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Minimal transitive factorizations of a permutation of type (p, q)

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Abstract. We give a combinatorial proof of Goulden and Jackson’s formula for the number of minimal transitive factorizations of a permutation when the permutation has two cycles. We use the recent result of Goulden, Nica, and Oancea on the number of maximal chains of annular noncrossing partitions of type B .

Résumé. Nous donnons une preuve combinatoire de formule de Goulden et Jackson pour le nombre de factorisations transitives minimales d’une permutation lorsque la permutation a deux cycles. Nous utilisons le résultat récent de Goulden, Nica, et Oancea sur le nombre de chaînes maximales des partitions non-croisées annulaires de type B .

Keywords: minimal transitive factorizations, annular noncrossing partitions, bijective proof

1 Introduction

Given an integer partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of n , denote by α_λ the permutation

$$(1 \dots \lambda_1)(\lambda_1 + 1 \dots \lambda_1 + \lambda_2) \dots (n - \lambda_\ell + 1 \dots n)$$

of the set $\{1, 2, \dots, n\}$ in the cycle notation. Let \mathcal{F}_λ be the set of all $(n + \ell - 2)$ -tuples $(\eta_1, \dots, \eta_{n+\ell-2})$ of transpositions such that

- (1) $\eta_1 \cdots \eta_{n+\ell-2} = \alpha_\lambda$ and
- (2) $\{\eta_1, \dots, \eta_{n+\ell-2}\}$ generates the symmetric group \mathcal{S}_n .

Such tuples are called *minimal transitive factorizations* of the permutation α_λ of type λ , which are related to the branched covers of the sphere suggested by Hurwitz [Hur91, Str96].

In 1997, using algebraic methods Goulden and Jackson [GJ97] proved that

$$|\mathcal{F}_\lambda| = (n + \ell - 2)! n^{\ell-3} \prod_{i=1}^{\ell} \frac{\lambda_i^{\lambda_i}}{(\lambda_i - 1)!}. \quad (1)$$

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Bousquet-Mélou and Schaeffer [BMS00] proved a more general formula than (1) and obtained (1) using the principle of inclusion and exclusion. Irving [Irv09] studied the enumeration of minimal transitive factorizations into cycles instead of transpositions.

If $\lambda = (n)$, the formula (1) yields

$$|\mathcal{F}_{(n)}| = n^{n-2}, \quad (2)$$

and there are several combinatorial proofs of (2) [Bia02, GY02, Mos89].

If $\lambda = (p, q)$, the formula (1) yields

$$|\mathcal{F}_{(p,q)}| = \frac{pq}{p+q} \binom{p+q}{q} p^p q^q. \quad (3)$$

A few special cases of (3) have bijective proofs: by Kim and Seo [KS03] for the case $(p, q) = (1, n-1)$, and by Rattan [Rat06] for the cases $(p, q) = (2, n-2)$ and $(p, q) = (3, n-3)$. There are no simple combinatorial proofs for other (p, q) .

Recently, Goulden et al. [GNO11] showed that the number of maximal chains in the poset $NC^{(B)}(p, q)$ of annular noncrossing partitions of type B is

$$\binom{p+q}{q} p^p q^q + \sum_{c \geq 1} 2c \binom{p+q}{p-c} p^{p-c} q^{q+c}. \quad (4)$$

Interestingly it turns out that half the sum in (4) is equal to the number in (3):

$$\sum_{c \geq 1} c \binom{p+q}{p-c} p^{p-c} q^{q+c} = \frac{pq}{p+q} \binom{p+q}{q} p^p q^q.$$

In this paper we will give a combinatorial proof of (3) using the results in [GNO11]. The rest of this paper is organized as follows. In Section 2 we recall the poset $\mathcal{S}_{\text{nc}}^B(p, q)$ of annular noncrossing permutations of type B which is isomorphic to the poset $NC^{(B)}(p, q)$ of annular noncrossing partitions of type B , and show that the number of connected maximal chains in $\mathcal{S}_{\text{nc}}^B(p, q)$ is equal to $\frac{2pq}{p+q} \binom{p+q}{q} p^p q^q$. In Section 3 we prove that there is a 2-1 map from the set of connected maximal chains in $\mathcal{S}_{\text{nc}}^B(p, q)$ to $\mathcal{F}_{(p,q)}$, thus completing a combinatorial proof of (3).

