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Noncommutative symmetric functions with matrix parameters (extended abstract)

Alain Lascoux, Jean-Christophe Novelli, and Jean-Yves Thibon

Institut Gaspard Monge, Université Paris-Est Marne-la-Vallée, 5 Boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France

Abstract. We define new families of noncommutative symmetric functions and quasi-symmetric functions depending on two matrices of parameters, and more generally on parameters associated with paths in a binary tree. Appropriate specializations of both matrices then give back the two-vector families of Hivert, Lascoux, and Thibon and the noncommutative Macdonald functions of Bergeron and Zabrocki.

Résumé. Nous définissons de nouvelles familles de fonctions symétriques non-commutatives et de fonctions quasi-symétriques, dépendant de deux matrices de paramètres, et plus généralement, de paramètres associés à des chemins dans un arbre binaire. Pour des spécialisations appropriées, on retrouve les familles à deux vecteurs de Hivert-Lascoux-Thibon et les fonctions de Macdonald non-commutatives de Bergeron-Zabrocki.

Keywords: Noncommutative symmetric functions, Quasi-symmetric functions, Macdonald polynomials

1 Introduction

The theory of Hall-Littlewood, Jack, and Macdonald polynomials is one of the most interesting subjects in the modern theory of symmetric functions. It is well-known that combinatorial properties of symmetric functions can be explained by lifting them to larger algebras (the so-called combinatorial Hopf algebras), the simplest examples being **Sym** (Noncommutative symmetric functions [3]) and its dual *QSym* (Quasi-symmetric functions [5]).

There have been several attempts to lift Hall-Littlewood and Macdonald polynomials to **Sym** and *QSym* [1, 7, 8, 13, 14]. The analogues defined in [1] were similar to, though different from, those of [8]. These last ones admitted multiple parameters q_i and t_i , which however could not be specialized to recover the version of [1].

The aim of this article is to show that many more parameters can be introduced in the definition of such bases. Actually, one can have a pair of $n \times n$ matrices (Q_n, T_n) for each degree n . The main properties established in [1] and [8] remain true in this general context, and one recovers the BZ and HLT polynomials for appropriate specializations of the matrices.

In the last section, another possibility involving quasideterminants is explored.

2 Notations

Our notations for noncommutative symmetric functions will be as in [3, 10]. The Hopf algebra of noncommutative symmetric functions is denoted by \mathbf{Sym} , or by $\mathbf{Sym}(A)$ if we consider the realization in terms of an auxiliary alphabet. Bases of \mathbf{Sym}_n are labelled by compositions I of n . The noncommutative complete and elementary functions are denoted by S_n and Λ_n , and the notation S^I means $S_{i_1} \dots S_{i_r}$. The ribbon basis is denoted by R_I .

The notation $I \vDash n$ means that I is a composition of n . The conjugate composition is denoted by I^\sim . The graded dual of \mathbf{Sym} is $QSym$ (quasi-symmetric functions). The dual basis of (S^I) is (M_I) (monomial), and that of (R_I) is (F_I) . The *descent set* of $I = (i_1, \dots, i_r)$ is $\text{Des}(I) = \{i_1, i_1 + i_2, \dots, i_1 + \dots + i_{r-1}\}$.

Finally, there are two operations on compositions: if $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$, the composition $I.J$ is $(i_1, \dots, i_r, j_1, \dots, j_s)$ and $I \triangleright J$ is $(i_1, \dots, i_r + j_1, \dots, j_s)$.

3 \mathbf{Sym}_n as a Grassmann algebra

Since for $n > 0$, \mathbf{Sym}_n has dimension 2^{n-1} , it can be identified (as a vector space) with a Grassmann algebra on $n - 1$ generators $\eta_1, \dots, \eta_{n-1}$ (that is, $\eta_i \eta_j = -\eta_j \eta_i$, so that in particular $\eta_i^2 = 0$). This identification is meaningful, for example, in the context of the representation theory of the 0-Hecke algebras $H_n(0)$ (see [2]).

