



HAL
open science

Noncommutative symmetric functions with matrix parameters

Alain Lascoux, Jean-Christophe Novelli, Jean-Yves Thibon

► **To cite this version:**

Alain Lascoux, Jean-Christophe Novelli, Jean-Yves Thibon. Noncommutative symmetric functions with matrix parameters. 24th International Conference on Formal Power Series and Algebraic Combinatorics (FPSAC 2012), 2012, Nagoya, Japan. pp.515-526, 10.46298/dmtcs.3059 . hal-01283108

HAL Id: hal-01283108

<https://hal.science/hal-01283108>

Submitted on 5 Mar 2016

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Noncommutative symmetric functions with matrix parameters

(extended abstract)

Alain Lascoux, Jean-Christophe Novelli, and Jean-Yves Thibon

Institut Gaspard Monge, Université Paris-Est Marne-la-Vallée, 5 Boulevard Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France

Abstract. We define new families of noncommutative symmetric functions and quasi-symmetric functions depending on two matrices of parameters, and more generally on parameters associated with paths in a binary tree. Appropriate specializations of both matrices then give back the two-vector families of Hivert, Lascoux, and Thibon and the noncommutative Macdonald functions of Bergeron and Zabrocki.

Résumé. Nous définissons de nouvelles familles de fonctions symétriques non-commutatives et de fonctions quasi-symétriques, dépendant de deux matrices de paramètres, et plus généralement, de paramètres associés à des chemins dans un arbre binaire. Pour des spécialisations appropriées, on retrouve les familles à deux vecteurs de Hivert-Lascoux-Thibon et les fonctions de Macdonald non-commutatives de Bergeron-Zabrocki.

Keywords: Noncommutative symmetric functions, Quasi-symmetric functions, Macdonald polynomials

1 Introduction

The theory of Hall-Littlewood, Jack, and Macdonald polynomials is one of the most interesting subjects in the modern theory of symmetric functions. It is well-known that combinatorial properties of symmetric functions can be explained by lifting them to larger algebras (the so-called combinatorial Hopf algebras), the simplest examples being **Sym** (Noncommutative symmetric functions [3]) and its dual *QSym* (Quasi-symmetric functions [5]).

There have been several attempts to lift Hall-Littlewood and Macdonald polynomials to **Sym** and *QSym* [1, 7, 8, 13, 14]. The analogues defined in [1] were similar to, though different from, those of [8]. These last ones admitted multiple parameters q_i and t_i , which however could not be specialized to recover the version of [1].

The aim of this article is to show that many more parameters can be introduced in the definition of such bases. Actually, one can have a pair of $n \times n$ matrices (Q_n, T_n) for each degree n . The main properties established in [1] and [8] remain true in this general context, and one recovers the BZ and HLT polynomials for appropriate specializations of the matrices.

In the last section, another possibility involving quasideterminants is explored.

2 Notations

Our notations for noncommutative symmetric functions will be as in [3, 10]. The Hopf algebra of noncommutative symmetric functions is denoted by \mathbf{Sym} , or by $\mathbf{Sym}(A)$ if we consider the realization in terms of an auxiliary alphabet. Bases of \mathbf{Sym}_n are labelled by compositions I of n . The noncommutative complete and elementary functions are denoted by S_n and Λ_n , and the notation S^I means $S_{i_1} \dots S_{i_r}$. The ribbon basis is denoted by R_I .

The notation $I \vDash n$ means that I is a composition of n . The conjugate composition is denoted by I^\sim . The graded dual of \mathbf{Sym} is $QSym$ (quasi-symmetric functions). The dual basis of (S^I) is (M_I) (monomial), and that of (R_I) is (F_I) . The *descent set* of $I = (i_1, \dots, i_r)$ is $\text{Des}(I) = \{i_1, i_1 + i_2, \dots, i_1 + \dots + i_{r-1}\}$.

Finally, there are two operations on compositions: if $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_s)$, the composition $I.J$ is $(i_1, \dots, i_r, j_1, \dots, j_s)$ and $I \triangleright J$ is $(i_1, \dots, i_r + j_1, \dots, j_s)$.

3 \mathbf{Sym}_n as a Grassmann algebra

Since for $n > 0$, \mathbf{Sym}_n has dimension 2^{n-1} , it can be identified (as a vector space) with a Grassmann algebra on $n - 1$ generators $\eta_1, \dots, \eta_{n-1}$ (that is, $\eta_i \eta_j = -\eta_j \eta_i$, so that in particular $\eta_i^2 = 0$). This identification is meaningful, for example, in the context of the representation theory of the 0-Hecke algebras $H_n(0)$ (see [2]).