We note that the present paper is part of [KSS12]. In the full version [KSS12] we give another combinatorial proof of (3) by introducing marked annular noncrossing permutations of type A .

2 Connected maximal chains

A *signed permutation* is a permutation σ on $\{\pm 1, \dots, \pm n\}$ satisfying $\sigma(-i) = -\sigma(i)$ for all $i \in \{1, \dots, n\}$. We denote by B_n the set of signed permutations on $\{\pm 1, \dots, \pm n\}$.

We will use the two notations

$$\begin{aligned} [a_1 a_2 \dots a_k] &= (a_1 a_2 \dots a_k - a_1 - a_2 \dots - a_k), \\ ((a_1 a_2 \dots a_k)) &= (a_1 a_2 \dots a_k)(-a_1 - a_2 \dots - a_k), \end{aligned}$$

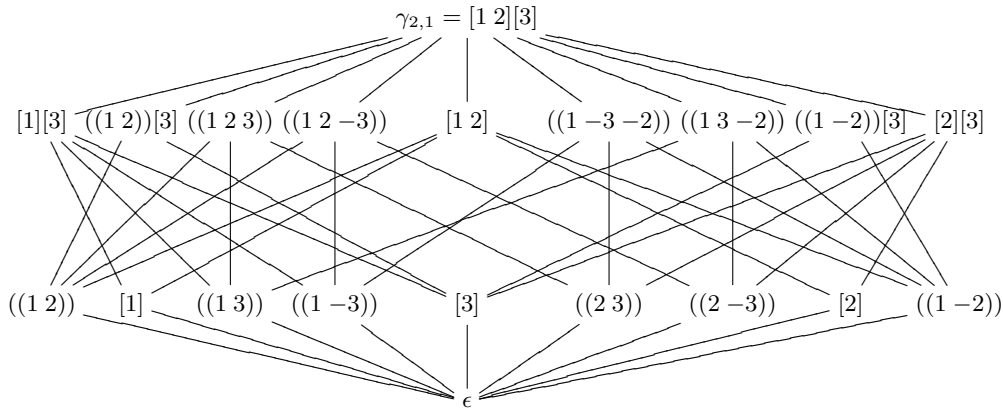


Fig. 1: The Hasse diagram for $\mathcal{S}_{nc}^B(2, 1)$.

and call $[a_1 a_2 \dots a_k]$ a *zero cycle* and $((a_1 a_2 \dots a_k))$ a *paired nonzero cycle*. We also call the cycles $\epsilon_i := [i] = (i - i)$ and $((i j))$ *type B transpositions*, or simply *transpositions* if there is no possibility of confusion.

For $\pi \in B_n$, the *absolute length* $\ell(\pi)$ is defined to be the smallest integer k such that π can be written as a product of k type B transpositions. The *absolute order* on B_n is defined by

$$\pi \leq \sigma \iff \ell(\sigma) = \ell(\pi) + \ell(\pi^{-1}\sigma).$$

From now, we fix positive integers p and q . The poset $\mathcal{S}_{nc}^B(p, q)$ of *annular noncrossing permutations of type B* is defined by

$$\mathcal{S}_{nc}^B(p, q) := [\epsilon, \gamma_{p,q}] = \{\sigma \in B_{p+q} : \epsilon \leq \sigma \leq \gamma_{p,q}\} \subseteq B_{p+q},$$

where ϵ is the identity in B_{p+q} and $\gamma_{p,q} = [1 \dots p][p+1 \dots p+q]$. Figure 1 shows the Hasse diagram for $\mathcal{S}_{nc}^B(2, 1)$. Then $\mathcal{S}_{nc}^B(p, q)$ is a graded poset with rank function

$$\text{rank}(\sigma) = (p + q) - (\# \text{ of paired nonzero cycles of } \sigma). \tag{5}$$

Nica and Oancea [NO09] showed that $\sigma \in \mathcal{S}_{nc}^B(p, q)$ if and only if σ can be drawn without crossing arrows inside an annulus in which the outer circle has integers $1, 2, \dots, p, -1, -2, \dots, -p$ in clockwise order and the inner circle has integers $p+1, p+2, \dots, p+q, -p-1, -p-2, \dots, -p-q$ in counterclockwise order, see Figure 2. They also showed that $\mathcal{S}_{nc}^B(p, q)$ is isomorphic to the poset $NC^{(B)}(p, q)$ of annular noncrossing partitions of type B .

