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Symmetries of the k -bounded partition lattice

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Abstract. We generalize the symmetry on Young's lattice, found by Suter, to a symmetry on the k -bounded partition lattice of Lapointe, Lascoux and Morse.

Résumé. Nous généralisons la symétrie sur le treillis de Young, découvert par Suter, à une symétrie sur le treillis des partages bornés par k et étudié par Lapointe, Lascoux and Morse.

Keywords: core partitions, k -Schur functions, cyclic symmetry

1 Introduction

In [Su1], Suter found a dihedral symmetry which exists in Young's lattice, by taking all partitions whose bounding rectangle is contained within the staircase $(k, k - 1, k - 2, \dots, 2, 1)$. He recognized that these partitions would have the same symmetries as the affine Dynkin diagram of type A_k .

While studying k -Schur functions, we noticed that the rectangles which Suter uses are the same rectangles that appear in Morse and Lapointe's paper [LM3]. The rectangles in this picture correspond to special elements of the homology of the affine Grassmannian [L1, L2, L3]. For this reason, the lattice of k -bounded partitions related to the algebra of k -Schur functions is a natural place to view a generalization of the symmetry observed by Suter.

Recent results of Berg, Bergeron, Thomas and Zabrocki [BBTZ] developed some geometric properties of the affine hyperplane arrangement. We use this geometric picture to generalize the symmetry that Suter found to the k -bounded partition lattice of Lapointe, Lascoux and Morse [LLM]. We do this by recognizing that the k -bounded partitions which are contained in a concatenation of m rectangles with a k hook is isomorphic to an m -dilation in the geometric picture.

Recently, Nathan Williams [W] has identified an isomorphism between the geometric picture presented here and the set of words of length $k + 1$ on $\{0, 1, 2, \dots, m\}$ which sum to 0 (mod $m + 1$) and a cyclic group action on these words.

1.1 From root systems in type A_k to the the affine Grassmannian

Let $\alpha_1, \dots, \alpha_k$ denote the simple roots of type A_k , which form a basis for a vector space V . V has a symmetric bilinear form given by:

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{else.} \end{cases}$$

and we let $\{\Lambda_i\}_{1 \leq i \leq k}$ denote the basis dual to $\{\alpha_i\}_{1 \leq i \leq k}$ under this bilinear form. The \mathbb{Z} span of the $\{\Lambda_i\}_{1 \leq i \leq k}$ will be called the *weights*.

For $v \in V$, we let H_v denote the hyperplane through the origin, perpendicular to v . We write H_i for H_{α_i} and $H_{v,p}$ for the points x satisfying $\langle v, x \rangle = p$.

Let s_i represent the reflection of a vector v through the hyperplane H_i so that the set of reflections s_1, \dots, s_k corresponding to the roots $\alpha_1, \dots, \alpha_k$ generate a reflection group W_0 which is isomorphic to the symmetric group S_{k+1} . The corresponding (finite) root system is Φ_0 is the closure of the set of vectors $\{\alpha_i\}_{1 \leq i \leq k}$ under the action of W_0 . The element $\phi = \alpha_1 + \dots + \alpha_k$ is known as the *highest root* of the the root system.

The *affine arrangement* is the collection of all hyperplanes $H_{\alpha,p}$ for $\alpha \in \Phi_0$ and $p \in \mathbb{Z}$.

The *dominant chamber* is the (closed) collection of points in V which are bounded by the hyperplanes $H_{\alpha_i,0}$. We denote it by C . A weight is called *dominant* if it lies in the dominant chamber.

The *fundamental alcove* is bounded by the walls of the dominant chamber, together with the hyperplane $H_{\phi,1}$. We denote it by A_\emptyset .

The affine reflection group, W , has an additional generator s_0 , which acts as reflection in $H_{\phi,1}$. The generators s_0, s_1, \dots, s_k satisfy the affine type A Coxeter relations:

$$\begin{aligned} s_i^2 &= 1 \text{ for } i \in \{0, 1, \dots, k\} \\ s_i s_j &= s_j s_i \text{ if } i - j \neq \pm 1 \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \text{ for } i \in \{0, 1, \dots, k\} \end{aligned}$$

where $i - j$ and $i + 1$ are understood to be taken modulo $k + 1$.

There is an action of W on V defined by s_i reflecting across the hyperplane H_i for $i \in \{1, 2, \dots, k\}$ and s_0 reflecting across the hyperplane $H_{\phi,1}$.