If I is a composition of n with descent set $D = \{d_1, \dots, d_k\}$, we make the identification

$$R_I \longleftrightarrow \eta_D := \eta_{d_1} \eta_{d_2} \dots \eta_{d_k} . \tag{1}$$

For example, $R_{213} \leftrightarrow \eta_2 \eta_3$. We then have

$$S^I \longleftrightarrow (1 + \eta_{d_1})(1 + \eta_{d_2}) \dots (1 + \eta_{d_k}) \tag{2}$$

and

$$\Lambda^I \longleftrightarrow \prod_{i=1}^{n-1} \theta_i, \tag{3}$$

where $\theta_i = \eta_i$ if $i \notin D$ and $\theta_i = 1 + \eta_i$ otherwise. Other bases have simple expression under this identification, e.g., Ψ_n , Φ_n and Hivert's Hall-Littlewood basis [7].

3.1 Structure on the Grassmann algebra

Let $*$ be the anti-involution given by $\eta_i^* = (-1)^i \eta_i$. The Grassmann integral of any function f is defined by

$$\int d\eta f := f_{12\dots n-1}, \quad \text{where } f = \sum_k \sum_{i_1 < \dots < i_k} f_{i_1 \dots i_k} \eta_{i_1} \dots \eta_{i_k}. \tag{4}$$

We define a bilinear form on \mathbf{Sym}_n by

$$(f, g) = \int d\eta f^* g. \tag{5}$$

Then,

$$(R_I, R_J) = (-1)^{\ell(I)-1} \delta_{I, J^\sim}, \tag{6}$$

so that this is (up to an unessential sign) the Bergeron-Zabrocki scalar product [1, Eq. (4)].

3.2 Factorized elements in the Grassman algebra

Now, for a sequence of parameters $Z = (z_1, \dots, z_{n-1})$, let

$$K_n(Z) = (1 + z_1\eta_1)(1 + z_2\eta_2) \dots (1 + z_{n-1}\eta_{n-1}). \quad (7)$$

We then have

Lemma 3.1

$$\langle K_n(X), K_n(Y) \rangle = \prod_{i=1}^{n-1} (y_i - x_i). \quad (8)$$

3.3 Bases of Sym

We shall investigate bases of \mathbf{Sym}_n of the form

$$\tilde{H}_I = K_n(Z_I) = \sum_J \tilde{\mathbf{k}}_{IJ} R_J, \quad (9)$$

where Z_I is a sequence of parameters depending on the composition I of n .

The bases defined in [8] and [1] are of the previous form and for both of them, the determinant of the Kostka matrix $\mathcal{K} = (\tilde{\mathbf{k}}_{IJ})$ is a product of linear factors (as for ordinary Macdonald polynomials). This is explained by the fact that these matrices have the form

$$\begin{pmatrix} A & xA \\ B & yB \end{pmatrix} \quad (10)$$

where A and B have a similar structure, and so on recursively. Indeed, for such matrices,

Lemma 3.2 *Let A, B be two $m \times m$ matrices. Then,*

$$\begin{vmatrix} A & xA \\ B & yB \end{vmatrix} = (y - x)^m \det A \cdot \det B. \quad (11)$$

3.4 Duality

Similarly, the dual vector space $QSym_n = \mathbf{Sym}_n^*$ can be identified with a Grassmann algebra on another set of generators ξ_1, \dots, ξ_{n-1} . Encoding the fundamental basis F_I of Gessel [5] by

$$\xi_D := \xi_{d_1} \xi_{d_2} \dots \xi_{d_k}, \quad (12)$$

the usual duality pairing such that the F_I are dual to the R_I is given in this setting by

$$\langle \xi_D, \eta_E \rangle = \delta_{DE}. \quad (13)$$

Let

$$L_n(Z) = (z_1 - \xi_1) \dots (z_{n-1} - \xi_{n-1}). \quad (14)$$

Then, as above, we have a factorization identity:

Lemma 3.3

$$\langle L_n(X), K_n(Y) \rangle = \prod_{i=1}^{n-1} (x_i - y_i). \quad (15)$$

4 Bases associated with paths in a binary tree

Let $\mathbf{y} = \{y_u\}$ be a family of indeterminates indexed by all boolean words of length $\leq n-1$. For example, for $n = 3$, we have the six parameters $y_0, y_1, y_{00}, y_{01}, y_{10}, y_{11}$.

