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# Hecke algebra characters and quantum chromatic symmetric functions

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**Abstract.** We evaluate induced sign characters of  $H_n(q)$  at certain elements of  $H_n(q)$  and conjecture an interpretation for the resulting polynomials as generating functions for  $P$ -tableaux by a certain statistic. Our conjecture relates the quantum chromatic symmetric functions of Shareshian and Wachs to  $H_n(q)$  characters.

**Résumé.** Nous évaluons les caractères de signe induits de  $H_n(q)$  à certains éléments de  $H_n(q)$  et nous conjecturons une interprétation des polynômes résultants comme fonctions génératrices pour les tableaux- $P$  par une certaine statistique. Cette conjecture établit un lien entre les fonctions chromatiques symétriques de Shareshian et Wachs et les caractères de  $H_n(q)$ .

**Keywords:** Hecke algebra character, unit interval order,  $P$ -tableau, chromatic symmetric function, quantum analog.

## 1 Introduction

Let  $S_n$  be the symmetric group, let  $s_i = (i, i + 1)$ ,  $i = 1, \dots, n - 1$  denote its standard generators, and for all  $v \in S_n$ , let  $\ell(v)$  denote the length of any reduced (short as possible) expression  $s_{i_1} \cdots s_{i_\ell}$  for  $v$ . (See [12] for more information on this material.) It is well known that each conjugacy class of  $S_n$  consists of the permutations whose cycle sizes in weakly decreasing order are equal to some fixed partition  $\lambda = (\lambda_1, \dots, \lambda_r) \vdash n$ . Define the *Young diagram* of  $\lambda$  to be the left-justified array of  $n$  boxes having  $\lambda_i$  boxes in row  $i$ . The  $\mathbb{Z}$ -module of  $S_n$  class functions, those functions  $f : S_n \rightarrow \mathbb{Z}$  satisfying  $f(w^{-1}vw) = f(v)$  for all  $v, w \in S_n$ , has several well-studied bases:

1. the irreducible characters  $\{\chi^\lambda \mid \lambda \vdash n\}$ ,
2. the induced sign characters  $\{\epsilon^\lambda = \text{sgn} \uparrow_{S_\lambda}^{S_n} \mid \lambda \vdash n\}$ ,
3. the induced trivial characters  $\{\eta^\lambda = \text{triv} \uparrow_{S_\lambda}^{S_n} \mid \lambda \vdash n\}$ ,
4. the monomial class functions  $\{\phi^\lambda \mid \lambda \vdash n\}$ ,

where  $S_\lambda$  is the Young subgroup of  $S_n$  of type  $\lambda$ . These bases are related to one another just as are the Schur, elementary, (complete) homogeneous, and monomial bases of the space of homogeneous degree  $n$

symmetric functions. Specifically, we have

$$\begin{aligned} h_\lambda &= \sum_{\mu} K_{\mu,\lambda} s_\mu, & e_\lambda &= \sum_{\mu} K_{\mu^\top,\lambda} s_\mu, & s_\lambda &= \sum_{\mu} K_{\lambda,\mu} m_\mu, \\ \eta^\lambda &= \sum_{\mu} K_{\mu,\lambda} \chi^\mu, & \epsilon^\lambda &= \sum_{\mu} K_{\mu^\top,\lambda} \chi^\mu, & \chi^\lambda &= \sum_{\mu} K_{\lambda,\mu} \phi^\mu, \end{aligned} \tag{1}$$

where  $K = (K_{\lambda,\mu})$  is the invertible matrix of *Kostka numbers*, and  $\lambda^\top$  is the partition whose Young diagram is transpose (or *conjugate*) to that of  $\lambda$ . For each of these class functions  $f$  which is not an irreducible character, the integer  $f(v)$  has a simple combinatorial formula; when  $f = \chi^\lambda$ , the number  $\chi^\lambda(v)$  is computed by a cumbersome algorithm called the Murnaghan-Nakayama rule. This fact is somewhat unfortunate, since the irreducible characters are the most important of the above class functions.

The above class functions extend linearly to the group algebra  $\mathbb{Z}[S_n]$ . Work of Goulden and Jackson [4] and Greene [5] has led to the study of combinatorially interpreting  $f(z)$  for  $f$  a class function and  $z$  in  $\mathbb{Z}[S_n]$ . In particular, these authors used  $S_n$  subgroups of the form  $S_{[i,j]}$ , which fix all letters outside of the interval  $[i,j] = \{i, \dots, j\}$  of  $[n] \stackrel{\text{def}}{=} \{1, \dots, n\}$ , to construct elements

$$C'_{[i,j]} \stackrel{\text{def}}{=} \sum_{v \in S_{[i,j]}} v \tag{2}$$

of  $\mathbb{Z}[S_n]$ , and products of these. It is not difficult to show that we have

$$f(C'_{[i_1,j_1]} \cdots C'_{[i_m,j_m]}) \geq 0 \tag{3}$$

when  $f$  is  $\eta^\lambda$  or  $\epsilon^\lambda$ , and to give a combinatorial interpretation of the resulting nonnegative integers in these cases. (See Theorems 2.2-2.3.) Stembridge showed furthermore [19, Cor. 3.3] that the inequality (3) holds for  $f = \chi^\lambda$ , and conjectured [20, Conj. 2.1] that it holds also for  $f = \phi^\lambda$ . In neither of these cases is there a proposed combinatorial interpretation, however. Stanley [15] initiated the study of the special case of (3) in which we have  $i_1 < \cdots < i_m$  and  $j_1 < \cdots < j_m$ , relating this special case of the equation to certain *chromatic symmetric functions*, and to posets and graphs which avoid certain substructures. In this special case Gasharov [2, Thm. 2] found an interpretation when  $f = \chi^\lambda$ , but when  $f = \phi^\lambda$  we still have no proof of positivity and no conjectured combinatorial interpretation.