If I is a composition of n with descent set $D = \{d_1, \dots, d_k\}$, we make the identification

$$R_I \longleftrightarrow \eta_D := \eta_{d_1} \eta_{d_2} \dots \eta_{d_k}. \quad (1)$$

For example, $R_{213} \leftrightarrow \eta_2 \eta_3$. We then have

$$S^I \longleftrightarrow (1 + \eta_{d_1})(1 + \eta_{d_2}) \dots (1 + \eta_{d_k}) \quad (2)$$

and

$$\Lambda^I \longleftrightarrow \prod_{i=1}^{n-1} \theta_i, \quad (3)$$

where $\theta_i = \eta_i$ if $i \notin D$ and $\theta_i = 1 + \eta_i$ otherwise. Other bases have simple expression under this identification, e.g., Ψ_n , Φ_n and Hivert's Hall-Littlewood basis [7].

3.1 Structure on the Grassmann algebra

Let $*$ be the anti-involution given by $\eta_i^* = (-1)^i \eta_i$. The Grassmann integral of any function f is defined by

$$\int d\eta f := f_{12\dots n-1}, \quad \text{where } f = \sum_k \sum_{i_1 < \dots < i_k} f_{i_1 \dots i_k} \eta_{i_1} \dots \eta_{i_k}. \quad (4)$$

We define a bilinear form on \mathbf{Sym}_n by

$$(f, g) = \int d\eta f^* g. \quad (5)$$

Then,

$$(R_I, R_J) = (-1)^{\ell(I)-1} \delta_{I, J^\sim}, \quad (6)$$

so that this is (up to an unessential sign) the Bergeron-Zabrocki scalar product [1, Eq. (4)].

3.2 Factorized elements in the Grassman algebra

Now, for a sequence of parameters $Z = (z_1, \dots, z_{n-1})$, let

$$K_n(Z) = (1 + z_1\eta_1)(1 + z_2\eta_2) \dots (1 + z_{n-1}\eta_{n-1}). \quad (7)$$

We then have

Lemma 3.1

$$\langle K_n(X), K_n(Y) \rangle = \prod_{i=1}^{n-1} (y_i - x_i). \quad (8)$$

3.3 Bases of Sym

We shall investigate bases of \mathbf{Sym}_n of the form

$$\tilde{H}_I = K_n(Z_I) = \sum_J \tilde{\mathbf{k}}_{IJ} R_J, \quad (9)$$

where Z_I is a sequence of parameters depending on the composition I of n .

The bases defined in [8] and [1] are of the previous form and for both of them, the determinant of the Kostka matrix $\mathcal{K} = (\tilde{\mathbf{k}}_{IJ})$ is a product of linear factors (as for ordinary Macdonald polynomials). This is explained by the fact that these matrices have the form

$$\begin{pmatrix} A & xA \\ B & yB \end{pmatrix} \quad (10)$$

where A and B have a similar structure, and so on recursively. Indeed, for such matrices,

Lemma 3.2 *Let A, B be two $m \times m$ matrices. Then,*

$$\begin{vmatrix} A & xA \\ B & yB \end{vmatrix} = (y - x)^m \det A \cdot \det B. \quad (11)$$

3.4 Duality

Similarly, the dual vector space $QSym_n = \mathbf{Sym}_n^*$ can be identified with a Grassmann algebra on another set of generators ξ_1, \dots, ξ_{n-1} . Encoding the fundamental basis F_I of Gessel [5] by

$$\xi_D := \xi_{d_1} \xi_{d_2} \dots \xi_{d_k}, \quad (12)$$

the usual duality pairing such that the F_I are dual to the R_I is given in this setting by

$$\langle \xi_D, \eta_E \rangle = \delta_{DE}. \quad (13)$$

Let

$$L_n(Z) = (z_1 - \xi_1) \dots (z_{n-1} - \xi_{n-1}). \quad (14)$$

Then, as above, we have a factorization identity:

Lemma 3.3

$$\langle L_n(X), K_n(Y) \rangle = \prod_{i=1}^{n-1} (x_i - y_i). \quad (15)$$

4 Bases associated with paths in a binary tree

Let $\mathbf{y} = \{y_u\}$ be a family of indeterminates indexed by all boolean words of length $\leq n-1$. For example, for $n = 3$, we have the six parameters $y_0, y_1, y_{00}, y_{01}, y_{10}, y_{11}$.

