



**HAL**  
open science

# Classification of Ehrhart polynomials of integral simplices

Akihiro Higashitani

► **To cite this version:**

Akihiro Higashitani. Classification of Ehrhart polynomials of integral simplices. 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012), 2012, Nagoya, Japan. pp.587-594, 10.46298/dmtcs.3065 . hal-01283113

**HAL Id: hal-01283113**

**<https://hal.science/hal-01283113v1>**

Submitted on 5 Mar 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Classification of Ehrhart polynomials of integral simplices

Akihiro Higashitani <sup>†</sup>

Department of Pure and Applied Mathematics, Graduate School of Information Science and Technology, Osaka University, Toyonaka, Osaka 560-0043, Japan

**Abstract.** Let  $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$  be the  $\delta$ -vector of an integral convex polytope  $\mathcal{P}$  of dimension  $d$ . First, by using two well-known inequalities on  $\delta$ -vectors, we classify the possible  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i \leq 3$ . Moreover, by means of Hermite normal forms of square matrices, we also classify the possible  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i = 4$ . In addition, for  $\sum_{i=0}^d \delta_i \geq 5$ , we characterize the  $\delta$ -vectors of integral simplices when  $\sum_{i=0}^d \delta_i$  is prime.

**Résumé.** Soit  $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$  le  $\delta$ -vecteur d'un polytope intégrante de dimension  $d$ . Tout d'abord, en utilisant deux bien connus des inégalités sur  $\delta$ -vecteurs, nous classons les  $\delta$ -vecteurs possibles avec  $\sum_{i=0}^d \delta_i \leq 3$ . En outre, par le biais de Hermite formes normales, nous avons également classer les  $\delta$ -vecteurs avec  $\sum_{i=0}^d \delta_i = 4$ . De plus, pour  $\sum_{i=0}^d \delta_i \geq 5$ , nous caractérisons les  $\delta$ -vecteurs des simplex inégalités lorsque  $\sum_{i=0}^d \delta_i$  est premier.

**Keywords:** Ehrhart polynomial,  $\delta$ -vector, integral convex polytope, integral simplex.

## 1 Introduction

One of the most attractive problems on enumerative combinatorics of convex polytopes is to find a combinatorial characterization of the Ehrhart polynomials of integral convex polytopes. In particular, the  $\delta$ -vectors of integral simplices play an important and interesting role.

Let  $\mathcal{P} \subset \mathbb{R}^N$  be an *integral* polytope, i.e., a convex polytope any of whose vertices has integer coordinates, of dimension  $d$ , and let  $\partial\mathcal{P}$  denote the boundary of  $\mathcal{P}$ . Given a positive integer  $n$ , we define

$$i(\mathcal{P}, n) = |n\mathcal{P} \cap \mathbb{Z}^N| \quad \text{and} \quad i^*(\mathcal{P}, n) = |n(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|,$$

where  $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}$ ,  $n(\mathcal{P} \setminus \partial\mathcal{P}) = \{n\alpha : \alpha \in \mathcal{P} \setminus \partial\mathcal{P}\}$  and  $|X|$  is the cardinality of a finite set  $X$ . The systematic study of  $i(\mathcal{P}, n)$  originated in the work of Ehrhart [2], who established the following fundamental properties:

(0.1)  $i(\mathcal{P}, n)$  is a polynomial in  $n$  of degree  $d$ ;

(0.2)  $i(\mathcal{P}, 0) = 1$ ;

<sup>†</sup>The author is supported by JSPS Research Fellowship for Young Scientists.

(0.3) (loi de réciprocité)  $i^*(\mathcal{P}, n) = (-1)^d i(\mathcal{P}, -n)$  for every integer  $n > 0$ .

We say that  $i(\mathcal{P}, n)$  is the *Ehrhart polynomial* of  $\mathcal{P}$ . We refer the reader to [1, Chapter 3], [3, Part II] or [10, pp. 235–241] for the introduction to the theory of Ehrhart polynomials.