A paired nonzero cycle $((a_1 a_2 \dots a_k))$ is called *connected* if the set $\{a_1, \dots, a_k\}$ intersects with both $\{\pm 1, \dots, \pm p\}$ and $\{\pm(p+1), \dots, \pm(p+q)\}$, and *disconnected* otherwise. A zero cycle is always considered to be disconnected. For $\sigma \in \mathcal{S}_{nc}^B(p, q)$, the *connectivity* of σ is the number of connected paired nonzero cycles of σ .

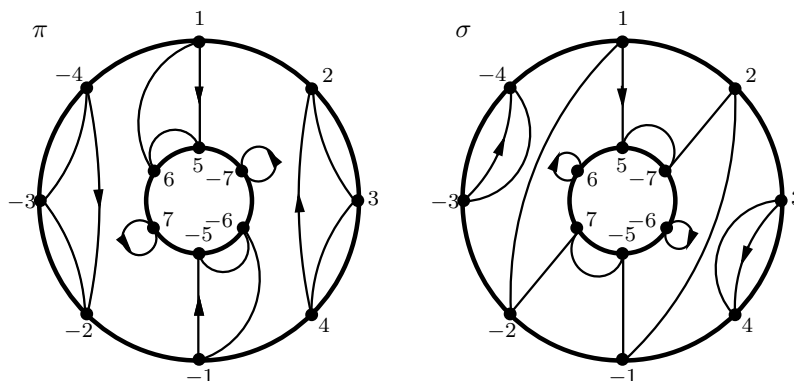


Fig. 2: $\pi = ((1\ 5\ 6))((2\ 3\ 4))$ and $\sigma = [1\ 5\ -7\ 2]((3\ 4))$ in $\mathcal{S}_{nc}^B(4, 3)$

We say that a maximal chain $C = \{\epsilon = \pi_0 < \pi_1 < \dots < \pi_{p+q} = \gamma_{p,q}\}$ of $\mathcal{S}_{nc}^B(p, q)$ is *disconnected* if the connectivity of each π_i is zero. Otherwise, C is called *connected*. Denote by $\mathcal{CM}(\mathcal{S}_{nc}^B(p, q))$ the set of connected maximal chains of $\mathcal{S}_{nc}^B(p, q)$.

For a maximal chain $C = \{\pi_0 < \pi_1 < \dots < \pi_n\}$ of the interval $[\pi_0, \pi_n]$, we define $\varphi(C) = (\tau_1, \tau_2, \dots, \tau_n)$, where $\tau_i = \pi_i^{-1}\pi_{i+1}$. Note that each τ_i is a type B transposition and $\pi_i = \tau_1\tau_2 \dots \tau_i$ for all $i = 1, 2, \dots, n$.

Lemma 1 *If C is a connected maximal chain of $\mathcal{S}_{nc}^B(p, q)$, then $\varphi(C)$ has no transpositions of the form $\epsilon_i = [i]$ and has at least one connected transposition. If C is a disconnected maximal chain of $\mathcal{S}_{nc}^B(p, q)$, then $\varphi(C)$ has only disconnected transpositions.*

Proof: By (5), σ covers π in $\mathcal{S}_{nc}^B(p, q)$ if and only if one of the following conditions holds, see [NO09, Proposition 2.2]:

- (a) $\pi^{-1}\sigma = \epsilon_i$ and the cycle containing i in π is nonzero, i.e., π has $((i \dots))$ and σ has $[i \dots]$.
- (b) $\pi^{-1}\sigma = ((i\ j))$ and no two of $i, -i, j, -j$ belong to the same cycle in π with $|i| \neq |j|$, i.e., π has $((i \dots))(j \dots)$ and σ has $((i \dots j \dots))$.
- (c) $\pi^{-1}\sigma = ((i\ j))$ and the cycle containing i in π is nonzero and the cycle containing j in π is zero with $|i| \neq |j|$, i.e., π has $((i \dots))[j \dots]$ and σ has $[i \dots j \dots]$.
- (d) $\pi^{-1}\sigma = ((i\ j))$ and i and $-j$ belong to the same nonzero cycle in π with $|i| \neq |j|$, i.e., π has $((i \dots -j \dots))$ and σ has $[i \dots][-j \dots]$.