We can encode a composition I with descent set D by the boolean word $u = (u_1, \dots, u_{n-1})$ such that $u_i = 1$ if $i \in D$ and $u_i = 0$ otherwise.

Let us denote by $u_{m\dots p}$ the sequence $u_m u_{m+1} \dots u_p$ and define

$$P_I := (1 + y_{u_1} \eta_1)(1 + y_{u_1\dots 2} \eta_2) \dots (1 + y_{u_1\dots n-1} \eta_{n-1}) \quad (16)$$

or, equivalently,

$$P_I := K_n(Y_I) \quad \text{with} \quad Y_I = (y_{u_1}, y_{u_1\dots 2}, \dots, y_u) =: (y_k(I)). \quad (17)$$

Similarly, let

$$Q_I := (y_{w_1} - \xi_1)(y_{w_1\dots 2} - \xi_2) \dots (y_{w_1\dots n-1} - \xi_{n-1}) =: L_n(Y^I), \quad (18)$$

where $w_{1\dots k} = u_1 \dots u_{k-1} \overline{u_k}$ where $\overline{u_k} = 1 - u_k$, so that

$$Y^I := (y_{w_1}, y_{w_1\dots 2}, \dots, y_{w_1\dots n-1}) =: (y^k(I)). \quad (19)$$

4.1 Kostka matrices

The Kostka matrix is defined as the transpose of the transition matrices from P_I to R_J . This matrix is recursively of the form of Eq. (10). Thus, its determinant factors completely. For $n = 4$, it is

$$(y_1 - y_0)^4 (y_{01} - y_{00})^2 (y_{11} - y_{10})^2 (y_{001} - y_{000})(y_{011} - y_{010})(y_{101} - y_{100})(y_{111} - y_{110}). \quad (20)$$

Proposition 4.1 *The bases (P_I) and (Q_I) are adjoint to each other, up to normalization:*

$$\langle Q_I, P_J \rangle = \langle L_n(Y^I), K_n(Y_J) \rangle = \prod_{k=1}^{n-1} (y^k(I) - y_k(J)), \quad (21)$$

which is indeed zero unless $I = J$.

From this, it is easy to derive a product formula for the basis P_I .

Proposition 4.2 *Let I and J be two compositions of respective sizes n and m . The product $P_I P_J$ is a sum over an interval of the lattice of compositions*

$$P_I P_J = \sum_{K \in [I \triangleright (m), I \cdot (1^m)]} c_{IJ}^K P_K \quad (22)$$

where

$$c_{IJ}^K = \frac{\langle L_{n+m}(Y^K), K_{n+m}(Y_I \cdot 1 \cdot Y_J) \rangle}{\langle Q_K, P_K \rangle}, \quad (23)$$

where $Y_I \cdot 1 \cdot Y_J$ stands for the sequence $(y_1(I), \dots, y_n(I), 1, y_1(J), \dots, y_m(J))$.

4.2 The quasi-symmetric side

As we have seen before, the (Q_I) being dual to the (P_I) , the inverse Kostka matrix is given by the simple construction:

Proposition 4.3 *The inverse of the Kostka matrix is given by*

$$(\mathcal{K}_n^{-1})_{IJ} = (-1)^{\ell(I)-1} \prod_{d \in \text{Des}(I \sim)} y^d(J) \prod_{p=1}^{n-1} \frac{1}{y^p(J) - y_p(J)}. \tag{24}$$

4.3 Some specializations

Let us now consider the specialization sending all y_w to 1 if w ends with a 1 and denote by \mathcal{K}' the matrix obtained by this specialization. Then, as in [8, p. 10],

Proposition 4.4 *Let n be an integer. Then*

$$S_n = \mathcal{K}_n \mathcal{K}'_n^{-1} \tag{25}$$

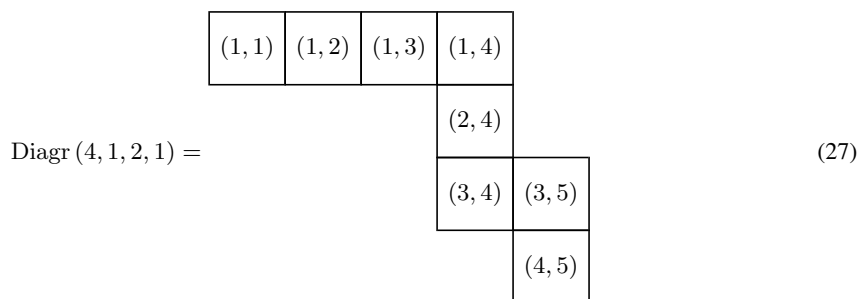
is lower triangular. More precisely, let Y'_J be the image of Y_J by the previous specialization and define Y'^J in the same way. Then the coefficient s_{IJ} indexed by (I, J) is

$$s_{IJ} = \prod_{k=1}^{n-1} \frac{y_k(I) - y'^k(J)}{y'_k(J) - y'^k(J)}. \tag{26}$$