We define the sequence  $\delta_0, \delta_1, \delta_2, \dots$  of integers by the formula

$$(1 - \lambda)^{d+1} \sum_{n=0}^{\infty} i(\mathcal{P}, n) \lambda^n = \sum_{i=0}^{\infty} \delta_i \lambda^i. \quad (1)$$

Then the basic facts (0.1) and (0.2) on  $i(\mathcal{P}, n)$  together with a fundamental result on generating function [10, Corollary 4.3.1] guarantee that  $\delta_i = 0$  for every  $i > d$ . We say that the sequence (resp. the polynomial)

$$\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d) \quad \left( \text{resp. } \delta_{\mathcal{P}}(t) = \sum_{i=0}^d \delta_i t^i \right)$$

which appears in 1 is called the  $\delta$ -vector (resp. the  $\delta$ -polynomial) of  $\mathcal{P}$ . Alternate names of  $\delta$ -vectors are, for example, *Ehrhart  $h$ -vector*, *Ehrhart  $\delta$ -vector* or  *$h^*$ -vector*. By the reciprocity law (0.3), one has

$$\sum_{n=1}^{\infty} i^*(\mathcal{P}, n) \lambda^n = \frac{\sum_{i=0}^d \delta_{d-i} \lambda^{i+1}}{(1 - \lambda)^{d+1}}. \quad (2)$$

The following properties on  $\delta$ -vectors are well known:

- By 1, one has  $\delta_0 = i(\mathcal{P}, 0) = 1$  and  $\delta_1 = i(\mathcal{P}, 1) - (d + 1) = |\mathcal{P} \cap \mathbb{Z}^N| - (d + 1)$ .
- By 2, one has  $\delta_d = i^*(\mathcal{P}, 1) = |(\mathcal{P} \setminus \partial\mathcal{P}) \cap \mathbb{Z}^N|$ . In particular, we have  $\delta_1 \geq \delta_d$ .
- Each  $\delta_i$  is nonnegative [9].
- If  $\delta_d \neq 0$ , then one has  $\delta_1 \leq \delta_i$  for every  $1 \leq i \leq d - 1$  [4].
- It follows from 2 that

$$\max\{j : \delta_j \neq 0\} + \min\{k : k(\mathcal{P} - \partial\mathcal{P}) \cap \mathbb{Z}^N \neq \emptyset\} = d + 1.$$

- When  $d = N$ , the leading coefficient of  $i(\mathcal{P}, n)$ , which coincides with  $\sum_{i=0}^d \delta_i / d!$ , is equal to the usual volume of  $\mathcal{P}$  [10, Proposition 4.6.30]. In general, the positive integer  $\sum_{i=0}^d \delta_i$  is called the *normalized volume* of  $\mathcal{P}$ .

On the classification problem of the  $\delta$ -vectors of integral convex polytopes, when we consider the possible  $\delta$ -vectors of small dimensions, all of them are essentially given in [8] when  $d = 2$ . However, the possible  $\delta$ -vectors are presumably open when  $d \geq 3$ .

In this article, we discuss the  $\delta$ -vectors of small normalized volumes. In particular, the  $\delta$ -vectors of integral simplices play a crucial role when  $\sum_{i=0}^d \delta_i \leq 4$ .

A brief overview of this article is as follows. After reviewing some well-known technique how to compute the  $\delta$ -vectors of integral simplices in Section 1, we study the possible  $\delta$ -vectors of  $\sum_{i=0}^d \delta_i \leq 3$

by using two well-known inequalities on  $\delta$ -vectors in Section 2. In Section 3, we consider the case where  $\sum_{i=0}^d \delta_i = 4$  by considering all the  $\delta$ -vectors of all the integral simplices up to some equivalence. In Section 4, we discuss the  $\delta$ -vectors of integral simplices when  $\sum_{i=0}^d \delta_i$  is prime and we classify the possible  $\delta$ -vectors of integral simplices of  $\sum_{i=0}^d \delta_i = 5$  and 7.

