, \dots, R_k$ . Hence the number of such rectangles is the number of ways to pick a set of  $m-1$  objects from a set of  $k$  elements with repetition.  $\square$

**Lemma 5.9** *The alcove  $A$  corresponding to the  $k$ -bounded partition  $R = R_{i_1} \cup \dots \cup R_{i_{m-1}}$  shares a facet with the wall  $H_{\phi, m}$ .*

**Proof:** The fundamental alcove  $A_\emptyset$  shares a facet with the hyperplane  $H_{\phi,1}$ . The fundamental weights  $\Lambda_i$  all satisfy  $\langle \Lambda_i, \phi \rangle = 1$  and are the coordinates of the vertices of this facet. Since  $A$  is a translate of the fundamental alcove,  $A = A_\emptyset + (\Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}})$ , the vertices of  $A$  which are not translates of the origin will have weight  $v_d = \Lambda_d + \Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}}$ . They will satisfy

$$\langle v_d, \phi \rangle = 1 + \sum_{j=1}^{m-1} \langle \Lambda_{i_j}, \phi \rangle = m,$$

and so will lie on the wall  $H_{\phi,m}$ . □

**Lemma 5.10** *The number of vertices on the hyperplane  $H_{\phi,m-1}$  which are in the fundamental chamber is  $\binom{m-1+k-1}{k-1}$ . There is a bijection between the alcoves corresponding to products of rectangles and these vertices; we identify an alcove with its unique vertex on  $H_{\phi,m-1}$ .*

**Proof:** Each vertex on  $H_{\phi,m-1}$  has the form  $\sum_i a_i \Lambda_i$  with  $a_i$  all non-negative integers and  $\sum_i a_i = m-1$ .

We conclude then that the vertices are then in bijection with non-negative integer solutions ( $a_i \geq 0$ ) to the equation  $\sum_{i=1}^k a_i = m-1$  and this is well known to be  $\binom{m-1+k-1}{k-1}$ .

For the last statement it is sufficient to remark that each alcove corresponding to partition  $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$  contains the vertex  $\Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}}$  which lies on  $H_{\phi,m-1}$  and by Lemma 5.8 these sets have the same number of elements. □

**Lemma 5.11** *Let  $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$  and let  $\mathcal{R}$  be the  $(k+1)$ -core which corresponds to the  $k$ -bounded partition  $R$ . Then  $\mathcal{R}$  has only one addable residue, that is there exists a unique  $i$  for which  $u_i \mathcal{R} \neq \emptyset$ .*

**Proof:** The only residue which is addable is  $i = i_1 + \cdots + i_{m-1}$ . The core  $\mathcal{R}$  is obtained by appending rectangles ordered by their widths in a skew fashion, stacking the rectangles so that adjacent rectangles share neither row nor column. Cells which are on the opposite sides of a rectangle in the core will have the same residue because they are separated by a hook of  $k$  therefore only one residue is addable. The length of the first row of  $\mathcal{R}$  will be  $i = i_1 + \cdots + i_{m-1}$  and so it is also the residue of the addable corner. □

**Corollary 5.12** *Let  $\lambda$  be a  $k$ -bounded partition and suppose that  $\lambda$  corresponds to an alcove  $A_w$  in the fundamental chamber which is bounded by  $H_{\phi,m}$ . Then there exists an  $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$  such that  $\lambda \subset R$ .*

**Proof:** The proof is by induction on  $m$ . When  $m = 1$  the statement is trivial; the only dominant alcove bounded by  $H_{\phi,1}$  is the fundamental alcove, which corresponds to the empty partition  $\emptyset$ , which is contained in an empty product of rectangles.

Now we fix  $m$ . If  $A_w$  is bounded by  $H_{\phi,m-1}$  then the statement follows by induction; we know that there is an  $R' = R_{i_1} \cup R_{i_2} \cup \cdots \cup R_{i_{m-2}}$  with  $\lambda \subset R'$ . Therefore  $\lambda \subset R' \cup R_{i_{m-1}}$  for any other  $i_{m-1} \in \{1, 2, \dots, k\}$ .

Now we may assume that  $A_w$  is between  $H_{\phi, m-1}$  and  $H_{\phi, m}$ . Therefore  $A_w$  has at least one vertex on  $H_{\phi, m-1}$ . Let such a vertex be  $\lambda = \sum_{j=1}^{m-1} \Lambda_{i_j}$ . Let  $R = R_{i_1} \cup \dots \cup R_{i_{m-1}}$  as in Lemma 5.10. We claim that  $\lambda \subset R$ .