We let $A_w := w^{-1}A_\emptyset$. The collection of A_w are called the *alcoves* of the affine arrangement. The hyperplanes $H_{\alpha_i,n}$ will intersect with A_w either in the empty set, at a single weight, or in a facet of the alcove (the convex hull of k of the vertices of A_w). An alcove $A_w \subset C$ if and only if w is a minimal length coset representative of W/W_0 . The set of minimal length coset representatives is denoted W^0 . A permutation $w \in W^0$ is called an *affine Grassmannian permutation*.

Example 1.1 Let $k = 4$. Then $s_4 s_1 s_0 = s_1 s_4 s_0$ is affine Grassmannian because all its reduced words end in s_0 , but $s_0 s_1 s_0 = s_1 s_0 s_1$ is not.

A partition λ is called a $(k + 1)$ -core if λ has no removable $(k + 1)$ -rim hook. Define the size of a $(k + 1)$ -core, $|\lambda|$, to be the number of cells (i, j) with hook smaller than $k + 1$ where the hook of a cell is $\lambda_i + \lambda'_j - i - j + 1$. Let $\mathcal{C}^{(k+1)}$ denote the set of all $(k + 1)$ -cores.

Example 1.2 Let $k = 3$ and $\lambda = (4, 2, 2)$. Then λ has no removable 4-rim hooks and the size of λ is 6.

W has an action on $\mathcal{C}^{(k+1)}$. Let the *content* of a cell (i, j) in the Young diagram of λ be the integer $j - i \pmod{k+1}$. If λ is a $(k+1)$ -core then $s_i\lambda$ is λ union all addable cells of content i , if λ has such an addable cell, $s_i\lambda$ is λ minus all removable boxes of content i from λ if λ has such a removable box (a $(k+1)$ -core cannot have both a removable box and an addable position of the same content), and $s_i\lambda = \lambda$ otherwise.

Example 1.3 Let $k = 3$ and $\lambda = (4, 2, 2)$ as above. Then $s_1\lambda = (4, 3, 2, 1)$ and $s_3\lambda = (3, 2, 1)$.

0	1	2	3
3	0		
2	3		

Proposition 1.4 [Lascoux] There is a bijection between affine Grassmannian permutations of length r and the set of $(k+1)$ -cores of size r by sending $w \in W^0$ to the $(k+1)$ core $w\emptyset$ obtained by w acting on the empty core.

2 Background: Suter symmetry

For a fixed positive integer k , let $R_1 = (1^k), R_2 = (2^{k-1}), \dots, R_k = (k)$ denote the rectangular partitions which have largest hook length equal to k . Let Y^k denote the (finite) sublattice of Young's lattice which contains everything smaller than R_1, R_2, \dots, R_k , i.e. $Y^k = \{\lambda : \lambda \subset R_i \text{ for some } i\}$.

Suter [Su1] noticed that Y^k had a dihedral symmetry, coming from the usual symmetry of partition transposition, as well as a k -fold rotational symmetry, as pictured in Figure 1.

Suter defined a cyclic action on Y^k of order $k+1$, described on a Young diagram of a partition. We will not present this here; our generalization comes from a different description of this cyclic action which we now introduce.

2.1 Suter symmetry on alcoves

Since every partition in Y^k is a $(k+1)$ -core, we can associate each partition $\lambda \in Y^k$ with some affine Grassmannian permutation, or equivalently, to an alcove A_w in the dominant chamber. It was noticed by Suter in [Su2] that all partitions whose hook is smaller than or equal to k are in bijection with the alcoves in the fundamental chamber bounded by $H_{\phi, 2}$. The elements of Y^k , viewed as alcoves, now form a 2 fold dilation of the fundamental alcove. The fundamental alcove has a $k+1$ cyclic symmetry (cycling the vertices of the dilated alcove) and so the elements of Y^k also have this symmetry. We will generalize this version of Suter symmetry in Section 4.

3 Combinatorics of k -bounded partitions

Lapointe and Morse [LM2] introduced a bijection between $(k+1)$ -cores and k -bounded partitions (a partition is k -bounded if all of its parts are less than or equal to k). The bijection sends a $(k+1)$ -core μ to the k -bounded partition λ whose i^{th} part is equal to the number of cells (i, j) in μ with hook less than $k+1$. For a $(k+1)$ -core μ , we let $\mathfrak{p}(\mu)$ denote the corresponding k -bounded partition, and we will let \mathfrak{c} denote the inverse map (so $\mathfrak{c}(\mathfrak{p}(\mu)) = \mu$).