## 2 Review on the computation of the $\delta$ -vectors of integral simplices

Before proving our theorems, we recall a combinatorial technique to compute the  $\delta$ -vector of an integral simplex.

Given an integral simplex  $\mathcal{F} \subset \mathbb{R}^N$  of dimension  $d$  with the vertices  $v_0, v_1, \dots, v_d$ , we set

$$S = \left\{ \sum_{i=0}^d r_i(v_i, 1) \in \mathbb{R}^{N+1} : 0 \leq r_i < 1 \right\} \cap \mathbb{Z}^{N+1} \text{ and}$$

$$S^* = \left\{ \sum_{i=0}^d r_i(v_i, 1) \in \mathbb{R}^{N+1} : 0 < r_i \leq 1 \right\} \cap \mathbb{Z}^{N+1}.$$

We define the degree of an integer point  $(\alpha, n) \in S$  ( $(\alpha, n) \in S^*$ ) with  $\deg(\alpha, n) = n$ , where  $\alpha \in \mathbb{Z}^N$  and  $n \in \mathbb{Z}_{\geq 0}$ . Let  $\delta_i = |\{\alpha \in S : \deg \alpha = i\}|$  and  $\delta_i^* = |\{\alpha \in S^* : \deg \alpha = i\}|$ . Then we have

**Lemma 2.1** *Work with the same notation as above. Then we have*

(a)

$$\sum_{n=0}^{\infty} i(\mathcal{F}, n)\lambda = \frac{\delta_0 + \delta_1\lambda + \dots + \delta_d\lambda^d}{(1 - \lambda)^{d+1}};$$

(b)

$$\sum_{n=0}^{\infty} i(\mathcal{F}^*, n)\lambda = \frac{\delta_1^*\lambda + \dots + \delta_{d+1}^*\lambda^{d+1}}{(1 - \lambda)^{d+1}};$$

(c)

$$\delta_i^* = \delta_{d+1-i} \text{ for } 1 \leq i \leq d + 1.$$

We also recall the following

**Lemma 2.2** [1, Theorem 2.4] *Suppose that  $(\delta_0, \delta_1, \dots, \delta_d)$  is the  $\delta$ -vector of an integral convex polytope of dimension  $d$ . Then there exists an integral convex polytope of dimension  $d + 1$  whose  $\delta$ -vector is  $(\delta_0, \delta_1, \dots, \delta_d, 0)$ .*

Note that the required  $\delta$ -vector is obtained by forming the pyramid over the integral convex polytope.

### 3 Two well-known inequalities on $\delta$ -vectors

In this section, we present two well-known inequalities on  $\delta$ -vectors. By using them, we give the complete classification of the possible  $\delta$ -vectors of integral convex polytopes with  $\sum_{i=0}^d \delta_i \leq 3$ .

Let  $s = \max\{i : \delta_i \neq 0\}$ . Stanley [11] shows the inequalities

$$\delta_0 + \delta_1 + \cdots + \delta_i \leq \delta_s + \delta_{s-1} + \cdots + \delta_{s-i}, \quad 0 \leq i \leq \lfloor s/2 \rfloor \quad (3)$$

by using the theory of Cohen–Macaulay rings. On the other hand, the inequalities

$$\delta_d + \delta_{d-1} + \cdots + \delta_{d-i} \leq \delta_1 + \delta_2 + \cdots + \delta_{i+1}, \quad 0 \leq i \leq \lfloor (d-1)/2 \rfloor \quad (4)$$

appear in [4, Remark (1.4)]. A proof of the inequalities 4 is given by using combinatorics on convex polytopes.