It is conceivable that quantum analogs of  $S_n$  characters may aid in the formulation of the unknown combinatorial interpretations mentioned above. In particular, the (type  $A$ ) Hecke algebra  $H_n(q)$  and certain functions  $\chi_q^\lambda, \epsilon_q^\lambda, \eta_q^\lambda, \phi_q^\lambda : H_n(q) \rightarrow \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ , known as  $H_n(q)$  *traces*, serve as analogs of  $S_n$  and its class functions. Furthermore, there are well-studied quantum analogs  $C'_{[i,j]}(q) \in H_n(q)$  of the elements (2) in  $\mathbb{Z}[S_n]$ , and Haiman [6, Lem. 1.1] has proved that we have

$$f_q(C'_{[i_1,j_1]}(q) \cdots C'_{[i_m,j_m]}(q)) \in \mathbb{N}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}] \tag{4}$$

when  $f_q$  is  $\eta_q^\lambda, \epsilon_q^\lambda$ , or  $\chi_q^\lambda$ . He conjectured that the same holds for  $f_q = \phi_q^\lambda$ . Unfortunately, no simple combinatorial formulas are known or even conjectured for the evaluation of  $H_n(q)$  traces at standard basis elements of  $H_n(q)$  (in general), or at the special elements in (4). We will state and prove the first such formula for elements of the form (4) in the special case that  $i_1 < \cdots < i_r, j_1 < \cdots < j_r$ , and will relate this to the quantum chromatic symmetric functions introduced recently by Shareshian and Wachs [13].

In Section 2 we associate an element  $\beta(P)$  of  $\mathbb{Z}[S_n]$  to each labeled poset  $P$ . We evaluate induced sign characters at  $\beta(P)$  and interpret the resulting nonnegative integers in terms of  $P$ -tableaux. In Section 3 we restrict our attention to special classes of posets  $P$ , especially unit interval orders. We evaluate other class functions at the associated elements  $\beta(P)$ , again interpreting the resulting nonnegative integers in terms of  $P$ -tableaux. We also discuss the appearance of these interpretations in Stanley’s chromatic symmetric functions [15]. In Section 4 we present quantum analogs of much of the earlier material, including an element  $\beta_q(P)$  of  $H_n(q)$  which we associate to each labeled poset  $P$ . In Section 5 we present our main results. For each unit interval order  $P$ , we evaluate induced  $H_n(q)$  sign characters at  $\beta_q(P)$  and interpret the resulting elements of  $\mathbb{N}[q]$  as generating functions for  $P$ -tableaux by a certain inversion statistic. Finally, we express the Shareshian-Wachs quantum chromatic symmetric functions [13] in terms of induced  $H_n(q)$  sign characters.

## 2 $P$ -tableaux and induced $S_n$ sign characters

Given a poset  $P$ , define a  $P$ -tableau of shape  $\lambda$  to be a Young diagram of shape  $\lambda$  whose boxes contain the elements of  $P$ . A  $P$ -tableau is called *column-strict* if whenever elements  $i_1, \dots, i_r$  appear from top to bottom in a column, then we have that  $i_1 <_P \dots <_P i_r$  is a chain in  $P$ . A  $P$ -tableau is called *row-semistrict* if whenever  $i_1, i_2$  appear consecutively (from left to right) in a row, then we have  $i_1 <_P i_2$  or  $i_1$  incomparable to  $i_2$  in  $P$ . A  $P$ -tableau is called *semistandard* if it is both column-strict and row-semistrict. When  $P$  is simply the chain of integers  $1 < \dots < n$ , we will refer to a  $P$ -tableau as a *Young tableau*.

It is often convenient to label the elements of a poset and to encode the labeled poset as a certain 0-1 matrix. If a labeled poset  $P$  has  $n$  elements, we will always assume the labels to be  $\{1, \dots, n\}$ . We call  $P$  *naturally labeled* if whenever  $i <_P j$ , we have  $i < j$  as integers. Every poset has at least one natural labeling. Given a labeled poset  $P$ , we define the *antiadjacency matrix*  $A = A(P) = (a_{i,j})$  of  $P$  by

$$a_{i,j} = \begin{cases} 0 & \text{if } i <_P j, \\ 1 & \text{otherwise.} \end{cases} \tag{5}$$

Two different labelings of a poset give antiadjacency matrices which are conjugate to one another by a permutation matrix. If  $P$  is naturally labeled, then in its antiadjacency matrix, all entries on or below the diagonal are 1. We may use the antiadjacency matrix of  $P$  to count column-strict  $P$ -tableaux as follows. Call a sequence  $(I_1, \dots, I_r)$  of disjoint subsets of  $[n]$  an *ordered set partition of type  $\lambda$*  if we have  $|I_k| = \lambda_k$  for  $k = 1, \dots, r$  and  $I_1 \cup \dots \cup I_r = [n]$ . For any  $n \times n$  matrix  $A$  and any subset  $I \subset [n]$ , define  $A_{I,I}$  to be the  $|I| \times |I|$  submatrix  $(a_{i,j})_{i,j \in I}$  of  $A$ .