Let  $B$  denote the alcove corresponding to  $R$ . By Lemma 5.11,  $B$  has a unique addable residue, which we shall denote  $r$ . This residue corresponds to crossing the hyperplane  $H_{\phi, m}$ , since crossing the hyperplane will increase the length of the corresponding core and we know there is only one reflection which will add box to  $R$ , by Lemma 5.11. Applications of all other generators  $s_i$  for  $i \neq r$  must therefore decrease the size of the partition. Since  $B$  shares a vertex with  $A_w$ , there is an element  $s_{a_1} s_{a_2} \dots s_{a_x}$  of  $W_r$  which takes  $A_w$  to  $B$  (i.e.  $A_{s_{a_1} s_{a_2} \dots s_{a_x} w} = B$  for some  $a_j \neq r$ ). Therefore  $\lambda \subset R$ , since  $\lambda = s_{a_x} \dots s_{a_1} R$ .  $\square$  As a consequence of Corollary 5.12 we have the following results.

**Theorem 5.13** *The set  $Y_m^k$  has a cyclic  $k + 1$  action.*

**Proof:** By Corollary 5.12,  $Y_m^k$  corresponds precisely with alcoves in the dominant chamber which are bounded by  $H_{\phi, m}$ . The region in the dominant chamber bounded by  $H_{\phi, m}$  has the same shape as the fundamental alcove; the lengths of the edges of the fundamental alcove have been multiplied by  $m$  in  $H_{\phi, m}$ . Since the fundamental alcove has a cyclic  $k + 1$  action which is inherited from the affine Dynkin diagram, the collection of alcoves in this region inherits the cyclic  $k + 1$  action.  $\square$

As a corollary we also have as a consequence an enumeration of the elements in  $Y_m^k$ .

**Proposition 5.14** *The number of partitions in  $Y_m^k$  is  $m^k$ .*

**Proof:** As noted in the previous result,  $Y_m^k$  is in bijection with the alcoves which lie inside of an  $m$ -dilation of the fundamental alcove. In a  $k$  dimensional space the volume of a region dilated by  $m$  on a side will be  $m^k$  times the original, hence there are  $m^k$  alcoves within this region.  $\square$  Others (e.g. Sommers [Som,

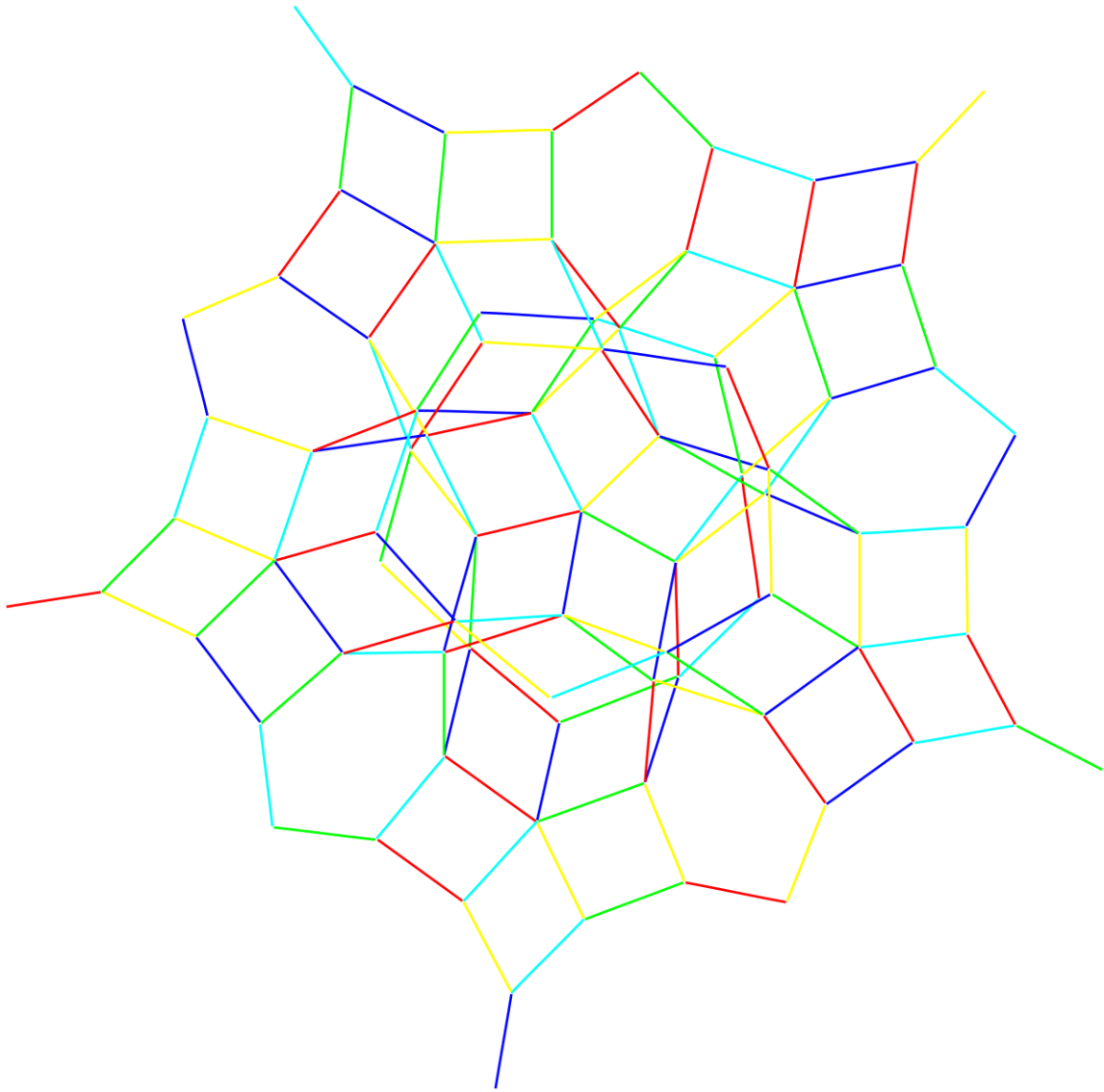
Theorem 5.7]) have considered this dilated alcove for reasons other than the connection with  $k$ -bounded partitions and  $(k + 1)$ -cores and so this lattice may have unexpected algebraic applications.