5 The two-matrix family

5.1 A specialization of the paths in a binary tree

The above bases can now be specialized to bases $\tilde{H}(A; Q, T)$, depending on two infinite matrices of parameters. Label the cells of the ribbon diagram of a composition I of n with their matrix coordinates as follows:



We associate a variable z_{ij} with each cell except $(1, 1)$ by setting $z_{ij} := q_{i,j-1}$ if (i, j) has a cell on its left, and $z_{ij} := t_{i-1,j}$ if (i, j) has a cell on its top. The alphabet $Z(I) = (z_j(I))$ is the sequence of the z_{ij} in their natural order.

Next, if J is a composition of the same integer n , form the monomial

$$\tilde{\mathbf{k}}_{IJ}(Q, T) = \prod_{d \in \text{Des}(J)} z_d(I). \tag{28}$$

For example, with $I = (4, 1, 2, 1)$ and $J = (2, 1, 1, 2, 2)$, we have $\text{Des}(J) = \{2, 3, 4, 6\}$ and $\tilde{\mathbf{k}}_{IJ} = q_{12}q_{13}t_{14}q_{34}$.

Definition 5.1 Let $Q = (q_{ij})$ and $T = (t_{ij})$ ($i, j \geq 1$) be two infinite matrices of commuting indeterminates. For a composition I of n , the noncommutative (Q, T) -Macdonald polynomial $\tilde{\mathbf{H}}_I(A; Q, T)$ is

$$\tilde{\mathbf{H}}_I(A; Q, T) = K_n(A; Z(I)) = \sum_{J \models n} \tilde{\mathbf{k}}_{IJ}(Q, T) R_J(A). \tag{29}$$

Note that $\tilde{\mathbf{H}}_I$ depends only on the q_{ij} and t_{ij} with $i + j \leq n$.

5.2 (Q, T) -Kostka matrices

The factorization property of the determinant of the (Q, T) -Kostka matrix, which is valid for the usual Macdonald polynomials as well as for the noncommutative analogues of [8] and [1] still holds since the $\tilde{\mathbf{H}}_I$ are specializations of the P_I .

Theorem 5.2 Let n be an integer. Then

$$\det \mathcal{K}_n = \prod_{i+j \leq n} (q_{ij} - t_{ij})^{e(i,j)}, \tag{30}$$

where $e(i, j) = \binom{i+j-2}{i-1} 2^{n-i-j}$.

5.3 Specializations

For appropriate specializations, we recover (up to indexation) the Bergeron-Zabrocki polynomials $\tilde{\mathbf{H}}_I^{BZ}$ of [1] and the multiparameter Macdonald functions $\tilde{\mathbf{H}}_I^{HLLT}$ of [8]:

Proposition 5.3 Let $(q_i), (t_i), i \geq 1$ be two sequences of indeterminates. For a composition I of n ,

(i) Let ν be the anti-involution of \mathbf{Sym} defined by $\nu(S_n) = S_n$. Under the specialization $q_{ij} = q_{i+j-1}$, $t_{ij} = t_{n+1-i-j}$, $\tilde{\mathbf{H}}_I(Q, T)$ becomes a multiparameter version of $i\nu(\tilde{\mathbf{H}}_I^{BZ})$, to which it reduces under the further specialization $q_i = q^i$ and $t_i = t^i$.

(ii) Under the specialization $q_{ij} = q_j, t_{ij} = t_i$, $\tilde{\mathbf{H}}_I(Q, T)$ reduces to $\tilde{\mathbf{H}}_I^{HLLT}$.

5.4 The quasi-symmetric side

Families of (Q, T) -quasi-symmetric functions can now be defined by duality by specialization of the (Q_I) defined in the general case. The dual basis of $(\tilde{\mathbf{H}}_J)$ in $QSym$ will be denoted by $(\tilde{\mathbf{G}}_I)$. We have

$$\tilde{\mathbf{G}}_I(X; Q, T) = \sum_J \tilde{\mathbf{g}}_{IJ}(q, t) F_J(X) \tag{31}$$

where the coefficients are given by the transposed inverse of the Kostka matrix: $(\tilde{\mathbf{g}}_{IJ}) = {}^t(\tilde{\mathbf{k}}_{IJ})^{-1}$.