Somewhat surprisingly, when  $\sum_{i=0}^d \delta_i \leq 3$ , the above inequalities 3 together with 4 give a characterization of the possible  $\delta$ -vectors. In fact,

**Theorem 3.1** *Given a finite sequence  $(\delta_0, \delta_1, \dots, \delta_d)$  of nonnegative integers, where  $\delta_0 = 1$ , which satisfies  $\sum_{i=0}^d \delta_i \leq 3$ , there exists an integral convex polytope  $\mathcal{P} \subset \mathbb{R}^d$  of dimension  $d$  whose  $\delta$ -vector coincides with  $(\delta_0, \delta_1, \dots, \delta_d)$  if and only if  $(\delta_0, \delta_1, \dots, \delta_d)$  satisfies all inequalities 3 and 4. Moreover, all integral convex polytopes can be chosen to be simplices.*

Note that the “Only if” part of Theorem 3.1 is obvious. Thus we may show the “If” part. Moreover, when  $\sum_{i=0}^d \delta_i = 1$ , it is obvious that the possible sequence is only  $(1, 0, \dots, 0)$  and this is a  $\delta$ -vector of some integral convex polytope, in particular, integral simplex. A sketch of a proof of the “If” part with  $\sum_{i=0}^d \delta_i = 2$  or 3 is as follows:

- When  $\sum_{i=0}^d \delta_i = 2$ , the possible integer sequence looks like  $(1, 0, \dots, 0, \underbrace{1}_i, 0, \dots, 0) \in \mathbb{Z}^{d+1}$ , where  $\underbrace{1}_i$  means that  $\delta_i = 1$ . On the other hand, we have  $i \leq \lfloor (d+1)/2 \rfloor$  by 4. Hence we may find an integral convex polytope, in particular, an integral simplex, whose  $\delta$ -vector coincides with that. Note that we may construct such simplex with  $i = \lfloor (d+1)/2 \rfloor$  by virtue of Lemma 2.2.
- When  $\sum_{i=0}^d \delta_i = 3$ , we have two candidates of the possible integer sequences.
  - When  $(1, 0, \dots, 0, \underbrace{2}_i, 0, \dots, 0) \in \mathbb{Z}^{d+1}$ , similar discussions to the previous case can be applied.
  - When  $(1, 0, \dots, 0, \underbrace{1}_i, 0, \dots, 0, \underbrace{1}_j, 0, \dots, 0) \in \mathbb{Z}^{d+1}$ , from 3 and 4, we have the inequalities
 
$$1 \leq i < j \leq d, \quad 2i \leq j \quad \text{and} \quad i + j \leq d + 1.$$

Once we can find an integral simplex whose  $\delta$ -vector coincides with that with  $2i = j$  and  $i + j = d + 1$ , we can also construct an integral simplex whose  $\delta$ -vector is that for any integers  $i$  and  $j$  with the above inequalities.

On the other hand, the following example shows that Theorem 3.1 is no longer true for the case where  $\sum_{i=0}^d \delta_i = 4$ .