**Proposition 2.1** *Let  $P$  be a labeled poset with antiadjacency matrix  $A$ , and let  $\lambda$  be a partition of  $n = |P|$ . Then the number of column-strict  $P$ -tableaux is given by*

$$\sum_{(I_1, \dots, I_r)} \det(A_{I_1, I_1}) \cdots \det(A_{I_r, I_r}), \tag{6}$$

where the sum is over all ordered set partitions of  $[n]$  of type  $\lambda$ .

**Proof:** Suppose that  $P$  is naturally labeled. It is straightforward to show that each minor in (6) satisfies

$$\det(A_{I_k, I_k}) = \begin{cases} 1 & \text{if } I_k \text{ is a chain in } P, \\ 0 & \text{otherwise,} \end{cases} \tag{7}$$

and thus the claim is true in this case. (See, e.g., [16].) Furthermore, it is also straightforward to show that the polynomial function  $\text{Mat}_{n \times n}(\mathbb{Z}) \rightarrow \mathbb{Z}$  defined by the expression (6) is invariant under conjugation by a permutation matrix.  $\square$

This relationship between the antiadjacency matrix and  $P$ -tableaux allows us to state a relationship between induced sign characters of  $S_n$  and  $P$ -tableaux. To any labeled  $n$ -element poset  $P$  we may associate an element  $\beta(P)$  of  $\mathbb{Z}[S_n]$  by

$$\beta(P) = \sum_v v, \quad (8)$$

where the sum is over all permutations  $v$  satisfying  $i \not\prec_P v_i$  for all  $i$ . Equivalently,  $\beta(P)$  is a sum of permutations which correspond to the placement of  $n$  nonattacking rooks on entries of  $A(P)$  which are equal to 1 [18]. Now we have the following [1, Cor. 4.2]. (See also [18, Eqn. (5.1)].)

**Theorem 2.2** *Let  $P$  be a labeled poset and let  $\lambda$  be a partition of  $|P|$ . Then  $\epsilon^\lambda(\beta(P))$  equals the number of column-strict  $P$ -tableaux of shape  $\lambda$ .*

**Proof:** Let  $n = |P|$  and let  $x = (x_{i,j})_{i,j \in [n]}$  be a matrix of  $n^2$  variables. The Littlewood-Merris-Watkins identity for induced sign characters [10, Sec. 6.5], [11, Sec. 1] states that in  $\mathbb{Z}[x_{1,1}, \dots, x_{n,n}]$  we have

$$\sum_{(I_1, \dots, I_r)} \det(x_{I_1, I_1}) \cdots \det(x_{I_r, I_r}) = \sum_{v \in S_n} \epsilon^\lambda(v) x_{1, v_1} \cdots x_{n, v_n}. \quad (9)$$

Now let  $A$  be the antiadjacency matrix of  $P$  and evaluate (9) at  $x = A$ . Since  $a_{1, v_1} \cdots a_{n, v_n} = 1$  precisely when  $i \not\prec_P v_i$  for all  $i$ , we obtain

$$\epsilon^\lambda \left( \sum_{v \in S_n} a_{1, v_1} \cdots a_{n, v_n} v \right) = \epsilon^\lambda(\beta(P)),$$

and our result follows from Proposition 2.1.  $\square$

A similar argument using permanents of submatrices applies to the induced trivial characters of  $S_n$ .

**Theorem 2.3** *Let  $P$  be a labeled poset and let  $\lambda$  be a partition of  $|P|$ . Then  $\eta^\lambda(\beta(P))$  equals the number of row-semistrict  $P$ -tableaux of shape  $\lambda$ .*

**Proof:** Omitted.  $\square$

### 3 $(\mathbf{3} + \mathbf{1})$ -free posets and chromatic symmetric functions

To state more results and conjectures concerning the interpretation of  $S_n$  class functions, we restrict our attention to special classes of posets. The following standard definitions concerning posets may be found in [17, Ch. 3] and [21, Sec. 9.2]. Given posets  $P, Q$ , let  $P \oplus Q$  be the  $(|P| + |Q|)$ -element poset whose Hasse diagram is that for  $P$  written below that for  $Q$ , i.e., with every element of  $P$  being less than every element of  $Q$ . Letting  $\mathbf{n}$  denote an  $n$ -element chain  $x_1 <_P \cdots <_P x_n$ , we have, e.g.,  $\mathbf{4} \cong \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}$ . Let  $P + Q$  be the  $(|P| + |Q|)$ -element poset which is a disjoint union of  $P$  and  $Q$ , i.e., the poset whose Hasse diagrams is that for  $P$  written beside that for  $Q$ , with all elements of  $P$  being incomparable to all

elements of  $Q$ . Thus  $\mathbf{a} + \mathbf{b}$  is the disjoint union of an  $a$ -element chain and a  $b$ -element chain. We say that  $P$  is  $(\mathbf{a} + \mathbf{b})$ -free if no induced subposet of  $P$  is isomorphic to  $\mathbf{a} + \mathbf{b}$ . Call  $P$  an *interval order* if it is  $(\mathbf{2} + \mathbf{2})$ -free, and a *unit interval order* if it is  $(\mathbf{2} + \mathbf{2})$ -free and  $(\mathbf{3} + \mathbf{1})$ -free.