Let $Z'(I)(Q, T) = Z(I)(T, Q) = Z(\tilde{I})(Q, T)$. Then, thanks to Proposition 4.3 and to the fact that changing the last bit of a binary word amounts to change a q into a t , we have

Proposition 5.4 *The inverse of the (Q, T) -Kostka matrix is given by*

$$(\mathcal{K}_n^{-1})_{IJ} = (-1)^{\ell(I)-1} \prod_{d \in \text{Des}(\bar{I})} z'_d(J) \prod_{p=1}^{n-1} \frac{1}{z_p(J) - z'_p(J)}. \quad (32)$$

6 Multivariate BZ polynomials

In this section, we restrict our attention to the multiparameter version of the Bergeron-Zabrocki polynomials, obtained by setting $q_{ij} = q_{i+j-1}$ and $t_{ij} = t_{n+1-i-j}$ in degree n .

6.1 Multivariate BZ polynomials

As in the case of the two matrices of parameters, Q and T , one can deduce the product in the \tilde{H} basis by some sort of specialization of the general case. However, since t_{ij} specializes to another t where n appears, one has to be a little more cautious to get the correct answer.

Theorem 6.1 *Let I and J be two compositions of respective sizes p and r . Let us denote by $K = I \cdot \bar{J}$ and $n = |K| = p + r$. Then*

$$\tilde{H}_I \tilde{H}_J = \frac{(-1)^{\ell(I)+|J|}}{\prod_{k \in \text{Des}(K)} (q_k - t_{n-k})} \sum_{K'} \prod_{k \in \text{Des}(K')} (-1)^{\ell(K')} (z_k(K') - z'_k(K')) \tilde{H}_{K'} \quad (33)$$

where the sum is computed as follows. Let I' and J' be the compositions such that $|I'| = |I|$ and either $K' = I' \cdot J'$, or $K' = I' \triangleright J'$. If I' is not coarser than I or if J' is not finer than J , then $\tilde{H}(K')$ does have coefficient 0. Otherwise, $z_k(K') = q_k$ if k is a descent of K' and t_{n-k} otherwise. Finally, $z'_k(K')$ does not depend on K' and is $(Z(I), 1, Z(J))$.

6.2 The ∇ operator

The ∇ operator of [1] can be extended by

$$\nabla \tilde{H}_I = \left(\prod_{d=1}^{n-1} z_d(I) \right) \tilde{H}_I. \quad (34)$$

Then,

Proposition 6.2 *The action of ∇ on the ribbon basis is given by*

$$\nabla R_I = (-1)^{|I|+\ell(I)} \prod_{d \in \text{Des}(I)} q_d \prod_{d \in \text{Des}(\bar{I})} t_d \sum_{J \geq \bar{I}} \prod_{i \in \text{Des}(I) \cap \text{Des}(J)} (t_i + q_{n-i}) R_J. \quad (35)$$

Note also that if $I = (1^n)$, one has

$$\nabla \Lambda_n = \sum_{J \neq n} \prod_{j \in \text{Des}(J)} (q_j + t_{n-j}) R_J = \sum_{J \neq n} \prod_{j \notin \text{Des}(J)} (q_j + t_{n-j} - 1) \Lambda^J. \quad (36)$$

As a positive sum of ribbons, this is the multigraded characteristic of a projective module of the 0-Hecke algebra. Its dimension is the number of packed words of length n (called preference functions in

[1]). Let us recall that a packed word is a word w over $\{1, 2, \dots\}$ so that if $i > 1$ appears in w , then $i - 1$ also appears in w . The set of all packed words of size n is denoted by PW_n .

Then the multigraded dimension of the previous module is

$$W_n(\mathbf{q}, \mathbf{t}) = \langle \nabla \Lambda_n, F_1^n \rangle = \sum_{w \in \text{PW}_n} \phi(w) \tag{37}$$

where the statistic $\phi(w)$ is obtained as follows.

Let $\sigma_w = \overline{\text{std}(\overline{w})}$, where \overline{w} denotes the mirror image of w . Then

$$\phi(w) = \prod_{i \in \text{Des}(\sigma_w^{-1})} x_i \tag{38}$$

where $x_i = q_i$ if $w_i^\uparrow = w_{i+1}^\uparrow$ and $x_i = t_{n-i}$ otherwise, where w^\uparrow is the nondecreasing reordering of w .

For example, with $w = 22135411$, $\sigma_w = 54368721$, $w^\uparrow = 11122345$, the recoils of σ_w are 1, 2, 3, 4, 7, and $\phi(w) = q_1 q_2 t_5 q_4 t_1$.