If σ covers π in $\mathcal{S}_{nc}^B(p, q)$, we have $zc(\sigma) \geq zc(\pi)$, where $zc(\sigma)$ is the the number of zero cycles in σ . More precisely we have

$$zc(\sigma) - zc(\pi) = \begin{cases} 0 & \text{if type (b) or (c),} \\ 1 & \text{if type (a),} \\ 2 & \text{if type (d).} \end{cases}$$

Since $\gamma_{p,q}$ has two zero cycles, each $\pi \in \mathcal{S}_{nc}^B(p, q)$ has at most two zero cycles. Moreover, if π has two zero cycles, then one of them belongs to $\{\pm 1, \dots, \pm p\}$ and the other belongs to $\{\pm(p+1), \dots, \pm(p+q)\}$. Consider a maximal chain C in $\mathcal{S}_{nc}^B(p, q)$.

- If C has a permutation π with $zc(\pi) = 1$, there are two cover relations of type (a) and no cover relations of type (d) in C . For each cover relation $\pi < \sigma$ of type (a), (b), or (c), σ is obtained by merging cycles in π . Since $\gamma_{p,q}$ has only disconnected cycles, all permutations in C are disconnected, which implies that C is disconnected.
- Otherwise, there is a cover relation $\pi < \sigma$ of type (d) in C . Then σ has two zero cycles $[i \dots]$ and $[-j \dots]$, one of which is contained in $\{\pm 1, \dots, \pm p\}$ and the other is contained in $\{\pm(p+1), \dots, \pm(p+q)\}$. Thus π has a connected nonzero cycle $((i \dots -j \dots))$, and C is connected. Since C has no cover relations of type (a), $\varphi(C)$ has no transposition of the form ϵ_i .

Therefore, if C is a disconnected maximal chain of $\mathcal{S}_{nc}^B(p, q)$, then $\varphi(C)$ has two transpositions of the form ϵ_i . So all transpositions of $\varphi(C)$ are disconnected. Also, if C is a connected maximal chain of $\mathcal{S}_{nc}^B(p, q)$, then $\varphi(C)$ has no transposition of the form ϵ_i and has at least one connected transposition. \square

The following proposition is a refinement of (4).

Proposition 2 *The number of disconnected maximal chains of $\mathcal{S}_{nc}^B(p, q)$ is equal to*

$$\binom{p+q}{q} p^p q^q \tag{6}$$

and the number of connected maximal chains of $\mathcal{S}_{nc}^B(p, q)$ is equal to

$$\sum_{c \geq 1} 2c \binom{p+q}{p-c} p^{p-c} q^{q+c}. \tag{7}$$

We now prove the following identity that appears in the introduction. The proof is due to Krattenthaler [Kra].

Lemma 3 *We have*

$$\sum_{c \geq 1} c \binom{p+q}{p-c} p^{p-c} q^{q+c} = \frac{pq}{p+q} \binom{p+q}{q} p^p q^q. \tag{8}$$

Proof: Since $c = p \cdot \frac{q+c}{p+q} - q \cdot \frac{p-c}{p+q}$, we have

$$\begin{aligned} \sum_{c=0}^p c \binom{p+q}{p-c} p^{p-c} q^{q+c} &= \sum_{c=0}^p \left(p \cdot \frac{q+c}{p+q} - q \cdot \frac{p-c}{p+q} \right) \binom{p+q}{p-c} p^{p-c} q^{q+c} \\ &= \sum_{c=0}^p \left(\binom{p+q-1}{p-c} p^{p-c+1} q^{q+c} - \binom{p+q-1}{p-c-1} p^{p-c} q^{q+c+1} \right) \\ &= \binom{p+q-1}{p} p^{p+1} q^q = \frac{pq}{p+q} \binom{p+q}{p} p^p q^q. \end{aligned}$$

