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# On the degree-chromatic polynomial of a tree

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**Abstract.** The *degree chromatic polynomial*  $P_m(G, k)$  of a graph  $G$  counts the number of  $k$ -colorings in which no vertex has  $m$  adjacent vertices of its same color. We prove Humpert and Martin’s conjecture on the leading terms of the degree chromatic polynomial of a tree.

**Résumé.** Le *polynôme degré chromatique*  $P_m(G, k)$  d’un graphe  $G$  compte le nombre de  $k$ -colorations dans lesquelles aucun sommet n’a  $m$  sommets adjacents de sa même couleur. On démontre la conjecture de Humpert et Martin sur les coefficients principaux du polynôme degré chromatique d’un arbre.

**Keywords:** chromatic polynomial, graph coloring, tree

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George David Birkhoff defined the chromatic polynomial of a graph to attack the renowned four color problem. The chromatic polynomial  $P(G, k)$  counts the  $k$ -colorings of a graph  $G$  in which no two adjacent vertices have the same color [Read(1968)].

Given a graph  $G$ , Humpert and Martin defined its  $m$ -chromatic polynomial  $P_m(G, k)$  to be the number of  $k$ -colorings of  $G$  such that no vertex has  $m$  adjacent vertices of its same color. They proved this is indeed a polynomial. When  $m = 1$ , we recover the usual chromatic polynomial of the graph.

The chromatic polynomial is of the form

$$P(G, k) = k^n - ek^{n-1} + o(k^{n-1}),$$

where  $n$  is the number of vertices and  $e$  the number of edges of  $G$ . For  $m > 1$  the formula is no longer true, but Humpert and Martin conjectured the following formula which we now prove:

**Theorem 1 ([Humpert and Martin(2010), Humpert and Martin(2011)], Conjecture)** *Let  $T$  be a tree with  $n$  vertices and let  $m$  be an integer with  $1 < m < n$ , then the following equation holds.*

$$P_m(T, k) = k^n - \sum_{v \in V(T)} \binom{d(v)}{m} k^{n-m} + o(k^{n-m})$$

**Proof:** For a given coloring of  $T$ , say vertices  $v_1$  and  $v_2$  are “friends” if they are adjacent and have the same color. For each  $v$ , let  $A_v$  be the set of colorings such that  $v$  has at least  $m$  friends. We want to find the number of colorings which are not in any  $A_v$ , and we will use the inclusion-exclusion principle. As the total number of  $k$ -colorings is  $k^n$ , we have

$$P_m(T, k) = k^n - \sum_{v \in V} |A_v| + \sum_{v_1, v_2 \in V} |A_{v_1} \cap A_{v_2}| - \dots$$

We first show that  $|A_v| = \binom{d(v)}{m} k^{n-m} + o(k^{n-m})$ . Let  $A_v^{(l)}$  be the set of  $k$ -colorings such that  $v$  has exactly  $l$  friends. In order to obtain a coloring in  $A_v^{(l)}$ , we may choose the  $l$  friends in  $\binom{d(v)}{l}$  ways, the color of  $v$  and its friends in  $k$  ways, the color of the remaining adjacent vertices to  $v$  in  $(k-1)^{d(v)-l}$  ways, and the color of the rest of the vertices in  $k^{n-1-d(v)}$  ways. Then

$$\begin{aligned} |A_v| &= \sum_{l=m}^{n-1} |A_v^{(l)}| = \sum_{l=m}^{n-1} \binom{d(v)}{l} k^{n-d(v)} (k-1)^{d(v)-l} \\ &= \binom{d(v)}{m} k^{n-m} + o(k^{n-m}). \end{aligned}$$

To complete the proof, it is sufficient to see that for any set  $S$  of at least 2 vertices  $|\bigcap_{v \in S} A_v| = o(k^{n-m})$ ; clearly we may assume  $S = \{v_1, v_2\}$ . Consider the following cases:

**Case 1**  $v_1$  and  $v_2$  are not adjacent.

Split  $A_{v_1}$  into equivalence classes with the equivalence relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1(w) = \sigma_2(w) \text{ for all } w \neq v_2.$$

Note that each equivalence class  $C$  consists of  $k$  colorings, which only differ in the color of  $v_2$ . In addition, for each  $C$  at most  $\frac{d(v_2)}{m}$  of its colorings are in  $A_{v_2}$ , as if  $\sigma \in A_{v_2}$  there must be  $m$  vertices adjacent to  $v_2$  with the color  $\sigma(v_2)$ . Therefore

$$|A_{v_1} \cap A_{v_2}| = \sum_C |C \cap A_{v_2}| \leq \sum_C \frac{d(v_2)}{m} = \frac{|A_{v_1}|}{k} \cdot \frac{d(v_2)}{m}.$$

It follows that  $\frac{|A_{v_1} \cap A_{v_2}|}{|A_{v_1}|}$  goes to 0 as  $k$  goes to infinity, so  $|A_{v_1} \cap A_{v_2}| = o(k^{n-m})$ .

**Case 2**  $v_1$  and  $v_2$  are adjacent.

Let  $W$  be the set of adjacent vertices to  $v_2$  other than  $v_1$ . They are not adjacent to  $v_1$  as  $T$  has no cycles. Split  $A_{v_1}$  into equivalence classes with the equivalence relation

$$\sigma_1 \sim \sigma_2 \Leftrightarrow \sigma_1(w) = \sigma_2(w) \text{ for all } w \notin W.$$

Each equivalence class  $C$  consists of  $k^{|W|}$  colorings, which may only differ in the colors of the vertices in  $W$ . If  $v_1$  and  $v_2$  are friends in the colorings of  $C$ , then a coloring in  $|C \cap A_{v_2}|$  must contain at least  $m-1$  vertices in  $W$  of the same color as  $v_2$ . Therefore

$$|C \cap A_{v_2}| = \sum_{l=m-1}^{|W|} \binom{|W|}{l} (k-1)^{|W|-l} < \sum_{l=0}^{|W|} \binom{|W|}{l} k^{|W|-1} = 2^{|W|} k^{|W|-1}.$$

Notice that here we are using  $m \geq 2$  so that  $l \geq 1$ . Otherwise, if  $v_1$  and  $v_2$  are not friends in the colorings of  $C$ , then

$$|C \cap A_{v_2}| = \sum_{l=m}^{|W|} \binom{|W|}{l} (k-1)^{|W|-l} < \sum_{l=0}^{|W|} \binom{|W|}{l} k^{|W|-1} = 2^{|W|} k^{|W|-1}.$$

Therefore

$$\begin{aligned} |A_{v_1} \cap A_{v_2}| &= \sum_C |C \cap A_{v_2}| < \sum_C 2^{|W|} k^{|W|-1} \\ &= \frac{|A_{v_1}|}{k^{|W|}} \cdot 2^{|W|} k^{|W|-1} = \frac{|A_{v_1}| \cdot 2^{|W|}}{k} \end{aligned}$$

and  $|A_{v_1} \cap A_{v_2}| = o(k^{n-m})$  follows as in the first case.

This completes the proof of the theorem.  $\square$

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## References

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