Theorem 6.3 Denote by d_I the number of permutations σ with descent composition $C(\sigma) = I$. Then, for any composition I of n ,

$$\nabla R_I = (-1)^{|I|+\ell(I)} \theta(\sigma) \sum_{w \in \text{PW}_n; \text{ev}(w) \leq I} \frac{R_{C(\sigma_w^{-1})}}{d_{C(\sigma_w^{-1})}}, \tag{39}$$

where σ is any permutation such that $C(\sigma^{-1}) = \bar{I}^\sim$, and

$$\theta(\sigma) = \prod_{d \in \text{Des}(\bar{I}^\sim)} t_d. \tag{40}$$

The behaviour of the multiparameter BZ polynomials with respect to the scalar product

$$[R_I, R_J] := (-1)^{|I|+\ell(I)} \delta_{I, \bar{J}^\sim} \tag{41}$$

is the natural generalization of [1, Prop. 1.7]:

$$[\tilde{H}_I, \tilde{H}_J] = (-1)^{|I|+\ell(I)} \delta_{I, \bar{J}^\sim} \prod_{i=1}^{n-1} (q_i - t_{n-i}). \tag{42}$$

7 Quasideterminantal bases

7.1 Quasideterminants of almost triangular matrices

Quasideterminants [4] are noncommutative analogs of the ratio of a determinant by one of its principal minors. Thus, the quasideterminants of a generic matrix are not polynomials, but complicated rational expressions living in the free field generated by the coefficients. However, for almost triangular matrices,

i.e., such that $a_{ij} = 0$ for $i > j + 1$, all quasideterminants are polynomials, with a simple explicit expression. We shall only need the formula (see [3], Prop.2.6):

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & \boxed{a_{1n}} \\ -1 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & -1 & a_{33} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{n-1n} \\ 0 & \cdots & 0 & -1 & a_{nn} \end{vmatrix} = a_{1n} + \sum_{1 \leq j_1 < \cdots < j_k < n} a_{1j_1} a_{j_1+1j_2} a_{j_2+1j_3} \cdots a_{j_k+1n}. \quad (43)$$

Recall that the quasideterminant $|A|_{pq}$ is invariant by scaling the rows of index different from p and the columns of index different from q . It is homogeneous of degree 1 with respect to row p and column q . Also, the quasideterminant is invariant under permutations of rows and columns.

The quasideterminant (43) coincides with the row-ordered expansion of an ordinary determinant

$$\text{rdet}(A) := \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma) a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)} \quad (44)$$

which will be denoted as an ordinary determinant in the sequel.

7.2 Quasideterminantal bases of Sym

Many interesting families of noncommutative symmetric functions can be expressed as quasi-determinants of the form

$$H(W, G) = \begin{vmatrix} w_{11}G_1 & w_{12}G_2 & \cdots & w_{1n-1}G_{n-1} & \boxed{w_{1n}G_n} \\ w_{21} & w_{22}G_1 & \cdots & w_{2n-1}G_{n-2} & w_{2n}G_{n-1} \\ 0 & w_{32} & \cdots & w_{3n-3}G_{n-3} & w_{3n}G_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & w_{nn-1} & w_{nn}G_1 \end{vmatrix} \quad (45)$$

(or of the transposed form), where G_k is some sequence of free generators of **Sym**, and W an almost-triangular ($w_{ij} = 0$ for $i > j + 1$) scalar matrix. For example, S_n over the Λ^I and the Ψ^I (see [3, (37)-(41)]), or over the Θ^I , where $\Theta_n(q) = (1 - q)^{-1} S_n((1 - q)A)$ (see [10, Eq. (78)]). These examples illustrate relations between sequences of free generators. Quasi-determinantal expressions for some linear bases can be recast in this form as well. For example, the formula for ribbons [3, (50)] can be rewritten as follows. Let U and V be the $n \times n$ almost-triangular matrices

$$U = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & -1 & \cdots & -1 & -1 \\ 0 & -1 & \cdots & -1 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & -1 \end{bmatrix} \quad V = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 \end{bmatrix} \quad (46)$$

Given the pair (U, V) , define, for each composition I of n , a matrix $W(I)$ by

$$w_{ij}(I) = \begin{cases} u_{ij} & \text{if } i - 1 \in \text{Des}(I), \\ v_{ij} & \text{otherwise,} \end{cases} \quad (47)$$

and set

$$H_I(U, V; A) := H(W(I), S(A)). \quad (48)$$

Then,

$$(-1)^{\ell(I)-1} R_I = H_I(U, V). \quad (49)$$

Indeed, $H_I(U, V; A)$ is obtained by substituting in (43)

$$a_{j_p+1, j_p+1} = \begin{cases} -S_{j_p+1-j_p} & \text{if } j_p \in \text{Des}(I), \\ 0 & \text{otherwise.} \end{cases} \quad (50)$$