While no true analog of Theorem 2.2 and 2.3 is known for  $\chi^\lambda(\beta(P))$  and  $\phi^\lambda(\beta(P))$ , some results exist for special cases of  $\lambda$  and for  $P$  belonging to one of the above classes of posets.

**Proposition 3.1** *Let  $P$  be a labeled  $(\mathbf{3} + \mathbf{1})$ -free poset and let  $\lambda$  be a partition of  $|P|$ .*

- (1)  $\chi^\lambda(\beta(P))$  equals the number of standard  $P$ -tableaux of shape  $\lambda$  [2, Thm. 2].
- (2) If  $P$  is a unit interval order and  $\lambda_1 \leq 2$ , then  $\phi^\lambda(\beta(P))$  equals zero if there exists a column-strict  $P$ -tableaux of shape  $\mu \prec \lambda$  in the dominance order; otherwise it equals the number of column-strict  $P$ -tableaux of shape  $\lambda$  [1, Thm. 5.7].
- (3) If  $P$  is a unit interval order and  $\lambda = (k, \dots, k)$ , then  $\phi^\lambda(\beta(P))$  equals the number of ways to cover  $P$  with certain cycles [20, Thm. 2.8].

Furthermore, Stembridge’s [20, Conj. 2.1] has the following special case.

**Conjecture 3.2** *Let  $P$  be a labeled unit interval order and let  $\lambda$  be a partition of  $|P|$ . Then we have  $\phi^\lambda(\beta(P)) \geq 0$ .*

Even with this restriction on  $P$ , there is no conjectured combinatorial interpretation for  $\phi^\lambda(\beta(P))$  which applies to all  $\lambda$ .

One property of unit interval orders  $P$  which facilitates combinatorial interpretation is the fact that certain labelings of  $P$  cause the antiadjacency matrix  $A(P)$  and the  $\mathbb{Z}[S_n]$  element  $\beta(P)$  to have particularly nice forms. Define the *altitude* of a poset element  $i$  to be  $\alpha(i) = \#\{x \mid x <_P i\} - \#\{x \mid x >_P i\}$ . Call a labeling an *altitude respecting* or *ar-labeling* if  $i < j$  whenever  $\alpha(i) < \alpha(j)$ . It is known that every ar-labeling of a  $(\mathbf{3} + \mathbf{1})$ -free poset is natural. It is also known that in the antiadjacency matrix corresponding to an ar-labeling of an  $n$ -element unit interval order, the zero entries form a right justified Young diagram which fits inside the right justified shape  $(n - 1, \dots, 1)$ . (See [21, Sec. 8.2].) In Proposition 4.1 we will prove that this special form of the antiadjacency matrix implies a nice expression for  $\beta(P)$ ,

$$\beta(P) = \frac{C'_{[i_1, j_1]} \cdots C'_{[i_m, j_m]}}{k}, \quad i_1 < \cdots < i_m, \quad j_1 < \cdots < j_m, \quad k \in \mathbb{N}. \tag{10}$$





$S_n$  class functions may be used to define Stanley’s *chromatic symmetric function* [15] for a poset  $P$ ,

$$X_P = \sum_{\lambda \vdash n} c_\lambda m_\lambda, \tag{11}$$

where  $c_\lambda$  is the number of ways to partition  $P$  into a sequence of  $r$  chains of sizes  $(\lambda_1, \dots, \lambda_r)$  respectively, and to assign color  $k$  to the  $k$ th chain. Equivalently, we have  $c_\lambda = \epsilon^\lambda(\beta(P))$ . The transition matrices of Kostka numbers in (1) imply furthermore that we have

$$X_P = \sum_{\lambda \vdash n} \epsilon^\lambda(\beta(P)) m_\lambda = \sum_{\lambda \vdash n} \chi^{\lambda^\top}(\beta(P)) s_\lambda = \sum_{\lambda \vdash n} \phi^\lambda(\beta(P)) e_\lambda. \tag{12}$$

Monomial nonnegative by definition, the chromatic symmetric functions are sometimes Schur nonnegative and even elementary nonnegative as well. For example, the four posets  $P = \mathbf{3}, \mathbf{2} + \mathbf{1}, (\mathbf{1} + \mathbf{1}) \oplus \mathbf{1}, \mathbf{1} + \mathbf{1} + \mathbf{1}$  and their chromatic symmetric functions  $X_P$  are

				(13)
$\begin{aligned} X_P &= 6m_{111} + 3m_{21} + m_3 \\ &= s_{111} + 2s_{21} + s_3 \\ &= e_{111}, \end{aligned}$	$\begin{aligned} X_P &= 6m_{111} + m_{21} \\ &= 4s_{111} + s_{21} \\ &= 3e_3 + e_{21}, \end{aligned}$	$\begin{aligned} X_P &= 6m_{111} + 2m_{21} \\ &= 2s_{111} + 2s_{21} \\ &= 2e_{21}, \end{aligned}$	$\begin{aligned} X_P &= 6m_{111} \\ &= 6s_{111} \\ &= 6e_3. \end{aligned}$	

These nonnegativity phenomena are addressed in the following result of Gasharov [2, Thm. 2] and conjecture of Stanley and Stembridge [15, Conj. 5.1], [18, Conj. 5.5].