**Example 3.2** The integer sequence  $(1, 0, 1, 0, 1, 1, 0, 0)$  cannot be the  $\delta$ -vector of any integral convex polytope of dimension 7, although this satisfies the inequalities 3 and 4. In fact, suppose, on the contrary, that there exists an integral convex polytope  $\mathcal{P} \subset \mathbb{R}^N$  of dimension 7 with  $(\delta_0, \delta_1, \dots, \delta_7) = (1, 0, 1, 0, 1, 1, 0, 0)$  its  $\delta$ -vector. Since  $\delta_1 = 0$ , we know that  $\mathcal{P}$  is a simplex. Let  $v_0, v_1, \dots, v_7$  be the vertices of  $\mathcal{P}$ . By using Lemma 2.1, one has  $S = \{(0, \dots, 0), (\alpha, 2), (\beta, 4), (\gamma, 5)\}$  and  $S^* = \{(\alpha', 3), (\beta', 4), (\gamma', 6), (\sum_{i=0}^7 v_i, 7)\}$ . Write  $\alpha' = \sum_{i=0}^7 r_i v_i$  with each  $0 < r_i \leq 1$ . Since  $(\alpha', 3) \notin S$ , there is  $0 \leq j \leq 7$  with  $r_j = 1$ . If there are  $0 \leq k < \ell \leq 7$  with  $r_k = r_\ell = 1$ , say,  $r_0 = r_1 = 1$ , then  $0 < r_q < 1$  for each  $2 \leq q \leq 7$  and  $\sum_{i=2}^7 r_i = 1$ . Hence  $(\alpha' - v_0 - v_1, 1) \in S$ , a contradiction. Thus there is a unique  $0 \leq j \leq 7$  with  $r_j = 1$ , say,  $r_0 = 1$ . Then  $\alpha = \sum_{i=1}^7 r_i v_i$  and  $\gamma = \sum_{i=1}^7 (1 - r_i) v_i$ . Let  $\mathcal{F}$  denote the facet of  $\mathcal{P}$  whose vertices are  $v_1, v_2, \dots, v_7$  with  $\delta(\mathcal{F}) = (\delta'_0, \delta'_1, \dots, \delta'_6) \in \mathbb{Z}^7$ . Then  $\delta'_2 = \delta'_5 = 1$ . Since  $\delta'_i \leq \delta_i$  for each  $0 \leq i \leq 6$ , it follows that  $\delta(\mathcal{F}) = (1, 0, 1, 0, 0, 1, 0)$ . This contradicts the inequalities 3.

## 4 Hermite normal forms with a given $\delta$ -vector

In this section, we give the complete classification of the possible  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i = 4$  by means of Hermite normal forms. Moreover, it turns out that all the possible  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i = 4$  can be chosen to be integral simplices.

Let  $\mathbb{Z}^{d \times d}$  denote the set of  $d \times d$  integer matrices. Recall that a matrix  $A \in \mathbb{Z}^{d \times d}$  is *unimodular* if  $\det(A) = \pm 1$ . Given integral convex polytopes  $\mathcal{P}$  and  $\mathcal{Q}$  in  $\mathbb{R}^d$  of dimension  $d$ , we say that  $\mathcal{P}$  and  $\mathcal{Q}$  are *unimodularly equivalent* if there exists a unimodular matrix  $U \in \mathbb{Z}^{d \times d}$  and an integral vector  $w \in \mathbb{Z}^d$ , such that  $\mathcal{Q} = f_U(\mathcal{P}) + w$ , where  $f_U$  is the linear transformation in  $\mathbb{R}^d$  defined by  $U$ , i.e.,  $f_U(\mathbf{v}) = \mathbf{v}U$  for all  $\mathbf{v} \in \mathbb{R}^d$ . Clearly, if  $\mathcal{P}$  and  $\mathcal{Q}$  are unimodularly equivalent, then  $\delta(\mathcal{P}) = \delta(\mathcal{Q})$ . Conversely, given a vector  $v \in \mathbb{Z}_{\geq 0}^{d+1}$ , it is natural to ask what are all the integral convex polytopes  $\mathcal{P}$  under unimodular equivalence, such that  $\delta(\mathcal{P}) = v$ . We focus on this problem for simplices with one vertex at the origin. In addition, we do not allow any shifts in the equivalence, i.e., integral convex polytopes  $\mathcal{P}$  and  $\mathcal{Q}$  of dimension  $d$  are equivalent if there exists a unimodular matrix  $U$ , such that  $\mathcal{Q} = f_U(\mathcal{P})$ .

For discussing the representative under this equivalence of the integral simplices with one vertex at the origin, we consider Hermite normal forms of square matrices.