We can encode a composition I with descent set D by the boolean word $u = (u_1, \dots, u_{n-1})$ such that $u_i = 1$ if $i \in D$ and $u_i = 0$ otherwise.

Let us denote by $u_{m\dots p}$ the sequence $u_m u_{m+1} \dots u_p$ and define

$$P_I := (1 + y_{u_1} \eta_1)(1 + y_{u_1\dots 2} \eta_2) \dots (1 + y_{u_1\dots n-1} \eta_{n-1}) \quad (16)$$

or, equivalently,

$$P_I := K_n(Y_I) \quad \text{with} \quad Y_I = (y_{u_1}, y_{u_1\dots 2}, \dots, y_u) =: (y_k(I)). \quad (17)$$

Similarly, let

$$Q_I := (y_{w_1} - \xi_1)(y_{w_1\dots 2} - \xi_2) \dots (y_{w_1\dots n-1} - \xi_{n-1}) =: L_n(Y^I), \quad (18)$$

where $w_{1\dots k} = u_1 \dots u_{k-1} \overline{u_k}$ where $\overline{u_k} = 1 - u_k$, so that

$$Y^I := (y_{w_1}, y_{w_1\dots 2}, \dots, y_{w_1\dots n-1}) =: (y^k(I)). \quad (19)$$

4.1 Kostka matrices

The Kostka matrix is defined as the transpose of the transition matrices from P_I to R_J . This matrix is recursively of the form of Eq. (10). Thus, its determinant factors completely. For $n = 4$, it is

$$(y_1 - y_0)^4 (y_{01} - y_{00})^2 (y_{11} - y_{10})^2 (y_{001} - y_{000})(y_{011} - y_{010})(y_{101} - y_{100})(y_{111} - y_{110}). \quad (20)$$

Proposition 4.1 *The bases (P_I) and (Q_I) are adjoint to each other, up to normalization:*

$$\langle Q_I, P_J \rangle = \langle L_n(Y^I), K_n(Y_J) \rangle = \prod_{k=1}^{n-1} (y^k(I) - y_k(J)), \quad (21)$$

which is indeed zero unless $I = J$.

From this, it is easy to derive a product formula for the basis P_I .

Proposition 4.2 *Let I and J be two compositions of respective sizes n and m . The product $P_I P_J$ is a sum over an interval of the lattice of compositions*

$$P_I P_J = \sum_{K \in [I \triangleright (m), I \cdot (1^m)]} c_{IJ}^K P_K \quad (22)$$

where

$$c_{IJ}^K = \frac{\langle L_{n+m}(Y^K), K_{n+m}(Y_I \cdot 1 \cdot Y_J) \rangle}{\langle Q_K, P_K \rangle}, \quad (23)$$

where $Y_I \cdot 1 \cdot Y_J$ stands for the sequence $(y_1(I), \dots, y_n(I), 1, y_1(J), \dots, y_m(J))$.

4.2 The quasi-symmetric side

As we have seen before, the (Q_I) being dual to the (P_I) , the inverse Kostka matrix is given by the simple construction:

Proposition 4.3 *The inverse of the Kostka matrix is given by*

$$(\mathcal{K}_n^{-1})_{IJ} = (-1)^{\ell(I)-1} \prod_{d \in \text{Des}(I \sim)} y^d(J) \prod_{p=1}^{n-1} \frac{1}{y^p(J) - y_p(J)}. \tag{24}$$

4.3 Some specializations

Let us now consider the specialization sending all y_w to 1 if w ends with a 1 and denote by \mathcal{K}' the matrix obtained by this specialization. Then, as in [8, p. 10],