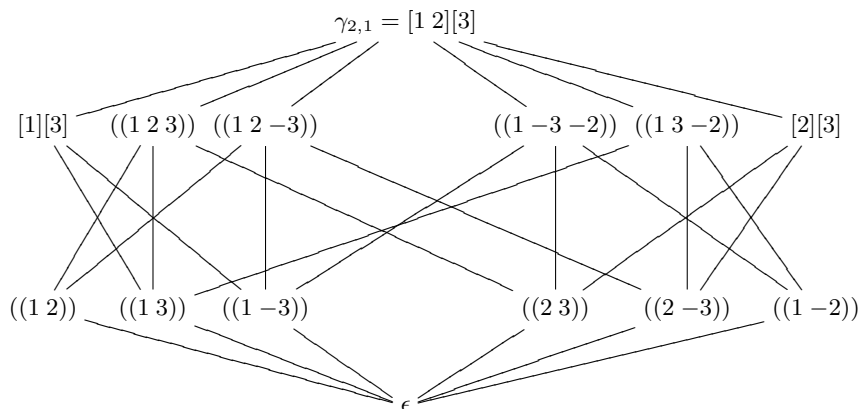


Fig. 3: Connected maximal chains in $\mathcal{S}_{nc}^B(2, 1)$.

□

By Proposition 2 and Lemma 3, we get the following.

Corollary 4 *The number of connected maximal chains of $\mathcal{S}_{nc}^B(p, q)$ is equal to*

$$\frac{2pq}{p+q} \binom{p+q}{q} p^p q^q. \tag{9}$$

For example, Figure 3 illustrates $16 = \frac{4}{3} \binom{3}{1} 2^2$ connected maximal chains of $\mathcal{S}_{nc}^B(2, 1)$.

By Corollary 4, in order to prove (3) combinatorially it is sufficient to find a 2-1 map from $\mathcal{CM}(\mathcal{S}_{nc}^B(p, q))$ to $\mathcal{F}_{(p,q)}$. We will find such a map in the next section.

Remark 1. One can check that the factorizations $\varphi(C)$ coming from connected maximal chains C in $\mathcal{S}_{nc}^B(p, q)$ are precisely the minimal factorizations of $\gamma_{p,q}$ in the Weyl group D_{p+q} . Thus Corollary 4 can be restated as follows: the number of minimal factorizations of $\gamma_{p,q}$ in D_{p+q} is equal to $\frac{2pq}{p+q} \binom{p+q}{q} p^p q^q$. Goupil [Gou95, Theorem 3.1] also proved this result by finding a recurrence relation.

Remark 2. Since the proof of Lemma 3 is a simple manipulation, it is easy and straightforward to construct a combinatorial proof for the identity in Lemma 3. Together with the result in Section 3 we get a combinatorial proof of (9). It would be interesting to find a direct bijective proof of (9) without using Lemma 3.

3 A 2-1 map from $\mathcal{CM}(\mathcal{S}_{nc}^B(p, q))$ to $\mathcal{F}_{(p,q)}$

Recall that a minimal transitive factorization of $\alpha_{p,q} = (1 \dots p)(p + 1 \dots p + q)$ is a sequence $(\eta_1, \dots, \eta_{p+q})$ of transpositions in \mathcal{S}_{p+q} such that

(1) $\eta_1 \cdots \eta_{p+q} = \alpha_{p,q}$ and

(2) $\{\eta_1, \dots, \eta_{p+q}\}$ generates \mathcal{S}_{p+q} ,

and $\mathcal{F}_{(p,q)}$ is the set of minimal transitive factorizations of $\alpha_{p,q}$.

In this section we will prove the following theorem.

Theorem 5 *There is a 2-1 map from the set of connected maximal chains in $\mathcal{S}_{\text{nc}}^B(p, q)$ to the set $\mathcal{F}_{(p,q)}$ of minimal transitive factorizations of $\alpha_{p,q}$.*

In order to prove Theorem 5 we need some definitions.

Definition 6 (Two maps $(\cdot)^+$ and $|\cdot|$) *We introduce the following two maps.*

(1) *The map $(\cdot)^+ : B_n \rightarrow B_n$ is defined by*

$$\sigma^+(i) = \begin{cases} |\sigma(i)| & \text{if } i > 0, \\ -|\sigma(i)| & \text{if } i < 0. \end{cases}$$

(2) *The map $|\cdot| : B_n \rightarrow \mathcal{S}_n$ is defined by $|\sigma|(i) = |\sigma(i)|$ for all $i \in \{1, \dots, n\}$.*

Definition 7 *A $(p+q)$ -tuple $(\tau_1, \dots, \tau_{p+q})$ of transpositions in B_{p+q} is called a minimal transitive factorization of type B of $\gamma_{p,q} = [1 \dots p][p+1 \dots p+q]$ if it satisfies*

(1) $\tau_1 \dots \tau_{p+q} = \gamma_{p,q}$

(2) $\{|\tau_1|, \dots, |\tau_{p+q}|\}$ generates \mathcal{S}_{p+q} .