**Theorem 3.3** *If  $P$  is  $(\mathbf{3} + \mathbf{1})$ -free, then  $X_P$  is Schur nonnegative.*

**Conjecture 3.4** *If  $P$  is  $(\mathbf{3} + \mathbf{1})$ -free, then  $X_P$  is elementary nonnegative.*

By (12), Theorem 3.3 is equivalent to Proposition 3.1 (1), Conjecture 3.4 is stronger than Conjecture 3.2, and two special cases of Conjecture 3.4 have been proved in Proposition 3.1 (2)-(3).

### 4 The Hecke algebra and quantum polynomial ring in $n^2$ variables

Let  $H_n(q)$  be the (type  $A$ ) Hecke algebra, the noncommutative  $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -algebra with multiplicative identity  $T_e = 1$ , generated by  $T_{s_1}, \dots, T_{s_{n-1}}$ , subject to relations

$$\begin{aligned} T_{s_i}^2 &= (q - 1)T_{s_i} + qT_e, & \text{for } i = 1, \dots, n - 1, \\ T_{s_i}T_{s_j}T_{s_i} &= T_{s_j}T_{s_i}T_{s_j}, & \text{if } |i - j| = 1, \\ T_{s_i}T_{s_j} &= T_{s_j}T_{s_i}, & \text{if } |i - j| \geq 2. \end{aligned} \tag{14}$$

If  $s_{i_1} \dots s_{i_\ell}$  is (any) reduced expression for  $v \in S_n$  we define  $T_v = T_{s_{i_1}} \dots T_{s_{i_\ell}}$ . The  $n!$  elements  $\{T_v \mid v \in S_n\}$  are a basis of  $H_n(q)$  as a  $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -module.

Let  $\chi_q^\lambda$  be the irreducible  $H_n(q)$  character indexed by  $\lambda$ , and let  $\epsilon_q^\lambda$  and  $\eta_q^\lambda$  be the sign and trivial characters induced from Young subalgebras of type  $\lambda$ . All of these  $H_n(q)$  characters are  $H_n(q)$  traces, i.e., they satisfy  $f_q(gh) = f_q(hg)$  for  $g, h \in H_n(q)$ . Like a class function, a trace  $f_q$  satisfies  $f_q(g^{-1}hg) = f_q(h)$  whenever  $g$  is invertible. Define  $\phi_q^\lambda = \sum_{\mu} K_{\lambda, \mu}^{-1} \chi_q^\mu$  to be the monomial trace indexed by  $\lambda$ . These sets of traces form four bases for the space of  $H_n(q)$  traces and are related to one another by the same transition matrices of Kostka numbers (1) that relate the  $S_n$  class function bases used in Sections 1-3:

$$\eta_q^\lambda = \sum_{\mu \vdash n} K_{\mu, \lambda} \chi_q^\mu, \quad \epsilon_q^\lambda = \sum_{\mu \vdash n} K_{\mu, \lambda} \chi_q^\mu, \quad \chi_q^\lambda = \sum_{\mu \vdash n} K_{\lambda, \mu} \phi_q^\mu. \tag{15}$$

(See, e.g., [3].) For none of these traces  $f_q$  do we have a known combinatorial interpretation of  $f_q(T_v)$  for all  $v \in S_n$ . Neither do we have a known combinatorial interpretation of

$$f_q(C'_{[i_1, j_1]}(q) \cdots C'_{[i_m, j_m]}(q)), \tag{16}$$

where

$$C'_{[i,j]}(q) \stackrel{\text{def}}{=} q^{-\frac{\ell(w)}{2}} \sum_{v \in S_{[i,j]}} T_v,$$

and  $w$  is the longest element in  $S_{[i,j]}$ . We therefore consider special cases of the elements (16) which are quantum analogs of the elements  $\beta(P)$  in  $\mathbb{Z}[S_n]$ . To any labeled  $n$ -element poset  $P$  we may associate an element  $\beta_q(P)$  of  $H_n(q)$  by

$$\beta_q(P) = \sum_{v \in S_n} T_v, \tag{17}$$

where the sum is over all permutations  $v \in S_n$  satisfying  $i \not\prec_P v_i$  for all  $i$ . The  $ar$ -labeling of a unit interval order  $P$  gives a nice expression for  $\beta_q(P)$ .

**Proposition 4.1** *Let  $P$  be an  $ar$ -labeled  $n$ -element unit interval order. Then there exist subintervals  $[i_1, j_1], \dots, [i_m, j_m]$  of  $[n]$  with  $i_1 < \dots < i_m, j_1 < \dots < j_m$ , a polynomial  $p \in \mathbb{N}[q]$ , a 3412-, 4231-avoiding permutation  $w = w(P)$  in  $S_n$ , and a nonnegative integer  $c$  such that*

$$\beta_q(P) = \frac{q^{\frac{\ell(w)+c}{2}} C'_{[i_1, j_1]}(q) \cdots C'_{[i_m, j_m]}(q)}{p(q)} = \sum_{v \leq w} T_v. \tag{18}$$

**Proof:** (Idea.) The  $n$ -element unit interval orders correspond bijectively to certain planar networks  $G$  having  $n$  source and  $n$  sink vertices, with each such network being composed of smaller networks indexed by intervals  $[i_1, j_1], \dots, [i_m, j_m]$ . For each such planar network  $G$ , one defines a matrix  $B = (b_{i,j})$  by setting  $b_{i,j} = 1$  if there exists a path from source  $i$  to sink  $j$ , and  $b_{i,j} = 0$  otherwise. One also defines an element  $\beta_q(G) = \sum_v b_{1,v_1} \cdots b_{n,v_n} T_v$  of  $H_n(q)$ . If in the above bijection  $G$  corresponds to  $P$ , then one can show that the antiadjacency matrix  $A$  of  $P$  satisfies  $a_{1,v_1} \cdots a_{n,v_n} = b_{1,v_1} \cdots b_{n,v_n}$  for all  $v \in S_n$  (although  $A$  and  $B$  are not in general equal). Thus we have  $\beta_q(P) = \beta_q(G)$ .