Proposition 4.4 *Let n be an integer. Then*

$$S_n = \mathcal{K}_n \mathcal{K}'_n^{-1} \tag{25}$$

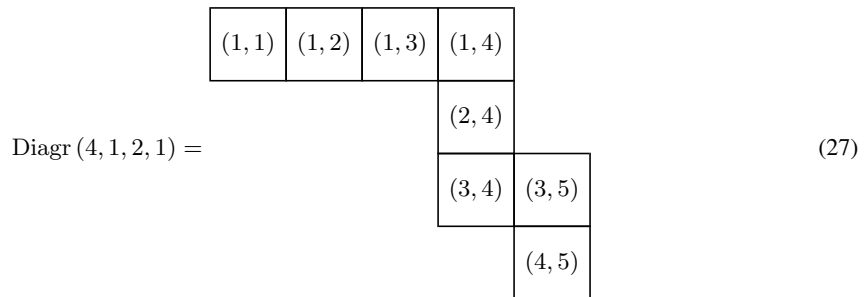
is lower triangular. More precisely, let Y'_J be the image of Y_J by the previous specialization and define Y'^J in the same way. Then the coefficient s_{IJ} indexed by (I, J) is

$$s_{IJ} = \prod_{k=1}^{n-1} \frac{y_k(I) - y'^k(J)}{y'_k(J) - y'^k(J)}. \tag{26}$$

5 The two-matrix family

5.1 A specialization of the paths in a binary tree

The above bases can now be specialized to bases $\tilde{H}(A; Q, T)$, depending on two infinite matrices of parameters. Label the cells of the ribbon diagram of a composition I of n with their matrix coordinates as follows:



We associate a variable z_{ij} with each cell except $(1, 1)$ by setting $z_{ij} := q_{i,j-1}$ if (i, j) has a cell on its left, and $z_{ij} := t_{i-1,j}$ if (i, j) has a cell on its top. The alphabet $Z(I) = (z_j(I))$ is the sequence of the z_{ij} in their natural order.

Next, if J is a composition of the same integer n , form the monomial

$$\tilde{\mathbf{k}}_{IJ}(Q, T) = \prod_{d \in \text{Des}(J)} z_d(I). \tag{28}$$

For example, with $I = (4, 1, 2, 1)$ and $J = (2, 1, 1, 2, 2)$, we have $\text{Des}(J) = \{2, 3, 4, 6\}$ and $\tilde{\mathbf{k}}_{IJ} = q_{12}q_{13}t_{14}q_{34}$.

Definition 5.1 Let $Q = (q_{ij})$ and $T = (t_{ij})$ ($i, j \geq 1$) be two infinite matrices of commuting indeterminates. For a composition I of n , the noncommutative (Q, T) -Macdonald polynomial $\tilde{\mathbf{H}}_I(A; Q, T)$ is

$$\tilde{\mathbf{H}}_I(A; Q, T) = K_n(A; Z(I)) = \sum_{J \models n} \tilde{\mathbf{k}}_{IJ}(Q, T) R_J(A). \tag{29}$$

Note that $\tilde{\mathbf{H}}_I$ depends only on the q_{ij} and t_{ij} with $i + j \leq n$.

5.2 (Q, T) -Kostka matrices

The factorization property of the determinant of the (Q, T) -Kostka matrix, which is valid for the usual Macdonald polynomials as well as for the noncommutative analogues of [8] and [1] still holds since the $\tilde{\mathbf{H}}_I$ are specializations of the P_I .

Theorem 5.2 Let n be an integer. Then

$$\det \mathcal{K}_n = \prod_{i+j \leq n} (q_{ij} - t_{ij})^{e(i,j)}, \tag{30}$$

where $e(i, j) = \binom{i+j-2}{i-1} 2^{n-i-j}$.

5.3 Specializations

For appropriate specializations, we recover (up to indexation) the Bergeron-Zabrocki polynomials $\tilde{\mathbf{H}}_I^{BZ}$ of [1] and the multiparameter Macdonald functions $\tilde{\mathbf{H}}_I^{HLT}$ of [8]:

Proposition 5.3 Let $(q_i), (t_i), i \geq 1$ be two sequences of indeterminates. For a composition I of n ,

(i) Let ν be the anti-involution of \mathbf{Sym} defined by $\nu(S_n) = S_n$. Under the specialization $q_{ij} = q_{i+j-1}$, $t_{ij} = t_{n+1-i-j}$, $\tilde{\mathbf{H}}_I(Q, T)$ becomes a multiparameter version of $i\nu(\tilde{\mathbf{H}}_I^{BZ})$, to which it reduces under the further specialization $q_i = q^i$ and $t_i = t^i$.