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**Fig. 5:** Suter symmetry of type  $k = 4$  and  $m = 3$

## References

- [BBTZ] Berg, Bergeron, Thomas, Zabrocki, Expansion of  $k$ -Schur functions for maximal  $k$ -rectangles within the affine nilCoxeter algebra, arXiv:1107.3610
- [graphviz] Graphviz software. <http://www.research.att.com/sw/tools/graphviz/>
- [L1] T. Lam, Affine Stanley symmetric functions, *Amer. J. Math.* **128** (2006), no. 6, 1553–1586.
- [L2] T. Lam, Stanley symmetric functions and Peterson algebras, arXiv:1007.2871v1.
- [L3] T. Lam, Schubert polynomials for the affine Grassmannian, *J. Amer. Math. Soc.*, 21 (2008), 259–281.
- [LLMS] T. Lam, L. Lapointe, J. Morse, and M. Shimozono, *Affine insertion and Pieri rules for the affine Grassmannian*, *Memoirs of the AMS*, Volume 208, Number 977, November 2010.
- [LS] T. Lam and M. Shimozono, Quantum cohomology of  $G/P$  and homology of affine Grassmannian, *Acta Math.* 204 (2010), 49–90.
- [LS2] T. Lam and M. Shimozono, From double quantum Schubert polynomials to  $k$ -double Schur functions via the Toda lattice, arXiv:1109.2193.
- [LLM] L. Lapointe, A. Lascoux, and J. Morse, Tableau atoms and a new Macdonald positivity conjecture, *Duke Math. J.* **116** (2003), no. 1, 103–146.
- [LM1] L. Lapointe and J. Morse, Schur function analogs for a filtration of the symmetric function space, *J. Combin. Theory Ser. A*, (2003), no. 101, 191–224.
- [LM2] L. Lapointe and J. Morse, Tableaux on  $k + 1$ -cores, reduced words for affine permutations, and  $k$ -Schur expansions, *J. Combin. Theory Ser. A* **112** (2005), no. 1, 44–81.
- [LM3] L. Lapointe and J. Morse, A  $k$ -tableau characterization of  $k$ -Schur functions, *Adv. Math.* **213** (2007), no. 1, 183–204.
- [LM4] L. Lapointe, J. Morse, *Quantum Cohomology and the  $k$ -Schur Basis*, *Trans. Amer. Math. Soc.* **360** (2008), pp. 2021–2040.
- [Lascoux] A. Lascoux, Ordering the affine symmetric group, in *Algebraic Combinatorics and Applications* (Gossweinstein, 1999), 219231, Springer, Berlin (2001).
- [sage] W.A. Stein et al., *Sage Mathematics Software (Version 4.3.3)*, The Sage Development Team, 2010, <http://www.sagemath.org>.
- [sage-combinat] The Sage-Combinat community, *Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics*, <http://combinat.sagemath.org>, 2008.
- [Som] E. Sommers, B-stable ideals in the nilradical of a Borel subalgebra. *Canadian Mathematical Bulletin* 48.3 (2005): 460–472.
- [Su1] R. Suter, Young’s lattice and dihedral symmetries. *European J. Combin.* 23 (2002) 233–238.
- [Su2] R. Suter, Abelian ideals in a Borel subalgebra of a complex simple Lie algebra. *Invent. math.* 156 (2004) 175–221.
- [W] N. Williams, presentation at York University, February 13, 2012, *A Generalization of Suter’s Surprising Cyclic Symmetry and an Associated CSP*, available at <http://math.umn.edu/~will13089/docs/2.12.12CyclicSymmetryNFW>.