By [14, Thm. 4.3], there exist a certain nonnegative integer  $c$  and a 3412-, 4231-avoiding permutation  $w \in S_n$  associated to  $G$  such that  $q^{-\frac{\ell(w)}{2}} \beta_q(G)$  is equal to  $q^{\frac{c}{2}} C'_{[i_1, j_1]}(q) \cdots C'_{[i_m, j_m]}(q)$  divided by a polynomial in  $\mathbb{N}[q]$ . By [14, Lem. 5.3],  $w$  also satisfies

$$b_{1,v_1} \cdots b_{n,v_n} = \begin{cases} 1 & \text{if } v \leq w \\ 0 & \text{otherwise.} \end{cases} \tag{19}$$

Thus we have  $\beta_q(P) = \beta_q(G) = \sum_{v \leq w} T_v$ . □

For  $P$  an  $ar$ -labeled unit interval order, let  $w(P)$  denote the permutation appearing in the above proposition. We remark that by results in [9],  $\beta_q(P)$  is equal to  $q^{\frac{\ell(w)}{2}}$  times the Kazhdan-Lusztig basis element  $C'_{w(P)}(q)$  of  $H_n(q)$  [7], and the factors  $C'_{[i,j]}(q)$  also are Kazhdan-Lusztig basis elements.

Even for the special case (18) of (16), there is no published combinatorial interpretation for  $f_q(\beta_q(P))$  when  $f_q \in \{\chi_q^\lambda, \eta_q^\lambda, \epsilon_q^\lambda, \phi_q^\lambda\}$ . On the other hand, Haiman's [6, Lem. 1.1, Conj. 2.1] have the following special cases which suggest that combinatorial interpretations for some of these expressions should exist.

**Proposition 4.2** *Let  $P$  be an  $ar$ -labeled unit interval order and let  $\lambda$  be a partition of  $|P|$ . Then we have  $\chi_q^\lambda(\beta_q(P)) \in \mathbb{N}[q]$ , and therefore  $\epsilon_q^\lambda(\beta_q(P)) \in \mathbb{N}[q]$  and  $\eta_q^\lambda(\beta_q(P)) \in \mathbb{N}[q]$ .*



**Conjecture 4.3** *Let  $P$  be an ar-labeled unit interval order and let  $\lambda$  be a partition of  $|P|$ . Then we have  $\phi_q^\lambda(\beta_q(P)) \in \mathbb{N}[q]$ .*

In Section 5 we give a combinatorial interpretation for the polynomial  $\epsilon_q^\lambda(q_{e,w}\beta_q(P))$  for  $P$  an ar-labeled unit interval order. To do this, we employ a quantum analog of the identity (9) which holds in the quantum polynomial ring  $\mathcal{A}_n(q)$ . Let  $\mathcal{A}_n(q)$  be the noncommutative  $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -algebra generated by  $n^2$  variables  $x = (x_{1,1}, \dots, x_{n,n})$ , subject to the relations

$$\begin{aligned} x_{i,\ell}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{i,\ell}, & x_{j,k}x_{i,\ell} &= x_{i,\ell}x_{j,k}, \\ x_{j,k}x_{i,k} &= q^{\frac{1}{2}}x_{i,k}x_{j,k}, & x_{j,\ell}x_{i,k} &= x_{i,k}x_{j,\ell} + (q^{\frac{1}{2}} - q^{-\frac{1}{2}})x_{i,\ell}x_{j,k}, \end{aligned} \tag{20}$$

for all indices  $1 \leq i < j \leq n$  and  $1 \leq k < \ell \leq n$ .

$\mathcal{A}_n(q)$  is a multigraded  $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -module whose components  $\mathcal{A}_{L,M}(q)$  are spanned by monomials having a fixed multiset  $L$  of row indices and a fixed multiset  $M$  of column indices. In particular, the component  $\mathcal{A}_{[n],[n]}(q)$  contains monomials having total degree  $n$ , with all row indices and all column indices appearing exactly once. Defining  $x^{u,v} = x_{u_1,v_1} \cdots x_{u_n,v_n}$  for  $u, v \in S_n$ , we thus have

$$\mathcal{A}_{[n],[n]}(q) = \text{span}\{x^{u,v} \mid u, v \in S_n\}.$$

By the relations (20), each multigraded component has a  $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -basis of monomials in which variables appear in lexicographic order. Thus the component  $\mathcal{A}_{[n],[n]}(q)$  has the natural basis  $\{x^{e,v} \mid v \in S_n\}$ , and contains the quantum determinant, defined by

$$\det_q(x) \stackrel{\text{def}}{=} \sum_{v \in S_n} (-q^{-\frac{1}{2}})^{\ell(v)} x^{e,v}.$$

The quantized Littlewood-Merris-Watkins identity, due to Konvalinka and the second author [8, Thm. 5.4], gives generating functions for the induced sign characters of  $H_n(q)$ .