(ii) Under the specialization $q_{ij} = q_j, t_{ij} = t_i$, $\tilde{\mathbf{H}}_I(Q, T)$ reduces to $\tilde{\mathbf{H}}_I^{HLT}$.

5.4 The quasi-symmetric side

Families of (Q, T) -quasi-symmetric functions can now be defined by duality by specialization of the (Q_I) defined in the general case. The dual basis of $(\tilde{\mathbf{H}}_J)$ in $QSym$ will be denoted by $(\tilde{\mathbf{G}}_I)$. We have

$$\tilde{\mathbf{G}}_I(X; Q, T) = \sum_J \tilde{\mathbf{g}}_{IJ}(q, t) F_J(X) \tag{31}$$

where the coefficients are given by the transposed inverse of the Kostka matrix: $(\tilde{\mathbf{g}}_{IJ}) = {}^t(\tilde{\mathbf{k}}_{IJ})^{-1}$.

Let $Z'(I)(Q, T) = Z(I)(T, Q) = Z(\tilde{I})(Q, T)$. Then, thanks to Proposition 4.3 and to the fact that changing the last bit of a binary word amounts to change a q into a t , we have

Proposition 5.4 *The inverse of the (Q, T) -Kostka matrix is given by*

$$(\mathcal{K}_n^{-1})_{IJ} = (-1)^{\ell(I)-1} \prod_{d \in \text{Des}(\bar{I}^\sim)} z'_d(J) \prod_{p=1}^{n-1} \frac{1}{z_p(J) - z'_p(J)}. \quad (32)$$

6 Multivariate BZ polynomials

In this section, we restrict our attention to the multiparameter version of the Bergeron-Zabrocki polynomials, obtained by setting $q_{ij} = q_{i+j-1}$ and $t_{ij} = t_{n+1-i-j}$ in degree n .

6.1 Multivariate BZ polynomials

As in the case of the two matrices of parameters, Q and T , one can deduce the product in the \tilde{H} basis by some sort of specialization of the general case. However, since t_{ij} specializes to another t where n appears, one has to be a little more cautious to get the correct answer.

Theorem 6.1 *Let I and J be two compositions of respective sizes p and r . Let us denote by $K = I \cdot \bar{J}^\sim$ and $n = |K| = p + r$. Then*

$$\tilde{H}_I \tilde{H}_J = \frac{(-1)^{\ell(I)+|J|}}{\prod_{k \in \text{Des}(K)} (q_k - t_{n-k})} \sum_{K'} \prod_{k \in \text{Des}(K')} (-1)^{\ell(K')} (z_k(K') - z'_k(K')) \tilde{H}_{K'} \quad (33)$$

where the sum is computed as follows. Let I' and J' be the compositions such that $|I'| = |I|$ and either $K' = I' \cdot J'$, or $K' = I' \triangleright J'$. If I' is not coarser than I or if J' is not finer than J , then $\tilde{H}(K')$ does have coefficient 0. Otherwise, $z_k(K') = q_k$ if k is a descent of K' and t_{n-k} otherwise. Finally, $z'_k(K')$ does not depend on K' and is $(Z(I), 1, Z(J))$.

6.2 The ∇ operator

The ∇ operator of [1] can be extended by

$$\nabla \tilde{H}_I = \left(\prod_{d=1}^{n-1} z_d(I) \right) \tilde{H}_I. \quad (34)$$

Then,

Proposition 6.2 *The action of ∇ on the ribbon basis is given by*

$$\nabla R_I = (-1)^{|I|+\ell(I)} \prod_{d \in \text{Des}(I^\sim)} q_d \prod_{d \in \text{Des}(\bar{I}^\sim)} t_d \sum_{J \geq \bar{I}^\sim} \prod_{i \in \text{Des}(I) \cap \text{Des}(J)} (t_i + q_{n-i}) R_J. \quad (35)$$

Note also that if $I = (1^n)$, one has

$$\nabla \Lambda_n = \sum_{J \neq n} \prod_{j \in \text{Des}(J)} (q_j + t_{n-j}) R_J = \sum_{J \neq n} \prod_{j \notin \text{Des}(J)} (q_j + t_{n-j} - 1) \Lambda^J. \quad (36)$$

As a positive sum of ribbons, this is the multigraded characteristic of a projective module of the 0-Hecke algebra. Its dimension is the number of packed words of length n (called preference functions in

[1]). Let us recall that a packed word is a word w over $\{1, 2, \dots\}$ so that if $i > 1$ appears in w , then $i - 1$ also appears in w . The set of all packed words of size n is denoted by PW_n .