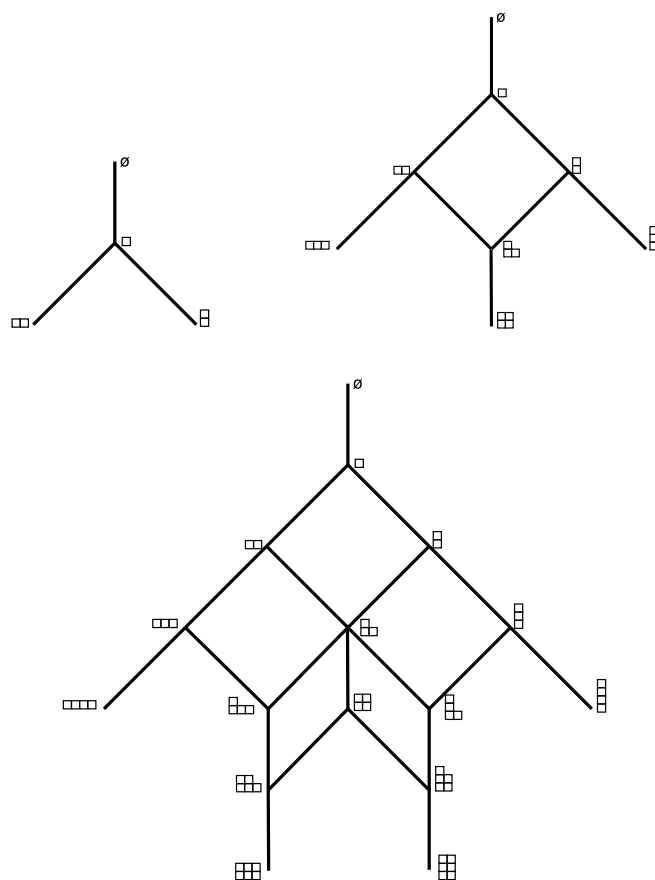


Fig. 1: Three examples of the $k + 1$ dihedral symmetry of Y^k for $k \in \{2, 3, 4\}$.

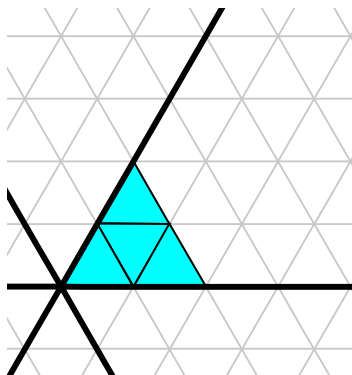


Fig. 2: A dilation of the fundamental alcove of \tilde{A}_2 by multiplying the edge lengths by 2. The highlighted cells are in bijection with the partitions $\{\emptyset, (1), (2), (1, 1)\}$.

Lapointe, Lascoux and Morse [LLM] introduced a k -version of Young's lattice. It is a sublattice of Young's lattice whose vertices are labeled by k -bounded partitions. It is the lattice generated by the covering relation $\lambda < \mu$ if $|\lambda| + 1 = |\mu|$ and $s_i c(\lambda) = c(\mu)$ for some $i = 0, 1, \dots, k$.

The rectangles R_1, \dots, R_k described above play an important role in the study of k -Schur functions. k -Schur functions, first introduced by Lapointe, Lascoux and Morse [LLM], were motivated in the study of Macdonald polynomials, but have since appeared in other contexts (see, in particular, [L2, L3, LS, LM3]). Each k -Schur function $s_\lambda^{(k)}$ is indexed by a k -bounded partition λ (or equivalently a $(k + 1)$ -core, or an affine Grassmannian permutation).

An important open problem in the study of k -Schur functions is to understand their multiplication rule. One special case is very explicitly understood, due to the following theorem of Lapointe and Morse. For two partitions λ and μ , let $\lambda \cup \mu$ denote the partition obtained by combining the parts of λ and μ and placing them into non-increasing order.

Theorem 3.1 (Lapointe, Morse [LM3]) $s_\lambda^{(k)} s_R^{(k)} = s_{\lambda \cup R}^{(k)}$ for a rectangle $R = R_1, \dots, R_k$.