Let  $\mathcal{P}$  be an integral simplex in  $\mathbb{R}^d$  of dimension  $d$  with the vertices  $v_0, v_1, \dots, v_d$ , where  $v_0 = (0, \dots, 0)$ . Define  $M(\mathcal{P}) \in \mathbb{Z}^{d \times d}$  to be the matrix with the row vectors  $v_1, \dots, v_d$ . Then we have the following connection between the matrix  $M(\mathcal{P})$  and the  $\delta$ -vector of  $\mathcal{P}$ :  $|\det(M(\mathcal{P}))| = \sum_{i=0}^d \delta_i$ . In this setting,  $\mathcal{P}$  and  $\mathcal{P}'$  are equivalent if and only if  $M(\mathcal{P})$  and  $M(\mathcal{P}')$  have the same Hermite normal form, where the *Hermite normal form* of a nonsingular integral square matrix  $B$  is a unique nonnegative lower triangular matrix  $A = (a_{ij}) \in \mathbb{Z}_{\geq 0}^{d \times d}$  such that  $A = BU$  for some unimodular matrix  $U \in \mathbb{Z}^{d \times d}$  and  $0 \leq a_{ij} < a_{ii}$  for all  $1 \leq j < i$ . (See, e.g., [7, Chapter 4].) In other words, we can pick the Hermite normal form as the representative in each equivalence class. By considering the  $\delta$ -vectors of all the integral simplices arising from the Hermite normal forms  $M$  with  $\det(M) = 4$ , we obtain the following

**Theorem 4.1** Let  $1 + t^{i_1} + t^{i_2} + t^{i_3}$  be a polynomial in  $t$  with  $1 \leq i_1 \leq i_2 \leq i_3 \leq d$ . Then there exists an integral convex polytope  $\mathcal{P} \subset \mathbb{R}^d$  of dimension  $d$  whose  $\delta$ -polynomial coincides with  $1 + t^{i_1} + t^{i_2} + t^{i_3}$  if and only if  $(i_1, i_2, i_3)$  satisfies

$$i_3 \leq i_1 + i_2, \quad i_1 + i_3 \leq d + 1, \quad i_2 \leq \lfloor (d + 1)/2 \rfloor \quad (5)$$

and an additional condition

$$2i_2 \leq i_1 + i_3 \quad \text{or} \quad i_2 + i_3 \leq d + 1. \quad (6)$$

Moreover, all integral convex polytopes can be chosen to be simplices.

Note that the inequalities 5 follow from the inequalities 3 and 4, that is to say, the condition 5 is automatically a necessary condition. Thus, the condition 6 is the new necessary condition on  $\delta$ -vectors when  $\sum_{i=0}^d \delta_i = 4$ .

A sketct of a proof of this theorem is as follows:

- On the “If” part, we may construct integral simplices whose  $\delta$ -polynomials look like  $1 + t^{i_1} + t^{i_2} + t^{i_3}$  satisfying 5 and 6. By characterizing the possible  $\delta$ -vectors of all the integral simplices arising from the Hermite normal forms  $M$  with  $\det(M) = 4$ , we can find such integral simplices.
- On the “Only if” part, we may show that if a polynomial  $1 + t^{i_1} + t^{i_2} + t^{i_3}$  satisfies 5 but does not satisfy 6, then  $i_1 > 1$ , i.e., an integral convex polytope with this  $\delta$ -polynomial is always a simplex. By characterizing the possible  $\delta$ -vectors of all the integral simplices arising from the Hermite normal forms  $M$  with  $\det(M) = 4$ , we can say that there exists no integral simplex whose  $\delta$ -polynomial is equal to  $1 + t^{i_1} + t^{i_2} + t^{i_3}$  not satisfying 6.

**Example 4.2** As we see in Example 3.2, the integer sequence  $(1, 0, 1, 0, 1, 1, 0, 0)$  cannot be the  $\delta$ -vector of any integral convex polytope of dimension 7. In fact, since  $8 = 2i_2 > i_1 + i_3 = 7$  and  $9 = i_2 + i_3 > 8$ , there exists no integral convex polytope of dimension 7 whose  $\delta$ -polynomial is  $1 + t^2 + t^4 + t^5$ . On the other hand, there exists an integral convex polytope of dimension 8 whose  $\delta$ -vector is  $(1, 0, 1, 0, 1, 1, 0, 0)$  since  $9 = i_2 + i_3 = d + 1$ .