Then the multigraded dimension of the previous module is

$$W_n(\mathbf{q}, \mathbf{t}) = \langle \nabla \Lambda_n, F_1^n \rangle = \sum_{w \in \text{PW}_n} \phi(w) \quad (37)$$

where the statistic $\phi(w)$ is obtained as follows.

Let $\sigma_w = \text{std}(\overline{w})$, where \overline{w} denotes the mirror image of w . Then

$$\phi(w) = \prod_{i \in \text{Des}(\sigma_w^{-1})} x_i \quad (38)$$

where $x_i = q_i$ if $w_i^\uparrow = w_{i+1}^\uparrow$ and $x_i = t_{n-i}$ otherwise, where w^\uparrow is the nondecreasing reordering of w .

For example, with $w = 22135411$, $\sigma_w = 54368721$, $w^\uparrow = 11122345$, the recoils of σ_w are 1, 2, 3, 4, 7, and $\phi(w) = q_1 q_2 t_5 q_4 t_1$.

Theorem 6.3 Denote by d_I the number of permutations σ with descent composition $C(\sigma) = I$. Then, for any composition I of n ,

$$\nabla R_I = (-1)^{|I|+\ell(I)} \theta(\sigma) \sum_{w \in \text{PW}_n; \text{ev}(w) \leq I} \frac{R_{C(\sigma_w^{-1})}}{d_{C(\sigma_w^{-1})}}, \quad (39)$$

where σ is any permutation such that $C(\sigma^{-1}) = \bar{I}^\sim$, and

$$\theta(\sigma) = \prod_{d \in \text{Des}(\bar{I}^\sim)} t_d. \quad (40)$$

The behaviour of the multiparameter BZ polynomials with respect to the scalar product

$$[R_I, R_J] := (-1)^{|I|+\ell(I)} \delta_{I, \bar{J}^\sim} \quad (41)$$

is the natural generalization of [1, Prop. 1.7]:

$$[\tilde{H}_I, \tilde{H}_J] = (-1)^{|I|+\ell(I)} \delta_{I, \bar{J}^\sim} \prod_{i=1}^{n-1} (q_i - t_{n-i}). \quad (42)$$

7 Quasideterminantal bases

7.1 Quasideterminants of almost triangular matrices

Quasideterminants [4] are noncommutative analogs of the ratio of a determinant by one of its principal minors. Thus, the quasideterminants of a generic matrix are not polynomials, but complicated rational expressions living in the free field generated by the coefficients. However, for almost triangular matrices,

i.e., such that $a_{ij} = 0$ for $i > j + 1$, all quasideterminants are polynomials, with a simple explicit expression. We shall only need the formula (see [3], Prop.2.6):

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & \boxed{a_{1n}} \\ -1 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & -1 & a_{33} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{n-1n} \\ 0 & \cdots & 0 & -1 & a_{nn} \end{vmatrix} = a_{1n} + \sum_{1 \leq j_1 < \cdots < j_k < n} a_{1j_1} a_{j_1+1j_2} a_{j_2+1j_3} \cdots a_{j_k+1n}. \quad (43)$$

Recall that the quasideterminant $|A|_{pq}$ is invariant by scaling the rows of index different from p and the columns of index different from q . It is homogeneous of degree 1 with respect to row p and column q . Also, the quasideterminant is invariant under permutations of rows and columns.

The quasideterminant (43) coincides with the row-ordered expansion of an ordinary determinant

$$\text{rdet}(A) := \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \quad (44)$$

which will be denoted as an ordinary determinant in the sequel.