4 Generalized Suter symmetry

We now fix an integer $m > 1$. With Theorem 3.1 in mind, we will study all partitions contained in a product of m rectangles. Let Y_m^k denote the subposet of the k -Young's lattice which contains all partitions contained in a stack of $m - 1$ of the k -rectangles (so $\lambda \in Y_m^k$ if $\lambda \subset R_{i_1} \cup R_{i_2} \cup \dots \cup R_{i_{m-1}}$ for some i_1, \dots, i_{m-1}). By this definition, $Y_2^k = Y^k$ from the beginning of Section 2. As exhibited in Figure 4, the set Y_m^k also has a $k + 1$ cyclic symmetry. We will prove this by appealing to the geometric description of Suter symmetry. The collection of alcoves in the dominant chamber which are bounded by the affine hyperplane $H_{\phi, m}$ again inherits the cyclic $k + 1$ symmetry of the fundamental alcove, thus proving that a cyclic $k + 1$ symmetry exists on the alcoves. It remains to be shown that the alcoves in the dominant chamber bounded by the hyperplane $H_{\phi, m}$ correspond to the partitions which are contained in a product of $m - 1$ rectangles. Once we have shown this, our main theorem, that Y_m^k has a cyclic $k + 1$ action, will follow.

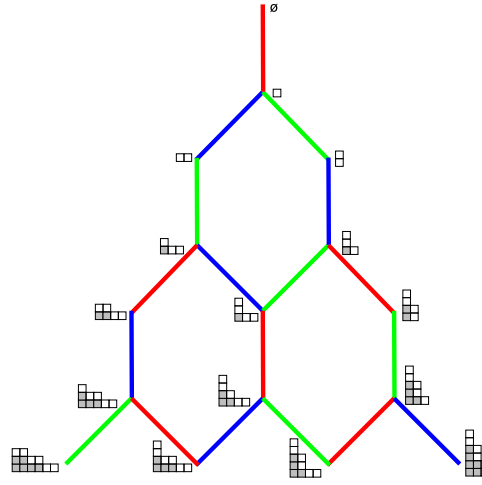


Fig. 3: The poset Y_4^2 labeled by cores which exhibits a dihedral 3-fold symmetry. A reflection in this symmetry is realized through conjugation of the 3-cores. The red indicates the cells added to the core are content $0 \pmod{3}$, blue at the cells are content $1 \pmod{3}$, green the cells are content $2 \pmod{3}$

5 The affine Nil-Coxeter algebra and rectangle k -Schur functions

The affine nilCoxeter algebra \mathbb{A} is the algebra generated by u_i for $i \in \{0, 1, \dots, k\}$, subject to the relations (see for instance [L1]):

$$\begin{aligned}
 u_i^2 &= 0 \text{ for } i \in \{0, 1, \dots, k\} \\
 u_i u_j &= u_j u_i \text{ if } i - j \neq \pm 1 \\
 u_i u_{i+1} u_i &= u_{i+1} u_i u_{i+1} \text{ for } i \in \{0, 1, \dots, k\}
 \end{aligned}$$

where $i - j$ and $i + 1$ are understood to be taken modulo $k + 1$.

If $s_{i_1} \dots s_{i_m}$ is a reduced word for an element $w \in W$, we let $\mathbf{u}(w) = u_{i_1} \dots u_{i_m}$, then $U := \{\mathbf{u}(w) : w \in W\}$ is a basis of \mathbb{A} .

The affine nilCoxeter algebra has an action on the free abelian group with basis the $(k + 1)$ -cores. Let $\nu \in \mathcal{C}^{(k+1)}$ and then define $u_i \nu$ to be the $(k + 1)$ -core formed by adding all addable boxes of content i if ν has at least one such addable box, and $u_i \nu$ is 0 otherwise.

Within the affine nilCoxeter algebra, Lam [L1] found elements \mathbf{h}_i for $1 \leq i \leq k$ which generate a subalgebra isomorphic to the subring of symmetric functions generated by the complete homogenous symmetric functions h_1, \dots, h_k .