**Remark 4.3** We see that all the possible  $\delta$ -vectors can be obtained by integral simplices when  $\sum_{i=0}^d \delta_i \leq 4$ . However, the  $\delta$ -vector  $(1, 3, 1)$  cannot be obtained from any integral simplex, while this is a possible  $\delta$ -vector of some integral convex polytope of dimension 2. In fact, suppose that  $(1, 3, 1)$  can be obtained from a simplex. Since  $\min\{i : \delta_i \neq 0, i > 0\} = 1$  and  $\max\{i : \delta_i \neq 0\} = 2$ , one has  $\min\{i : \delta_i \neq 0, i > 0\} = 3 - \max\{i : \delta_i \neq 0\}$ , which implies that the assumption of [5, Theorem 2.3] is satisfied. Thus the  $\delta$ -vector must be shifted symmetric, a contradiction.

## 5 Ehrhart polynomials of integral simplices with prime volumes

From the previous two sections, we know that all the possible  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i \leq 4$  can be obtained by integral simplices, while this does not hold when  $\sum_{i=0}^d \delta_i = 5$ . Therefore, for the further classifications of the  $\delta$ -vectors with  $\sum_{i=0}^d \delta_i \geq 5$ , it is natural to investigate the  $\delta$ -vectors of integral simplices. In this

section, we establish the new equalities and inequalities on  $\delta$ -vectors for integral simplices when  $\sum_{i=0}^d \delta_i$  is prime. Moreover, by using them, we classify all the possible  $\delta$ -vectors of integral simplices with  $\sum_{i=0}^d \delta_i = 5$  and 7.

The following equalities or inequalities are new constraints on the  $\delta$ -vectors of integral simplices when  $\sum_{i=0}^d \delta_i$  is prime.

**Theorem 5.1** *Let  $\mathcal{P} \subset \mathbb{R}^N$  be an integral simplex of dimension  $d$  and  $\delta(\mathcal{P}) = (\delta_0, \delta_1, \dots, \delta_d)$  its  $\delta$ -vector. Suppose that  $\sum_{i=0}^d \delta_i = p$  is an odd prime number. Let  $i_1, \dots, i_{p-1}$  be the positive integers such that  $\sum_{i=0}^d \delta_i t^i = 1 + t^{i_1} + \dots + t^{i_{p-1}}$  with  $1 \leq i_1 \leq \dots \leq i_{p-1} \leq d$ . Then*

(a) 
$$i_1 + i_{p-1} = i_2 + i_{p-2} = \dots = i_{(p-1)/2} + i_{(p+1)/2} \leq d + 1;$$

(b) 
$$i_k + i_\ell \geq i_{k+\ell} \text{ for } 1 \leq k \leq \ell \leq p - 1 \text{ with } k + \ell \leq p - 1.$$

**Example 5.2** When  $\sum_{i=0}^d \delta_i$  is not prime, Theorem 5.1 is not true. In fact, by virtue of Theorem 4.1,  $(1, 1, 0, 2, 0, 0)$  is the  $\delta$ -vector of some integral simplex of dimension 5. However, one has  $2 = i_1 + i_1 < i_2 = 3$ .

A proof of this theorem is given by considering the additive group  $S$  (appeared in Section 2) associated with an integral simplex with prime normalized volume. Since the order of  $S$  is equal to the normalized volume of  $\mathcal{P}$ ,  $S$  is nothing but a cyclic group  $\mathbb{Z}/p\mathbb{Z}$ . By studying  $S$  and the degrees of its elements, we obtain the statements (a) and (b). Note that (b) follows from [6, Theorem 2.2], known as *Cauchy–Davenport theorem*.