7.2 Quasideterminantal bases of Sym

Many interesting families of noncommutative symmetric functions can be expressed as quasi-determinants of the form

$$H(W, G) = \begin{vmatrix} w_{11}G_1 & w_{12}G_2 & \cdots & w_{1n-1}G_{n-1} & \boxed{w_{1n}G_n} \\ w_{21} & w_{22}G_1 & \cdots & w_{2n-1}G_{n-2} & w_{2n}G_{n-1} \\ 0 & w_{32} & \cdots & w_{3n-3}G_{n-3} & w_{3n}G_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & w_{nn-1} & w_{nn}G_1 \end{vmatrix} \quad (45)$$

(or of the transposed form), where G_k is some sequence of free generators of **Sym**, and W an almost-triangular ($w_{ij} = 0$ for $i > j + 1$) scalar matrix. For example, S_n over the Λ^I and the Ψ^I (see [3, (37)-(41)]), or over the Θ^I , where $\Theta_n(q) = (1 - q)^{-1} S_n((1 - q)A)$ (see [10, Eq. (78)]). These examples illustrate relations between sequences of free generators. Quasi-determinantal expressions for some linear bases can be recast in this form as well. For example, the formula for ribbons [3, (50)] can be rewritten as follows. Let U and V be the $n \times n$ almost-triangular matrices

$$U = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & -1 & \cdots & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & -1 \end{bmatrix} \quad V = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{bmatrix} \quad (46)$$

Given the pair (U, V) , define, for each composition I of n , a matrix $W(I)$ by

$$w_{ij}(I) = \begin{cases} u_{ij} & \text{if } i - 1 \in \text{Des}(I), \\ v_{ij} & \text{otherwise,} \end{cases} \quad (47)$$

and set

$$H_I(U, V; A) := H(W(I), S(A)). \quad (48)$$

Then,

$$(-1)^{\ell(I)-1} R_I = H_I(U, V). \quad (49)$$

Indeed, $H_I(U, V; A)$ is obtained by substituting in (43)

$$a_{j_p+1, j_p+1} = \begin{cases} -S_{j_p+1-j_p} & \text{if } j_p \in \text{Des}(I), \\ 0 & \text{otherwise.} \end{cases} \quad (50)$$

This yields

$$\begin{aligned} & S_n + \sum_k \sum_{\{j_1 < \dots < j_k\} \subseteq \text{Des}(I)} S_{j_1} (-S_{j_2-j_1}) \dots (-S_{n-j_k}) \\ &= \sum_{\text{Des}(K) \subseteq \text{Des}(I)} (-1)^{\ell(K)-1} S^K = (-1)^{\ell(I)-1} R_I. \end{aligned} \quad (51)$$

For a generic pair of almost-triangular matrices (U, V) , the H_I form a basis of \mathbf{Sym}_n . Without loss of generality, we may assume that $u_{1j} = v_{1j} = 1$ for all j . Then, the transition matrix M expressing the H_I on the S^J where $J = (j_1, \dots, j_p)$ satisfies:

$$M_{J,I} := x_{1j_1-1} x_{j_1 j_2-1} \dots x_{j_p n}. \quad (52)$$

where $x_{ij} = u_{ij}$ if $i-1$ is not a descent of I and v_{ij} otherwise.

As we shall sometimes need different normalizations, we also define for arbitrary almost triangular matrices U, V

$$H'(W, G) = \text{rdet} \begin{bmatrix} w_{11}G_1 & w_{12}G_2 & \dots & w_{1n-1}G_{n-1} & w_{1n}G_n \\ w_{21} & w_{22}G_1 & \dots & w_{2n-1}G_{n-2} & w_{2n}G_{n-1} \\ 0 & w_{32} & \dots & w_{3n-3}G_{n-3} & w_{3n}G_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & w_{nn-1} & w_{nn}G_1 \end{bmatrix} \quad (53)$$

and

$$H'_I(U, V) = H'(W(I), S(A)). \quad (54)$$

7.3 Expansion on the basis (S^I)

For a composition $I = (i_1, \dots, i_r)$ of n , let I^\sharp be the integer vector of length n obtained by replacing each entry k of I by the sequence $(k, 0, \dots, 0)$ ($k-1$ zeros):

$$I^\sharp = (i_1 0^{i_1-1} i_2 0^{i_2-1} \dots i_r 0^{i_r-1}). \quad (55)$$

Proposition 7.1 *The expansion of $H'(W, S)$ on the S -basis is given by*

$$H'(W, S) = \sum_{I=n} \varepsilon(\sigma_I) w_{1\sigma_I(1)} \dots w_{n\sigma_I(n)} S^I. \quad (56)$$