Definition 5.1 An element $u = u_{i_1} u_{i_2} \dots u_{i_m} \in U$ is said to be cyclically decreasing if each of i_1, \dots, i_m are distinct, and whenever $j = i_s$ and $j + 1 = i_t$ then $t < s$ (here $j + 1$ is taken modulo $k + 1$). To a strict subset $D \subset \{0, 1, \dots, k\}$, we let u_D denote the unique element of U which is cyclically decreasing and is a product of the generators u_m for $m \in D$.

Example 5.2 Let $k = 7$ and let $D = \{0, 1, 4, 7\}$. Then $u_D = u_1 u_0 u_7 u_4$.

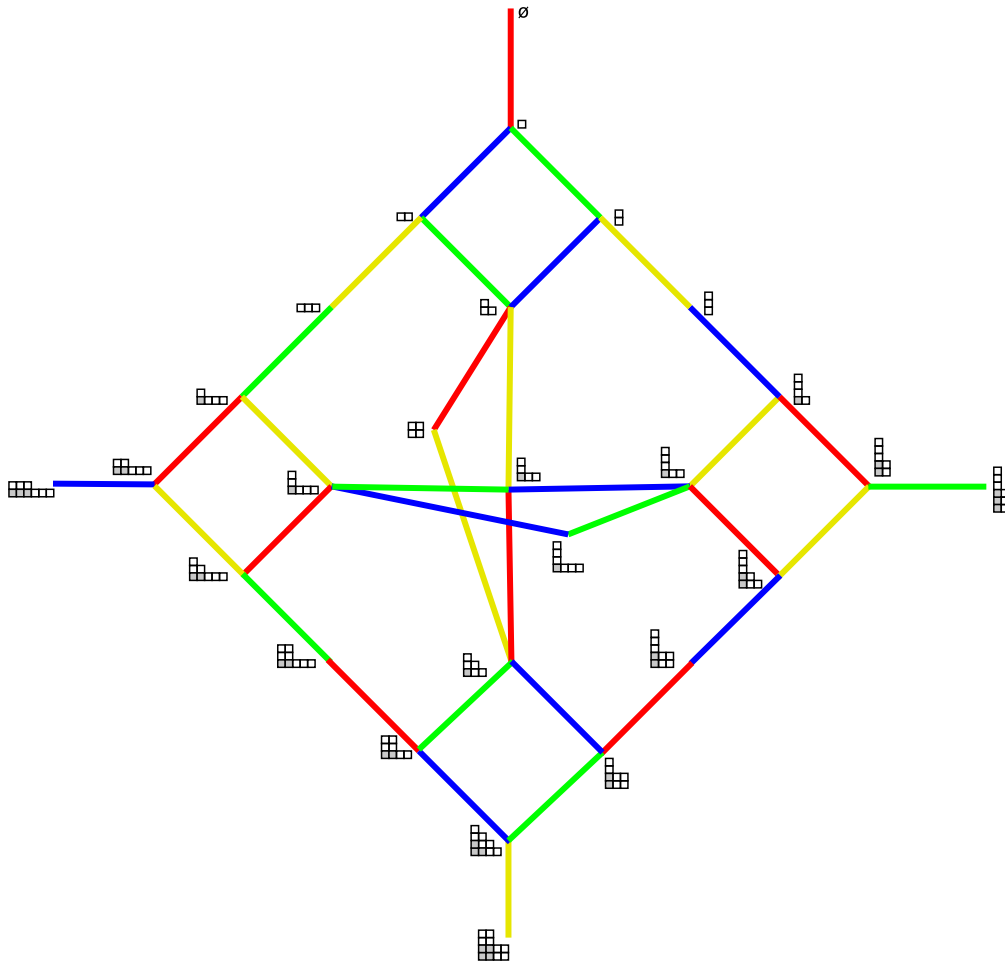


Fig. 4: The poset Y_3^3 exhibits a cyclic 4 symmetry. The vertices are labelled by 4-cores, and corresponding 3-bounded partitions are obtained by deleting shaded boxes and left justifying the partition. The edge colors correspond to the integer modulo 4 of the content of the cells being added; red is 0, blue is 1, yellow is 2 and green is 3.

Lam then defines elements $\mathbf{h}_i := \sum_{|D|=i} u_D \in \mathbb{A}$ for $i \in \{0, 1, \dots, k\}$.

Theorem 5.3 (Lam [L1] Corollary 14) *The \mathbf{h}_i for $i \in \{1, 2, \dots, k\}$ generate a subalgebra isomorphic to the ring generated by the complete homogeneous symmetric functions h_i for $i \in \{1, 2, \dots, k\}$. The isomorphism identifies \mathbf{h}_i and h_i .*

One can then define the k -Schur functions.