This yields

$$\begin{aligned} S_n + \sum_k \sum_{\{j_1 < \dots < j_k\} \subseteq \text{Des}(I)} S_{j_1} (-S_{j_2-j_1}) \dots (-S_{n-j_k}) \\ = \sum_{\text{Des}(K) \subseteq \text{Des}(I)} (-1)^{\ell(K)-1} S^K = (-1)^{\ell(I)-1} R_I. \end{aligned} \quad (51)$$

For a generic pair of almost-triangular matrices (U, V) , the H_I form a basis of \mathbf{Sym}_n . Without loss of generality, we may assume that $u_{1j} = v_{1j} = 1$ for all j . Then, the transition matrix M expressing the H_I on the S^J where $J = (j_1, \dots, j_p)$ satisfies:

$$M_{J,I} := x_{1j_1-1} x_{j_1 j_2-1} \dots x_{j_p n}. \quad (52)$$

where $x_{ij} = u_{ij}$ if $i-1$ is not a descent of I and v_{ij} otherwise.

As we shall sometimes need different normalizations, we also define for arbitrary almost triangular matrices U, V

$$H'(W, G) = \text{rdet} \begin{bmatrix} w_{11}G_1 & w_{12}G_2 & \dots & w_{1n-1}G_{n-1} & w_{1n}G_n \\ w_{21} & w_{22}G_1 & \dots & w_{2n-1}G_{n-2} & w_{2n}G_{n-1} \\ 0 & w_{32} & \dots & w_{3n-3}G_{n-3} & w_{3n}G_{n-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & w_{nn-1} & w_{nn}G_1 \end{bmatrix} \quad (53)$$

and

$$H'_I(U, V) = H'(W(I), S(A)). \quad (54)$$

7.3 Expansion on the basis (S^I)

For a composition $I = (i_1, \dots, i_r)$ of n , let I^\sharp be the integer vector of length n obtained by replacing each entry k of I by the sequence $(k, 0, \dots, 0)$ ($k-1$ zeros):

$$I^\sharp = (i_1 0^{i_1-1} i_2 0^{i_2-1} \dots i_r 0^{i_r-1}). \quad (55)$$

Proposition 7.1 *The expansion of $H'(W, S)$ on the S -basis is given by*

$$H'(W, S) = \sum_{I=n} \varepsilon(\sigma_I) w_{1\sigma_I(1)} \dots w_{n\sigma_I(n)} S^I. \quad (56)$$

7.4 Expansion of the basis (R_I)

Proposition 7.2 For $I = (i_1, \dots, i_r)$ be a composition of n , denote by W_I the product of diagonal minors of the matrix W taken over the first i_1 rows and columns, then the next i_2 ones and so on. Then,

$$H'(W, S) = \sum_{I \models n} W_I R_I. \quad (57)$$

7.5 Examples

7.5.1 A family with factoring coefficients

Theorem 7.3 Let U and V be defined by

$$u_{ij} = \begin{cases} x^j - y^j & \text{if } i = 1, \\ aq_{i-1}x^{j-i+1} - y^{j-i+1} & \text{if } 1 < i < j + 2, \\ 0 & \text{otherwise,} \end{cases} \quad (58)$$

$$v_{ij} = \begin{cases} x^j - y^j & \text{if } i = 1, \\ x^{j-i+1} - bu_{n+1-i}y^{j-i+1} & \text{if } 1 < i < j + 2, \\ 0 & \text{otherwise.} \end{cases} \quad (59)$$

Then the coefficients W_J of the expansion of $H'_I(U, V)$ on the ribbon basis all factor as products of binomials.