**Theorem 4.4** *Fix any partition  $\lambda \vdash n$ . Then in  $\mathcal{A}_{[n],[n]}(q)$  we have the identity*

$$\sum_{(I_1, \dots, I_r)} \det_q(x_{I_1, I_1}) \cdots \det_q(x_{I_r, I_r}) = \sum_{v \in S_n} \epsilon_q^\lambda(q^{-\frac{\ell(v)}{2}} T_v) x^{e,v}, \tag{21}$$

where the first sum is over all ordered set partitions of  $[n]$  of type  $\lambda$ .

## 5 Main results

Recall that for  $P$  a unit interval order with antiadjacency matrix  $A$ , the combinatorial interpretation of  $\beta(P)$  in Theorem 2.2 follows immediately from the evaluation of both sides of the classical Littlewood-Merris-Watkins identity (9) at  $x = A$ . However, the quantum analog (21) of this identity does not allow us to compute  $\epsilon^\lambda(\beta_q(P))$  as the evaluation of the left-hand side at  $x = A$ , since the noncommuting variables  $(x_{i,j})$  satisfy nontrivial relations. If we are to use the left-hand side of (21) to understand  $\epsilon^\lambda(\beta_q(P))$ , we must first express it in terms of the natural basis  $\{x^{e,v} \mid v \in S_n\}$ . We therefore define a family of maps as follows. For each matrix  $A \in \text{Mat}_{n \times n}(\mathbb{Z})$ , let  $\sigma_A : \mathcal{A}_{[n],[n]}(q) \rightarrow \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$  be the  $\mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$ -linear map defined by

$$\sigma_A(x^{e,v}) = q^{\frac{\ell(v)}{2}} a_{1,v_1} \cdots a_{n,v_n}.$$

This definition has the following immediate consequence.

**Proposition 5.1** *Let  $P$  be an ar-labeled poset with antiadjacency matrix  $A$ , let  $w = w(P)$ , and let  $\lambda$  be a partition of  $n = |P|$ . Then we have*

$$\sum_{(I_1, \dots, I_r)} \sigma_A(\det_q(x_{I_1, I_1}) \cdots \det_q(x_{I_r, I_r})) = \epsilon_q^\lambda(\beta_q(P)), \tag{22}$$

where the sum is over all ordered set partitions of  $[n]$  of type  $\lambda$ .

**Proof:** The left-hand side of (22) equals

$$\sigma_A\left(\sum_{v \in S_n} \epsilon_q^\lambda(q^{-\frac{\ell(v)}{2}} T_v) x^{e,v}\right) = \sum_{v \in S_n} \epsilon_q^\lambda(T_v) q^{-\frac{\ell(v)}{2}} \sigma_A(x^{e,v}) = \epsilon_q^\lambda\left(\sum_{v \leq w} T_v\right) = \epsilon_q^\lambda(\beta_q(P)),$$

since the proof of Proposition 4.1 implies that  $a_{1,v_1} \cdots a_{n,v_n}$  is 1 if  $v \leq w$  and is 0 otherwise. □

To combinatorially interpret each term on the left-hand side of (22), we introduce the following statistic on  $P$ -tableaux. Given a  $P$ -tableau  $U$ , define an *inversion* in  $U$  to be a pair  $(i, j)$  of incomparable elements in  $P$  satisfying  $i < j$ , with  $i$  appearing in a column of  $U$  to the right of the column containing  $j$ . Let  $\text{INV}(U)$  denote the number of inversions in  $U$ .

**Lemma 5.2** *Let  $P$  be a labeled unit interval order with antiadjacency matrix  $A$ , let  $\lambda$  be a partition of  $n = |P|$ , let  $(I_1, \dots, I_r)$  be an ordered set partition of  $[n]$  of type  $\lambda$ , and let  $U$  be the  $P$ -tableau whose  $j$ th column contains the poset elements labeled by  $I_j$ , with labels increasing upward in that column. If  $U$  is a column-strict  $P$ -tableau, then we have*

$$\sigma_A(\det_q(x_{I_1, I_1}) \cdots \det_q(x_{I_r, I_r})) = q^{\text{INV}(U)}. \tag{23}$$

**Proof:** Omitted. □

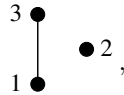
The authors believe that if we keep the hypotheses of Lemma 5.2, but assume that the  $P$ -tableau  $U$  is *not* column-strict, then  $\sigma_A(\det_q(x_{I_1, I_1}) \cdots \det_q(x_{I_r, I_r}))$  vanishes. In light of Proposition 5.1, we thus have the following conjecture.

**Conjecture 5.3** *Let  $P$  be an ar-labeled unit interval order, and let  $\lambda$  be a partition of  $|P|$ . Then we have*

$$\epsilon_q^\lambda(\beta_q(P)) = \sum_U q^{\text{INV}(U)}, \tag{24}$$

where the sum is over all column-strict  $P$ -tableaux of shape  $\lambda^\top$ .

For example, let  $P$  be the poset  $2 + 1$  with its unique ar-labeling



and its two pairs of incomparable elements  $(1, 2)$  and  $(2, 3)$ . Then there are six column-strict  $P$ -tableaux of shape  $111^\top = 3$ , one of shape  $21^\top = 21$ , and none of shape  $3^\top = 111$ . Enumerating these and indicating inversions by arrows, we have

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 1 & 3 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 1 & 3 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 2 & 3 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & 1 & 2 \\ \hline \end{array} \quad \begin{array}{|c|c|c|} \hline 3 & 2 & 1 \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 3 & 2 \\ \hline 1 & 2 \\ \hline \end{array}. \tag{25}$$

Comparing these tableaux with the known evaluations

$$\epsilon_q^{111}(\beta_q(P)) = 1 + 4q + q^2, \quad \epsilon_q^{21}(\beta_q(P)) = q, \quad \epsilon_q^3(\beta_q(P)) = 0, \quad (26)$$

we see that Conjecture 5.3 holds in this case.