7.4 Expansion of the basis (R_I)

Proposition 7.2 For $I = (i_1, \dots, i_r)$ be a composition of n , denote by W_I the product of diagonal minors of the matrix W taken over the first i_1 rows and columns, then the next i_2 ones and so on. Then,

$$H'(W, S) = \sum_{I \models n} W_I R_I. \quad (57)$$

7.5 Examples

7.5.1 A family with factoring coefficients

Theorem 7.3 Let U and V be defined by

$$u_{ij} = \begin{cases} x^j - y^j & \text{if } i = 1, \\ aq_{i-1}x^{j-i+1} - y^{j-i+1} & \text{if } 1 < i < j + 2, \\ 0 & \text{otherwise,} \end{cases} \quad (58)$$

$$v_{ij} = \begin{cases} x^j - y^j & \text{if } i = 1, \\ x^{j-i+1} - bu_{n+1-i}y^{j-i+1} & \text{if } 1 < i < j + 2, \\ 0 & \text{otherwise.} \end{cases} \quad (59)$$

Then the coefficients W_J of the expansion of $H'_I(U, V)$ on the ribbon basis all factor as products of binomials.

The formula for the coefficient of R_n is simple enough: if one orders the factors of $\det(U)$ and $\det(V)$ as

$$Z_n = (x - aq_1y, x - aq_2y, \dots, x - aq_{n-1}y) \quad (60)$$

and

$$Z'_n = (y - bu_{n-1}x, y - bu_{n-2}x, \dots, y - bu_1x), \quad (61)$$

then, the coefficient of R_n in $H'_I(U, V)$ is

$$(x - y) \prod_{d \in \text{Des}(I)} z'_d \prod_{e \notin \text{Des}(I)} z_e. \quad (62)$$

A more careful analysis allows one to compute directly the coefficient of R_J in H'_I . For example,

$$\begin{aligned} \frac{H'_3(U, V)}{(x - y)} &= (x - aq_1y)(x - aq_2y)R_3 + (x - aq_1y)(aq_2x - y)R_{21} \\ &\quad + a(x - y)(q_1x - q_2y)R_{12} + (aq_1x - y)(aq_2x - y)R_{111}. \end{aligned} \quad (63)$$

$$\begin{aligned} \frac{H'_{21}(U, V)}{(x - y)} &= (x - aq_1y)(bu_1x - y)R_3 + (x - aq_1y)(x - bu_1y)R_{21} \\ &\quad + (abq_1u_1x - y)(x - y)R_{12} + (aq_1x - y)(x - bu_1y)R_{111}. \end{aligned} \quad (64)$$

7.5.2 An analogue of the $(1-t)/(1-q)$ transform

Recall that for commutative symmetric functions, the $(1-t)/(1-q)$ transform is defined in terms of the power-sums by

$$p_n \left(\frac{1-t}{1-q} X \right) = \frac{1-t^n}{1-q^n} p_n(X). \quad (65)$$

With the specialization $x = 1, y = t, q_i = q^i, u_i = 1, a = b = 1$, one obtains a basis such that for a hook composition $I = (n-k, 1^k)$, the commutative image of $H'_I(U, V)$ becomes the $(1-t)/(1-q)$ transform of the Schur function $s_{n-k, 1^k}$.

7.5.3 An analogue of the Macdonald P -basis

With the specialization $x = 1, y = t, q_i = q^i, u_i = t^i, a = b = 1$, one obtains an analogue of the Macdonald P -basis, in the sense that for hook compositions $I = (n-k, 1^k)$, the commutative image of H'_I is proportional to the Macdonald polynomial $P_{n-k, 1^k}(q, t; X)$.

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