Definition 5.4 *Let λ be a k -bounded partition. Then we define $s_\lambda^{(k)}$ to be the unique elements of the subring generated by the \mathbf{h}_i which satisfy the following rule, known as the k -Pieri rule:*

$$\mathbf{h}_i s_\lambda^{(k)} = \sum_{\mu} s_\mu^{(k)}; \quad s_\emptyset^{(k)} = 1.$$

where $\mu = \mathbf{u}(y)\lambda$ and y is a cyclically decreasing element of length i .

Remark 5.5 *In general, expanding $s_\lambda^{(k)} = \sum_w c_w \mathbf{u}(w)$ is an open problem, and has been shown to be equivalent to understanding the structure coefficients of k -Schur functions (called the k -Littlewood Richardson coefficients).*

5.1 Expression of rectangle k -Schur functions as pseudo-translations

In [BBTZ], the authors introduced the notion of a pseudo-translation in order to describe the expansion of k -Schur functions corresponding to R_1, \dots, R_k in \mathbb{A} . Pseudo-translations have since been realized by Lam and Shimozono as being translations of the extended affine Weyl group (see [LS2]).

Definition 5.6 *Let η be a weight. We say $y \in W$ is a pseudo-translation of A_w in direction η if $A_{yw} = A_w + \eta$.*

For a weight γ we let z_γ denote the pseudo-translation of the fundamental alcove A_\emptyset in direction γ .

Theorem 5.7 (Berg, Bergeron, Thomas, Zabrocki [BBTZ]) *Inside \mathbb{A} ,*

$$s_{R_i}^{(k)} = \sum_{\gamma \in W_0 \Lambda_i} \mathbf{u}(z_\gamma).$$

5.2 Alcoves with a facet on the hyperplane $H_{\phi, m}$

Let R be the k -bounded partition $R_{i_1} \cup \dots \cup R_{i_{m-1}}$ with $i_j \in \{1, 2, \dots, k\}$. Then $s_R^{(k)} = s_{R_{i_{m-1}}}^{(k)} \cdots s_{R_{i_1}}^{(k)}$ by Theorem 3.1. By Theorem 5.7, the k -bounded partition R corresponds to the alcove $A_\emptyset + (\Lambda_{i_1} + \dots + \Lambda_{i_{m-1}})$.

Lemma 5.8 *There are $\binom{m-1+k-1}{k-1}$ distinct k -bounded partitions of the form $R = R_{i_1} \cup \dots \cup R_{i_{m-1}}$.*

Proof: The partition R will be the union some number (possibly 0) of each of the different rectangles R_1, R_2, \dots, R_k . Hence the number of such rectangles is the number of ways to pick a set of $m-1$ objects from a set of k elements with repetition. \square

Lemma 5.9 *The alcove A corresponding to the k -bounded partition $R = R_{i_1} \cup \dots \cup R_{i_{m-1}}$ shares a facet with the wall $H_{\phi, m}$.*

Proof: The fundamental alcove A_\emptyset shares a facet with the hyperplane $H_{\phi,1}$. The fundamental weights Λ_i all satisfy $\langle \Lambda_i, \phi \rangle = 1$ and are the coordinates of the vertices of this facet. Since A is a translate of the fundamental alcove, $A = A_\emptyset + (\Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}})$, the vertices of A which are not translates of the origin will have weight $v_d = \Lambda_d + \Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}}$. They will satisfy

$$\langle v_d, \phi \rangle = 1 + \sum_{j=1}^{m-1} \langle \Lambda_{i_j}, \phi \rangle = m,$$

and so will lie on the wall $H_{\phi,m}$. \square

Lemma 5.10 *The number of vertices on the hyperplane $H_{\phi,m-1}$ which are in the fundamental chamber is $\binom{m-1+k-1}{k-1}$. There is a bijection between the alcoves corresponding to products of rectangles and these vertices; we identify an alcove with its unique vertex on $H_{\phi,m-1}$.*

Proof: Each vertex on $H_{\phi,m-1}$ has the form $\sum_i a_i \Lambda_i$ with a_i all non-negative integers and $\sum_i a_i = m-1$.