Shareshian and Wachs [13] defined a quantum analog of Stanley's chromatic symmetric function. Given a labeled poset  $P$ , their *quantum chromatic quasisymmetric function* for  $P$  is the sum

$$X_{P,q} = \sum_{\kappa} q^{\#\{(i,j) \mid i \not\leq_P j, j \not\leq_P i, i < j, \kappa(i) > \kappa(j)\}} \prod_{k \in \mathbb{P}} x_k^{\#\{i \in P \mid \kappa(i) = k\}} \quad (27)$$

over all colorings  $\kappa : P \rightarrow \mathbb{N}$  of  $P$ , where a function  $\kappa$  is termed a *coloring* if it satisfies  $\kappa(i) \neq \kappa(j)$  whenever  $i$  and  $j$  are incomparable in  $P$ . By [13, Prop. 4.4], when  $P$  is an  $n$ -element *ar*-labeled unit interval order then  $X_{P,q}$  is in fact symmetric,

$$X_{P,q} = \sum_{\lambda \vdash n} c_{\lambda,q} m_{\lambda}, \quad (28)$$

and it follows that  $c_{\lambda,q} = \sum_U q^{\text{INV}(U)}$ , where the sum is over all column-strict  $P$ -tableaux of shape  $\lambda^{\top}$ . For example, when the four unit interval orders in (13) are *ar*-labeled, their quantum chromatic symmetric functions are

$$\begin{aligned} X_{\mathbf{3},q} &= 6m_{111} + 3m_{21} + m_3 & X_{\mathbf{2+1},q} &= (1 + 4q + q^2)m_{111} + qm_{21} \\ &= s_{111} + 2s_{21} + s_3 & &= (1 + 2q + q^2)s_{111} + qs_{21} \\ &= e_{111}, & &= (1 + q + q^2)e_3 + qe_{21}, \end{aligned}$$

$$\begin{aligned} X_{(\mathbf{1+1}) \oplus \mathbf{1},q} &= (3 + 3q)m_{111} + (1 + q)m_{21} & X_{\mathbf{1+1+1},q} &= (1 + 2q + 2q^2 + q^3)m_{111} \\ &= (1 + q)s_{111} + (1 + q)s_{21} & &= (1 + 2q + 2q^2 + q^3)s_{111} \\ &= (1 + q)e_{21}, & &= (1 + 2q + 2q^2 + q^3)e_3. \end{aligned}$$

Now Conjecture 5.3 and Equation (15) imply the following.

**Conjecture 5.4** *If  $P$  is an  $n$ -element *ar*-labeled unit interval order then the Shareshian-Wachs quantum chromatic symmetric function satisfies*

$$X_{P,q} = \sum_{\lambda \vdash n} \epsilon_q^{\lambda}(\beta_q(P)) m_{\lambda} = \sum_{\lambda \vdash n} \chi_q^{\lambda^{\top}}(\beta_q(P)) s_{\lambda} = \sum_{\lambda \vdash n} \phi_q^{\lambda}(\beta_q(P)) e_{\lambda}.$$

Shareshian and Wachs have formulated the following quantum analog [13, Conj. 4.8] of Stanley and Stembridge's [15, Conj. 5.1] ([18, Conj. 5.5]).

**Conjecture 5.5** *If  $P$  is an *ar*-labeled unit interval order, then the coefficients arising in the elementary expansion of  $X_{P,q}$  belong to  $\mathbb{N}[q]$ .*

By Conjecture 5.4, the authors believe this to be equivalent to the special case of Haiman's [6, Conj. 2.1] which we have stated in Conjecture 4.3. Furthermore, Shareshian and Wachs have proved the following weaker result [13, Thm. 4.11], which is a quantum analog of the special case of Gasharov's [2, Thm. 2] stated in Theorem 3.3.

**Theorem 5.6** *If  $P$  is an ar-labeled unit interval order, then the coefficient of  $s_\lambda$  in the Schur expansion of  $X_{P,q}$  is  $\sum_U q^{\text{INV}(U)}$ , where the sum is over all semistandard  $P$ -tableaux of shape  $\lambda^\top$ .*

This coincides with the authors' conjectured interpretation of the irreducible  $H_n(q)$  characters.

**Conjecture 5.7** *If  $P$  is an ar-labeled unit interval order and  $w = w(P)$ , then  $\chi_q^\lambda(\beta_q(P)) = \sum_U q^{\text{INV}(U)}$ , where the sum is over all semistandard  $P$ -tableaux of shape  $\lambda^\top$ .*

Since Theorem 3.3 and Conjecture 3.4 apply to  $(\mathbf{3} + \mathbf{1})$ -free posets, it would be interesting to see if the symmetry of  $X_{P,q}$  holds more generally for ar-labeled  $(\mathbf{3} + \mathbf{1})$ -free posets, and to interpret  $\epsilon_q^\lambda(\beta_q(P))$ ,  $\eta_q^\lambda(\beta_q(P))$ , and  $\chi_q^\lambda(\beta_q(P))$  in these cases.

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