Denote by $\mathcal{F}_{(p,q)}^{(B)}$ the set of minimal transitive factorizations of type B of $\gamma_{p,q}$.

Definition 8 *A $(p+q)$ -tuple $(\sigma_1, \dots, \sigma_{p+q})$ of transpositions in B_{p+q} is called a positive minimal transitive factorization of type B of $\beta_{p,q} = ((1 \dots p))(p+1 \dots p+q)$ if it satisfies*

(1) $\sigma_1 \dots \sigma_{p+q} = \beta_{p,q}$,

(2) $\{|\sigma_1|, \dots, |\sigma_{p+q}|\}$ generates \mathcal{S}_{p+q} ,

(3) $\sigma_i = \sigma_i^+$ for all $i = 1, \dots, p+q$.

Denote by $\mathcal{F}_{(p,q)}^+$ the set of positive minimal transitive factorizations of type B of $\beta_{p,q}$.

For the rest of this section we will prove the following:

1. The map $\varphi : \mathcal{CM}(\mathcal{S}_{\text{nc}}^B(p, q)) \rightarrow \mathcal{F}_{(p,q)}^{(B)}$ is a bijection. (Lemma 9)
2. There is a 2-1 map $(\cdot)^+ : \mathcal{F}_{(p,q)}^{(B)} \rightarrow \mathcal{F}_{(p,q)}^+$. (Lemma 11)
3. There is a bijection $|\cdot| : \mathcal{F}_{(p,q)}^+ \rightarrow \mathcal{F}_{(p,q)}$. (Lemma 10)

By the above three statements the composition $|\varphi^+| := |\cdot| \circ (\cdot)^+ \circ \varphi$ is a 2-1 map from $\mathcal{CM}(\mathcal{S}_{\text{nc}}^B(p, q))$ to $\mathcal{F}_{(p,q)}$, which completes the proof of Theorem 5. Since the proofs of the first and the third statements are simpler, we will present these first.

Lemma 9 *The map $\varphi : \mathcal{CM}(\mathcal{S}_{\text{nc}}^B(p, q)) \rightarrow \mathcal{F}_{(p, q)}^{(B)}$ is a bijection.*

Proof: Given a connected maximal chain $C = \{\epsilon = \pi_0 < \pi_1 < \dots < \pi_{p+q} = \gamma_{p, q}\}$ in $\mathcal{S}_{\text{nc}}^B(p, q)$, the elements in the sequence $\varphi(C) = (\tau_1, \dots, \tau_{p+q})$ are transpositions with $\tau_1 \cdots \tau_{p+q} = \gamma_{p, q}$. By Lemma 1, at least one of τ_i 's is connected. Thus $\{|\tau_1|, \dots, |\tau_{p+q}|\}$ generates \mathcal{S}_{p+q} , and $\varphi(C) \in \mathcal{F}_{(p, q)}^{(B)}$. Conversely, if $\tau = (\tau_1, \dots, \tau_{p+q}) \in \mathcal{F}_{(p, q)}^{(B)}$, then $\varphi^{-1}(\tau) = \{\epsilon = \pi_0 < \pi_1 < \dots < \pi_{p+q} = \gamma_{p, q}\}$, where $\pi_i = \tau_1 \cdots \tau_i$, is a connected maximal chain in $\mathcal{S}_{\text{nc}}^B(p, q)$ because $\{|\tau_1|, \dots, |\tau_{p+q}|\}$ generates \mathcal{S}_{p+q} . \square

Lemma 10 *There is a bijection $|\cdot| : \mathcal{F}_{(p, q)}^+ \rightarrow \mathcal{F}_{(p, q)}$.*

Proof: Let $(\sigma_1, \dots, \sigma_{p+q}) \in \mathcal{F}_{(p, q)}^+$. Each σ_i can be written as $\sigma_i = ((j \ k))$ for some positive integers j and k . In this case we let $\eta_i = |\sigma_i| = (j \ k) \in \mathcal{S}_{p+q}$. Then the map $|\cdot| : \mathcal{F}_{(p, q)}^+ \rightarrow \mathcal{F}_{(p, q)}$ sending $(\sigma_1, \dots, \sigma_{p+q})$ to $(\eta_1, \dots, \eta_{p+q})$ is a bijection. \square