We conclude then that the vertices are then in bijection with non-negative integer solutions $(a_i \geq 0)$ to the equation $\sum_{i=1}^k a_i = m-1$ and this is well known to be $\binom{m-1+k-1}{k-1}$.

For the last statement it is sufficient to remark that each alcove corresponding to partition $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ contains the vertex $\Lambda_{i_1} + \cdots + \Lambda_{i_{m-1}}$ which lies on $H_{\phi,m-1}$ and by Lemma 5.8 these sets have the same number of elements. \square

Lemma 5.11 *Let $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ and let \mathcal{R} be the $(k+1)$ -core which corresponds to the k -bounded partition R . Then \mathcal{R} has only one addable residue, that is there exists a unique i for which $u_i \mathcal{R} \neq \emptyset$.*

Proof: The only residue which is addable is $i = i_1 + \cdots + i_{m-1}$. The core \mathcal{R} is obtained by appending rectangles ordered by their widths in a skew fashion, stacking the rectangles so that adjacent rectangles share neither row nor column. Cells which are on the opposite sides of a rectangle in the core will have the same residue because they are separated by a hook of k therefore only one residue is addable. The length of the first row of \mathcal{R} will be $i = i_1 + \cdots + i_{m-1}$ and so it is also the residue of the addable corner. \square

Corollary 5.12 *Let λ be a k -bounded partition and suppose that λ corresponds to an alcove A_w in the fundamental chamber which is bounded by $H_{\phi,m}$. Then there exists an $R = R_{i_1} \cup \cdots \cup R_{i_{m-1}}$ such that $\lambda \subset R$.*

Proof: The proof is by induction on m . When $m = 1$ the statement is trivial; the only dominant alcove bounded by $H_{\phi,1}$ is the fundamental alcove, which corresponds to the empty partition \emptyset , which is contained in an empty product of rectangles.

Now we fix m . If A_w is bounded by $H_{\phi,m-1}$ then the statement follows by induction; we know that there is an $R' = R_{i_1} \cup R_{i_2} \cup \cdots \cup R_{i_{m-2}}$ with $\lambda \subset R'$. Therefore $\lambda \subset R' \cup R_{i_{m-1}}$ for any other $i_{m-1} \in \{1, 2, \dots, k\}$.

Now we may assume that A_w is between $H_{\phi, m-1}$ and $H_{\phi, m}$. Therefore A_w has at least one vertex on $H_{\phi, m-1}$. Let such a vertex be $\lambda = \sum_{j=1}^{m-1} \Lambda_{i_j}$. Let $R = R_{i_1} \cup \dots \cup R_{i_{m-1}}$ as in Lemma 5.10. We claim that $\lambda \subset R$.

Let B denote the alcove corresponding to R . By Lemma 5.11, B has a unique addable residue, which we shall denote r . This residue corresponds to crossing the hyperplane $H_{\phi, m}$, since crossing the hyperplane will increase the length of the corresponding core and we know there is only one reflection which will add box to R , by Lemma 5.11. Applications of all other generators s_i for $i \neq r$ must therefore decrease the size of the partition. Since B shares a vertex with A_w , there is an element $s_{a_1} s_{a_2} \dots s_{a_x}$ of W_r which takes A_w to B (i.e. $A_{s_{a_1} s_{a_2} \dots s_{a_x} w} = B$ for some $a_j \neq r$). Therefore $\lambda \subset R$, since $\lambda = s_{a_x} \dots s_{a_1} R$. \square As a consequence of Corollary 5.12 we have the following results.

Theorem 5.13 *The set Y_m^k has a cyclic $k + 1$ action.*

Proof: By Corollary 5.12, Y_m^k corresponds precisely with alcoves in the dominant chamber which are bounded by $H_{\phi, m}$. The region in the dominant chamber bounded by $H_{\phi, m}$ has the same shape as the fundamental alcove; the lengths of the edges of the fundamental alcove have been multiplied by m in $H_{\phi, m}$. Since the fundamental alcove has a cyclic $k + 1$ action which is inherited from the affine Dynkin diagram, the collection of alcoves in this region inherits the cyclic $k + 1$ action. \square

As a corollary we also have as a consequence an enumeration of the elements in Y_m^k .