The formula for the coefficient of R_n is simple enough: if one orders the factors of $\det(U)$ and $\det(V)$ as

$$Z_n = (x - aq_1y, x - aq_2y, \dots, x - aq_{n-1}y) \quad (60)$$

and

$$Z'_n = (y - bu_{n-1}x, y - bu_{n-2}x, \dots, y - bu_1x), \quad (61)$$

then, the coefficient of R_n in $H'_I(U, V)$ is

$$(x - y) \prod_{d \in \text{Des}(I)} z'_d \prod_{e \notin \text{Des}(I)} z_e. \quad (62)$$

A more careful analysis allows one to compute directly the coefficient of R_J in H'_I . For example,

$$\begin{aligned} \frac{H'_3(U, V)}{(x - y)} &= (x - aq_1y)(x - aq_2y)R_3 + (x - aq_1y)(aq_2x - y)R_{21} \\ &\quad + a(x - y)(q_1x - q_2y)R_{12} + (aq_1x - y)(aq_2x - y)R_{111}. \end{aligned} \quad (63)$$

$$\begin{aligned} \frac{H'_{21}(U, V)}{(x - y)} &= (x - aq_1y)(bu_1x - y)R_3 + (x - aq_1y)(x - bu_1y)R_{21} \\ &\quad + (abq_1u_1x - y)(x - y)R_{12} + (aq_1x - y)(x - bu_1y)R_{111}. \end{aligned} \quad (64)$$

7.5.2 An analogue of the $(1-t)/(1-q)$ transform

Recall that for commutative symmetric functions, the $(1-t)/(1-q)$ transform is defined in terms of the power-sums by

$$p_n \left(\frac{1-t}{1-q} X \right) = \frac{1-t^n}{1-q^n} p_n(X). \quad (65)$$

With the specialization $x = 1$, $y = t$, $q_i = q^i$, $u_i = 1$, $a = b = 1$, one obtains a basis such that for a hook composition $I = (n-k, 1^k)$, the commutative image of $H'_I(U, V)$ becomes the $(1-t)/(1-q)$ transform of the Schur function $s_{n-k, 1^k}$.

7.5.3 An analogue of the Macdonald P -basis

With the specialization $x = 1$, $y = t$, $q_i = q^i$, $u_i = t^i$, $a = b = 1$, one obtains an analogue of the Macdonald P -basis, in the sense that for hook compositions $I = (n-k, 1^k)$, the commutative image of H'_I is proportional to the Macdonald polynomial $P_{n-k, 1^k}(q, t; X)$.

References

- [1] N. BERGERON and M. ZABROCKI, q and q, t -analogs of non-commutative symmetric functions, *Discrete Math.* **298** (2005), no. 1-3, 79–103.
- [2] G. DUCHAMP, F. HIVERT and J.-Y. THIBON, *Noncommutative symmetric functions VI: free quasi-symmetric functions and related algebras* *Inter. J. of Alg. and Comput.* **12** (2002), 671–717.
- [3] I.M. GELFAND, D. KROB, A. LASCoux, B. LECLERC, V.S. RETAKH and J.-Y. THIBON, *Noncommutative symmetric functions*, *Adv. Math.* **112** (1995), 218–348.
- [4] I. GELFAND and V. RETAKH, Determinants of matrices over noncommutative rings, *Funct. Anal. Appl.*, **25** (1991), 91–102.
- [5] I. GESSEL, *Multipartite P -partitions and inner products of skew Schur functions*, *Contemp. Math.* **34** (1984), 289–301.
- [6] M. HAIMAN, *Macdonald polynomials and geometry*, in *New perspectives in algebraic combinatorics*, *MSRI Publications* **38** (1999), 207–254.
- [7] F. HIVERT, *Hecke algebras, difference operators, and quasi-symmetric functions*, *Adv. Math.* **155** (2000), 181–238.
- [8] F. HIVERT, A. LASCoux and J.-Y. THIBON, *Noncommutative symmetric functions with two and more parameters*, preprint arXiv: math.CO/0106191.
- [9] F. HIVERT, A. LASCoux and J.-Y. THIBON, *Macdonald polynomials with two vector parameters*, unpublished notes, 2001.
- [10] D. KROB, B. LECLERC and J.-Y. THIBON, *Noncommutative symmetric functions II: Transformations of alphabets*, *Internat. J. Alg. Comp.* **7** (1997), 181–264.
- [11] A. LASCoux, J.-C. NOVELLI and J.-Y. THIBON, *Noncommutative symmetric functions with matrix parameters*, arXiv:1110.3209.
- [12] I.G. MACDONALD, *Symmetric functions and Hall polynomials*, 2nd edition, Oxford, 1995.
- [13] J.-C. NOVELLI, J.-Y. THIBON and L.K. WILLIAMS, *Combinatorial Hopf algebras, noncommutative Hall-Littlewood functions, and permutation tableaux*, *Advances in Math.* **224** (2010), 1311–1348.
- [14] L. TEVLIN, *Noncommutative Symmetric Hall-Littlewood Polynomials*, *Proc. FPSAC 2011, DMTCS Proc.* **AO** 2011, 915–926.