Recall $\epsilon_i = [i] = (i \ -i)$. We write $\overline{((i \ j))} := ((i \ -j))$. It is easy to see that for $i, j \in \{\pm 1, \dots, \pm(p+q)\}$, we have

$$[i \ j] = \epsilon_i((i \ j)) = ((i \ j))\epsilon_j = \overline{((i \ j))}\epsilon_i = \epsilon_j\overline{((i \ j))}. \quad (10)$$

Lemma 11 *There is a 2-1 map $(\cdot)^+ : \mathcal{F}_{(p, q)}^{(B)} \rightarrow \mathcal{F}_{(p, q)}^+$.*

Here we only describe the map $(\cdot)^+$: For $(\tau_1, \tau_2, \dots, \tau_{p+q}) \in \mathcal{F}_{(p, q)}^{(B)}$, we define $(\tau_1, \tau_2, \dots, \tau_{p+q})^+ = (\tau_1^+, \tau_2^+, \dots, \tau_{p+q}^+)$. Since $\tau_1^+ \cdots \tau_{p+q}^+ = \gamma_{p, q}^+ = \beta_{p, q}$, we have $(\tau_1, \tau_2, \dots, \tau_{p+q})^+ \in \mathcal{F}_{(p, q)}^+$. Let us fix $\sigma = (\sigma_1, \sigma_2, \dots, \sigma_{p+q}) \in \mathcal{F}_{(p, q)}^+$. If $\tau = (\tau_1, \dots, \tau_{p+q}) \in \mathcal{F}_{(p, q)}^{(B)}$ satisfies $\tau^+ = \sigma$, then $\tau' = (\tau'_1 \cdots \tau'_{p+q}) \in \mathcal{F}_{(p, q)}^{(B)}$ defined by

$$\tau'_i = \begin{cases} \tau_i & \text{if } \tau_i \text{ is disconnected} \\ \overline{\tau_i} & \text{if } \tau_i \text{ is connected,} \end{cases} \quad (11)$$

also satisfies $(\tau')^+ = \sigma$. Thus the map $(\cdot)^+$ is two-to-one. For the detailed proof, see [KSS12].

For example, let $\sigma = ((1 \ 2)), ((2 \ 5)), ((2 \ 3)), ((4 \ 5)), ((3 \ 4)) \in \mathcal{F}_{(3, 2)}^+$ be the following factorization

$$\beta_{3, 2} = ((1 \ 2 \ 3))((4 \ 5)) = ((1 \ 2))((2 \ 5))((2 \ 3))((4 \ 5))((3 \ 4)).$$

Since $\gamma_{3, 2} = \epsilon_4 \epsilon_1 \beta_{3, 2}$, we can obtain a factorization of $\gamma_{3, 2}$ from σ as follows:

$$\begin{aligned} \gamma_{3, 2} &= [1 \ 2 \ 3][4 \ 5] = \epsilon_4 \epsilon_1 ((1 \ 2))((2 \ 5))((2 \ 3))((4 \ 5))((3 \ 4)) \\ &= \epsilon_4 \epsilon_2 \overline{((1 \ 2))}((2 \ 5))((2 \ 3))((4 \ 5))((3 \ 4)) \\ &= \epsilon_4 \epsilon_3 ((1 \ 2))\overline{((2 \ 5))}\overline{((2 \ 3))}((4 \ 5))((3 \ 4)) \\ &= \epsilon_4 \epsilon_4 ((1 \ 2))\overline{((2 \ 5))}((2 \ 3))\overline{((4 \ 5))}\overline{((3 \ 4))} \\ &= ((1 \ 2))\overline{((2 \ 5))}((2 \ 3))\overline{((4 \ 5))}\overline{((3 \ 4))}. \end{aligned}$$

Thus $\tau = \left(((1\ 2)), \overline{((2\ 5))}, ((2\ 3)), \overline{((4\ 5))}, \overline{((3\ 4))} \right) \in \mathcal{F}_{(3,2)}^{(B)}$ satisfies $\tau^+ = \sigma$. The factorization $\tau' = \left(((1\ 2)), ((2\ 5)), ((2\ 3)), \overline{((4\ 5))}, ((3\ 4)) \right)$ obtained by toggling the connected transpositions of τ also satisfies $(\tau')^+ = \sigma$.

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