Proposition 5.14 *The number of partitions in Y_m^k is m^k .*

Proof: As noted in the previous result, Y_m^k is in bijection with the alcoves which lie inside of an m -dilation of the fundamental alcove. In a k dimensional space the volume of a region dilated by m on a side will be m^k times the original, hence there are m^k alcoves within this region. \square Others (e.g. Sommers [Som,

Theorem 5.7]) have considered this dilated alcove for reasons other than the connection with k -bounded partitions and $(k + 1)$ -cores and so this lattice may have unexpected algebraic applications.

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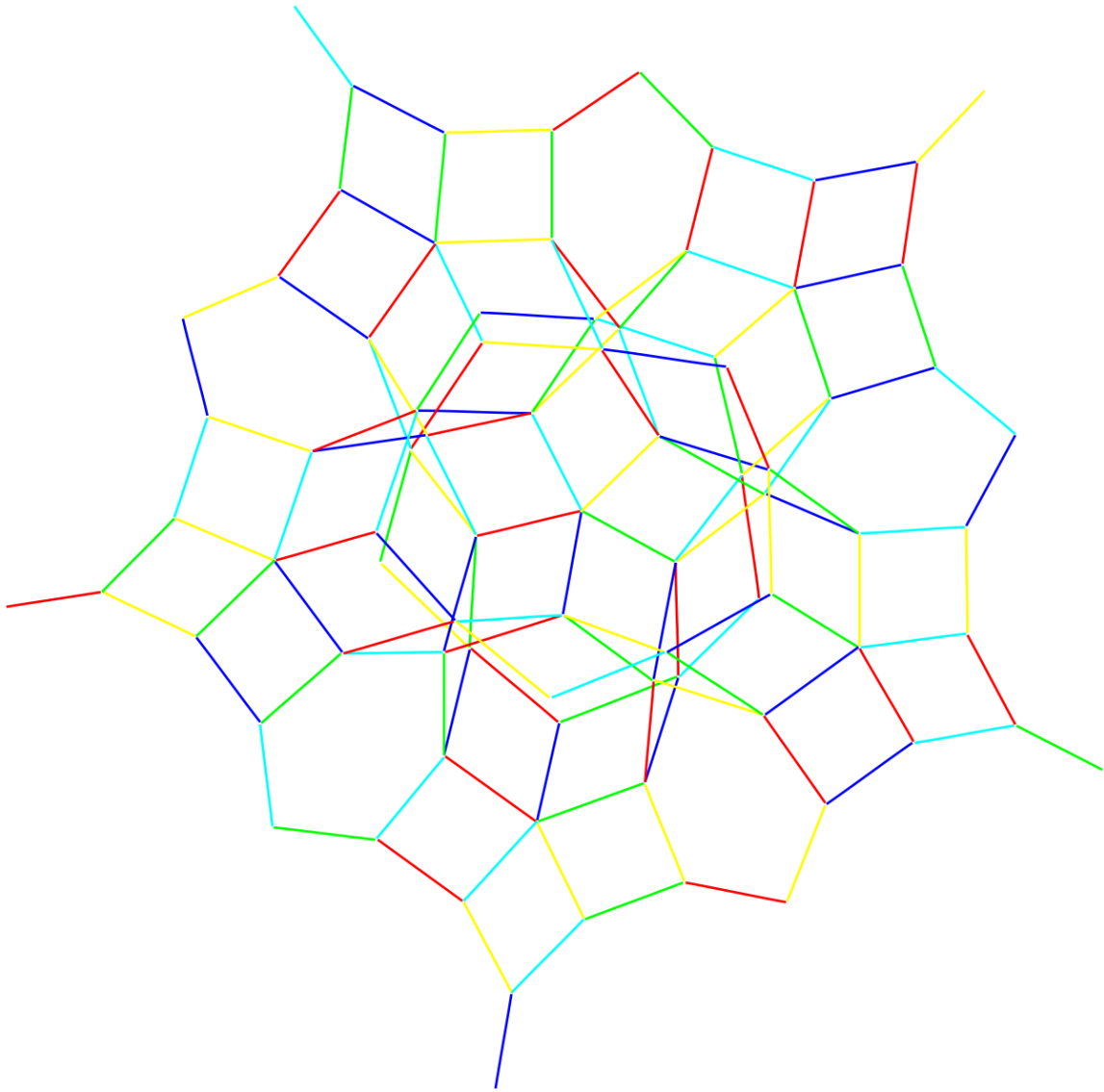


Fig. 5: Suter symmetry of type $k = 4$ and $m = 3$

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