As an application of Theorem 5.1, we give a complete characterization of the possible  $\delta$ -vectors of integral simplices when  $\sum_{i=0}^d \delta_i = 5$  and 7.

**Corollary 5.3** *Given a finite sequence  $(\delta_0, \delta_1, \dots, \delta_d)$  of nonnegative integers, where  $\delta_0 = 1$ , which satisfies  $\sum_{i=0}^d \delta_i = 5$ , there exists an integral simplex  $\mathcal{P} \subset \mathbb{R}^d$  of dimension  $d$  whose  $\delta$ -vector coincides with  $(\delta_0, \delta_1, \dots, \delta_d)$  if and only if  $i_1, \dots, i_4$  satisfy*

$$i_1 + i_4 = i_2 + i_3 \leq d + 1, \quad 2i_1 \geq i_2 \quad \text{and} \quad i_1 + i_2 \geq i_3,$$

where  $i_1, \dots, i_4$  are the positive integers such that  $\sum_{i=0}^d \delta_i t^i = 1 + t^{i_1} + \dots + t^{i_4}$  with  $1 \leq i_1 \leq \dots \leq i_4 \leq d$ .

**Corollary 5.4** *Given a finite sequence  $(\delta_0, \delta_1, \dots, \delta_d)$  of nonnegative integers, where  $\delta_0 = 1$ , which satisfies  $\sum_{i=0}^d \delta_i = 7$ , there exists an integral simplex  $\mathcal{P} \subset \mathbb{R}^d$  of dimension  $d$  whose  $\delta$ -vector coincides with  $(\delta_0, \delta_1, \dots, \delta_d)$  if and only if  $i_1, \dots, i_6$  satisfy*

$$i_1 + i_6 = i_2 + i_5 = i_3 + i_4 \leq d + 1, \quad i_1 + i_\ell \geq i_{\ell+1} \text{ for } 1 \leq \ell \leq 3 \text{ and } 2i_2 \geq i_4,$$

where  $i_1, \dots, i_6$  are the positive integers such that  $\sum_{i=0}^d \delta_i t^i = 1 + t^{i_1} + \dots + t^{i_6}$  with  $1 \leq i_1 \leq \dots \leq i_6 \leq d$ .

By virtue of Theorem 5.1, the “Only if” parts of Corollary 5.3 and Corollary 5.4 are obvious.



## References

- [1] M. Beck and S. Robins, “Computing the Continuous Discretely,” Undergraduate Texts in Mathematics, Springer, 2007.
- [2] E. Ehrhart, “Polynômes Arithmétiques et Méthode des Polyèdres en Combinatoire,” Birkhäuser, Boston/Basel/Stuttgart, 1977.
- [3] T. Hibi, “Algebraic Combinatorics on Convex Polytopes,” Carlaw Publications, Glebe NSW, Australia, 1992.
- [4] T. Hibi, A lower bound theorem for Ehrhart polynomials of convex polytopes, *Adv. in Math.* **105** (1994), 162 – 165.
- [5] A. Higashitani, Shifted symmetric  $\delta$ -vectors of convex polytopes, *Discrete Math.* **310** (2010), 2925–2934.
- [6] M. B. Nathanson, “Additive number theory, inverse problems and the geometry of subsets,” Springer–Verlag, 1996.
- [7] A. Schrijver, “Theory of Linear and Integer Programming,” John Wiley & Sons, 1986.
- [8] P. R. Scott, On convex lattice polygons, *Bull. Austral. Math. Soc.* **15** (1976), 395 – 399.
- [9] R. P. Stanley, Decompositions of rational convex polytopes, *Annals of Discrete Math.* **6** (1980), 333 – 342.
- [10] R. P. Stanley, “Enumerative Combinatorics, Volume 1,” Wadsworth & Brooks/Cole, Monterey, Calif., 1986.
- [11] R. P. Stanley, On the Hilbert function of a graded Cohen–Macaulay domain, *J. Pure and Appl. Algebra* **73** (1991), 307 – 314.